

M2.1 - Transformation Geometry

CLAUDIU C. REMSING

*‘Beauty is truth, truth beauty’ – that is all
Ye know on earth, and all ye need to know.*

JOHN KEATS

Imagination is more important than knowledge.

ALBERT EINSTEIN

*Do not just pay attention to the words;
Instead pay attention to meaning behind the words.
But, do not just pay attention to meanings behind the words;
Instead pay attention to your deep experience of those meanings.*

TENZIN GYATSO, THE 14TH DALAI LAMA

Where there is matter, there is geometry.

JOHANNES KEPLER

Geometry is the art of good reasoning from poorly drawn figures.

ANONYMOUS

What is a geometry ? The question [...] is metamathematical rather than mathematical, and consequently mathematicians of unquestioned competence may (and, indeed, do) differ in the answer they give to it. Even among those mathematicians called geometers there is no generally accepted definition of the term. It has been observed that the abstract, postulational method that has permeated nearly all parts of modern mathematics makes it difficult, if not meaningless, to mark with precision the boundary of that mathematical domain which should be called geometry. To some, geometry is not so much a subject as it is a point of view – a way of looking at a subject – so that geometry is the mathematics that a geometer does ! To others, geometry is a language that provides a very useful and suggestive means of discussing almost every part of mathematics (just as, in former days, French was the language of diplomacy); and there are, doubtless, some mathematicians who find such a query without any real significance and who, consequently, will disdain to vouchsafe any answer at all to it.

LEONARD BLUMENTHAL

Geometry is a uniquely favorable environment for young students to learn the spirit of pure mathematics and exercise their intuition.

JOHN D. SMITH

Any objective definition of geometry would probably include the whole of mathematics.

J.H.C. WHITEHEAD

Meaning is important in mathematics and geometry is an important source of that meaning.

DAVID HILBERT

What is Geometry ?

The Greek word for **geometry**, *γεωμετρία*, which means *measurement of the earth*, was used by the historian HERODOTUS (C.484-C.425 B.C.), who wrote that in ancient Egypt people used geometry to restore their land after the inundation of the Nile. Thus the theoretical use of figures for practical purposes goes back to pre-Greek antiquity. Tradition holds that THALES OF MILET (C.639-C.546 B.C.) knew some properties of congruent triangles and used them for indirect measurement, and that PYTHAGORAS (572-492 B.C.) and his school had the idea of systematizing this knowledge by means of *proofs*. Thus the Greeks made two vital contributions to geometry : they made geometry *abstract* and *deductive*. Starting from unquestionable premisses (or

axioms), and basic laws of thought, they would reason and prove their way towards previously unguessed knowledge. This whole process was codified by EUCLID (c.300 B.C.) in his book, the *Elements*, the most successful scientific textbook ever written. In this work, we can see the entire mathematical knowledge of the time presented as a *logical system*.

Geometry – in today’s usage – **means the branch of mathematics dealing with spatial figures**. Within mathematics, it plays a significant role. Geometry consists of a variety of intellectual structures, closely related to each other and to the original experiences of *space* and *motion*. A brief historical account of the subsequent development of this “science of space” from its Greek roots through modern times is given now.

In ancient Greece, however, all of mathematics was regarded as geometry. Algebra was introduced in Europe from the Middle East toward the end of the Middle Ages and was further developed during the Renaissance. In the 17th and 18th centuries, with the development of analysis, geometry achieved parity with algebra and analysis.

As RENÉ DESCARTES (1596-1650) pointed out, however, figures and numbers are closely related. Geometric figures can be treated algebraically (or analytically) by means of *coordinates*; conversely, algebraic facts can be expressed geometrically. **Analytic geometry** was developed in the 18th century, especially by LEONHARD EULER (1707-1783), who for the first time established a complete algebraic theory of curves of the second order. Previously, these curves had been studied by APOLLONIUS OF PERGA (262-C.200 B.C.) as *conic sections*.

The idea of DESCARTES was fundamental to the development of analysis in the 18th century. Toward the end of that century, analysis was again applied to geometry. GASPARD MONGE (1746-1818) can be regarded as a forerunner of **differential geometry**. CARL GAUSS (1777-1855) founded the *theory of surfaces* by introducing concepts of the geometry of surfaces. The influence that differential-geometric investigations of curves and surfaces have exerted

upon branches of mathematics, physics, and engineering has been profound.

However, we cannot say that the analytic method is always the best manner of dealing with geometric problems. The method of treating figures directly without using coordinates is called **synthetic geometry**. In this vein, a new field called **projective geometry** was created by GÉRARD DESARGUES (1593-1662) and BLAISE PASCAL (1623-1662) in the 17th century. It was further developed in the 19th century.

On the other hand, the *axiom of parallels* in EUCLID's *Elements* has been an object of criticism since ancient times. In the 19th century, by denying the a priori validity of Euclidean geometry, JÁNOS BOLYAI (1802-1860) and NIKOLAI LOBACHEVSKY (1793-1856) formulated **non-Euclidean geometry**.

In analytic geometry, physical spaces and planes, as we know them, are represented as 3-dimensional or 2-dimensional Euclidean spaces. It is easy to generalize these spaces to n -dimensional Euclidean space. The geometry of this new space is called the **n -dimensional Euclidean geometry**. We obtain **n -dimensional projective** and **non-Euclidean geometries** similarly. FELIX KLEIN (1849-1925) proposed systematizing all these geometries in group-theoretic terms : he called a “space” a set $\mathbf{S}$ on which a group $\mathfrak{G}$ operates and a “geometry” the study of properties of $\mathbf{S}$ invariant under the operations of $\mathfrak{G}$. KLEIN's idea not only synthetized the geometries known at that time, but also became a guiding principle for the development of new geometries.

BERNHARD RIEMANN (1826-1866) initiated another direction of geometric research when he investigated n -dimensional *manifolds* and, in particular, Riemannian manifolds and their geometries. Some aspects of **Riemannian geometry** fall outside of geometry in the sense of KLEIN. It was a starting point for the broad field of **modern differential geometry**, that is, the geometry of *differentiable manifolds* of various types. It became necessary to establish a theory that reconciled the ideas of KLEIN and RIEMANN; ELIE CARTAN (1869-1951) succeeded in this by introducing the notion of *connec-*

tion.

The reexamination of the system of axioms of EUCLID's *Elements* led to DAVID HILBERT's (1862-1943) foundations of geometry and to axiomatic tendency of present day mathematics. The study of algebraic curves, which started with the study of conic sections, developed into **algebraic geometry**. Another branch of geometry is **topology**, which has developed since the end of the 19th century. Its influence on the whole of mathematics today is considerable.

Geometry has now permeated all branches of mathematics, and it is sometimes difficult to distinguish it from algebra or analysis. Therefore, **geometry is not just a subdivision or a subject within mathematics, but a means of turning visual images into formal tools for the understanding of other mathematical phenomena.** The importance of geometric intuition, however, has not diminished from antiquity until today.

For accessible, informative materials about Geometry (and Mathematics, in general) – its past, its present and also its future – the following sources are highly recommended :

Expository papers

M. ATIYAH – What is geometry ?, *The Mathematical Gazette* **66**(1982), 179-184.

S-S. CHERN – From triangles to manifolds, *Amer. Math. Monthly* **86**(1979), 339-349.

S-S. CHERN – What is geometry ?, *Amer. Math. Monthly* **97**(1990), 679-686.

R.S. MILLMAN – Kleinian transformation geometry, *Amer. Math. Monthly* **84**(1977), 338-349.

Books

M.J. GREENBERG – *Euclidean and Non-Euclidean Geometries. Development and History*, Freeman, 1980.

T.Q. SIBLEY – *The Geometric Viewpoint. A Survey of Geometries*, Addison-Wesley, 1998.

H. WEYL – *Symmetry*, Princeton University Press, 1952.

I.M. YAGLOM – *Felix Klein and Sophus Lie. Evolution of the Idea of Symmetry in the 19th Century*, Birkhäuser, 1988.

Web sites

The MacTutor History of Mathematics archive [[http: //www-history.mcs.st-and.ac.uk](http://www-history.mcs.st-and.ac.uk)]

The Mathematical Atlas [[http: //www.math-atlas.org](http://www.math-atlas.org) or <http://www.math.niu.edu/~rusin/known-math/index/mathmap.html>]

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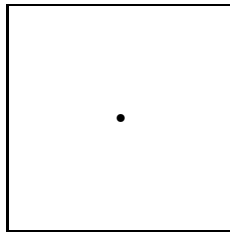
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Chapter 1

Geometric Transformations

Topics :

1. THE EUCLIDEAN PLANE $\mathbb{E}^2$
 2. TRANSFORMATIONS
 3. PROPERTIES OF TRANSFORMATIONS
-



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1.1 The Euclidean Plane $\mathbb{E}^2$

Consider the *Euclidean plane* (or two-dimensional *space*) $\mathbb{E}^2$ as studied in high school geometry.

NOTE : It is customary to assign different meanings to the terms *set* and *space*. Intuitively, a space is expected to possess a kind of arrangement or order that is not required of a set. The necessity of a *structure* in order for a set to qualify as a space may be rooted in the feeling that a notion of “proximity” (in some sense not necessarily quantitative) is inherent in our concept of a space. Thus a space differs from the mere set of its elements by possessing a structure which in some way (however vague) gives expression to that notion. A direct quantitative measure of proximity is introduced on an abstract set $\mathbf{S}$ by associating with each ordered pair (x, y) of its elements (called “points”) a non-negative real number, denoted by $d(x, y)$, and called the “distance” from x to y .

On this “geometric space” one introduces *Cartesian coordinates* which are used to define a one-to-one correspondence

$$P \mapsto (x_P, y_P)$$

between $\mathbb{E}^2$ and the set $\mathbb{R}^2$ of all ordered pairs of real numbers. This mapping preserves distances between *points* of $\mathbb{E}^2$ and their images in $\mathbb{R}^2$. It is the existence of such a coordinate system which makes the *identification* of $\mathbb{E}^2$ and $\mathbb{R}^2$ possible. Thus we can say that

$\mathbb{R}^2$ may be identified with $\mathbb{E}^2$ *plus a coordinate system*.

NOTE : The geometers before the 17th century did not think of the Euclidean plane $\mathbb{E}^2$ as a “space” of ordered pairs of real numbers. In fact it was defined *axiomatically* beginning with undefined objects such as *points* and *lines* together with a list of their properties – the *axioms* – from which the theorems of geometry were then deduced.

The identification of $\mathbb{E}^2$ and $\mathbb{R}^2$ (or, more generally, of $\mathbb{E}^n$ and $\mathbb{R}^n$) came about after the invention of *analytic geometry* by P. FERMAT (1601-1665) and R.

DESCARTES (1596-1650) and was eagerly seized upon since it is very tricky and difficult to give a suitable definition of Euclidean space (of any dimension) in the spirit of Euclid. This difficulty was certainly recognized for a long time, and has interested many great mathematicians. It lead in part to the discovery of *non-Euclidean geometries* (like spherical and hyperbolic geometries) and thus to *manifolds*.

We make the following definition.

1.1.1 DEFINITION. The **Euclidean plane** $\mathbb{E}^2$ is the set $\mathbb{R}^2$ together with the *Euclidean distance* between points $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$ given by

$$d(P, Q) = PQ := \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2}.$$

Since the Euclidean distance $d : \mathbb{E}^2 \times \mathbb{E}^2 \rightarrow \mathbb{R}$ is the only *distance function* to be considered in this course, we shall called it, simply, the *distance*.

Numbers will be denoted by lowercase Roman letters.

NOTE : The set $\mathbb{R}^2$ has the structure of a *vector space* (over $\mathbb{R}$). This means that the set $\mathbb{R}^2$ is endowed with a rule for *addition*

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

and a rule for *scalar multiplication*

$$r(x, y) := (rx, ry)$$

such that these operations satisfy the eight *axioms* below (for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$ and all $r, s \in \mathbb{R}$) :

$$(VS1) \quad ((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) = (x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) ;$$

$$(VS2) \quad (x_1, y_1) + (x_2, y_2) = (x_2, y_2) + (x_1, y_1) ;$$

$$(VS3) \quad (x_1, y_1) + (0, 0) = (x_1, y_1) ;$$

$$(VS4) \quad (x_1, y_1) + (-x_1, -y_1) = (0, 0) ;$$

$$(VS5) \quad r((x_1, y_1) + (x_2, y_2)) = r(x_1, y_1) + r(x_2, y_2) ;$$

$$(VS6) \quad (r + s)(x_1, y_1) = r(x_1, y_1) + s(x_1, y_1) ;$$

$$(VS7) \quad r(s(x_1, y_1)) = rs(x_1, y_1) ;$$

$$(VS8) \quad 1(x_1, y_1) = (x_1, y_1).$$

Hence, the Euclidean plane $\mathbb{E}^2$ is a (real, 2-dimensional) *vector space*.

A **point** P of $\mathbb{E}^2$ is an ordered pair (x, y) of real numbers.

Points will be denoted by uppercase Roman letters.

Exercise 1 Verify that (for $P, Q, R \in \mathbb{E}^2$) :

$$(M1) \quad PQ \geq 0, \quad \text{and} \quad PQ = 0 \iff P = Q ;$$

$$(M2) \quad PQ = QP ;$$

$$(M3) \quad PQ + QR \geq PR.$$

Relation (M3) is known as the *triangle inequality*.

NOTE : A set $\mathbf{S}$ equipped with a function $d : \mathbf{S} \times \mathbf{S} \rightarrow \mathbb{R}, \quad (P, Q) \mapsto PQ$ satisfying conditions (M1)-(M3) is called a *metric space* (with *metric* d). Hence *the Euclidean plane* $\mathbb{E}^2$ *is not only a vector space. It is also a metric space.*

It is important to realize that in order to do geometry we need a structure which provides lines. In Euclidean geometry, a (straight) *line* may be defined as either a *curve* with zero *acceleration* (i.e. such that the tangent vector to the curve is constant along the curve) or a curve which represents the shortest path between points.

In our (Cartesian) model of Euclidean plane it is convenient to define a line by specifying its (Cartesian) *equation*.

A **line** $\mathcal{L}$ in $\mathbb{E}^2$ is a *set of points* satisfying an equation $ax + by + c = 0$, where a, b, c are real numbers with not both $a = 0$ and $b = 0$ (i.e. $a^2 + b^2 \neq 0$).

NOTE : The triplets (a, b, c) and (ra, rb, rc) , $r \neq 0$ determine the same line. A point $P = (x_P, y_P)$ *lies* on the line with equation $ax + by + c = 0$ if (and only if) the coordinates of the point satisfy the equation of the line : $ax_P + by_P + c = 0$. If this is the case, we also say that the given line *passes through* the point P .

Lines will be denoted by uppercase calligraphic letters.

Exercise 2 PROVE or DISPROVE : Through any two different points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, there passes a unique line.

Notation and terminology (review)

We introduce some basic geometric notation and terminology. This should be read now to emphasize the basic notation and used later as a reference.

- By the *triangle inequality*, $AB + BC \geq AC$. $A - B - C$ is read “point B is between points A and C ”, and means A, B, C are three distinct points such that $AB + BC = AC$.
- $\overleftrightarrow{AB}$ is the unique *line* determined by two distinct points A and B .
- $\overline{AB}$ is a *line segment* and consists of A, B and all points between A and B .
- $AB\rightarrow$ is a *ray* from A through B and consists of all points in $\overline{AB}$ together with all points P such that $A - B - P$.
- $\angle ABC$ is an *angle* and is the union of noncollinear rays $BA\rightarrow$ and $BC\rightarrow$. $m(\angle ABC)$ is the *degree measure* of $\angle ABC$ and is a number between 0 and 180.
- $\triangle ABC$ is a *triangle* and is the union of noncollinear segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.
- “ $\cong$ ” is read “is congruent to” and has various meanings depending on context.
 - $\overline{AB} \cong \overline{CD} \iff AB = CD$;
 - $\angle ABC \cong \angle DEF \iff m(\angle ABC) = m(\angle DEF)$;
 - $\triangle ABC \cong \triangle DEF \iff \overline{AB} \cong \overline{DE}, \overline{BC} \cong \overline{EF}, \overline{AC} \cong \overline{DF}, \angle A \cong \angle D, \angle B \cong \angle E, \angle C \cong \angle F$. Not all six corresponding parts must be checked to show triangles congruent. The familiar congruence theorems for triangles $\triangle ABC$ and $\triangle DEF$ are :
 - * (SAS) : If $\overline{AB} \cong \overline{DE}$, $\angle A \cong \angle D$, and $\overline{AC} \cong \overline{DF}$, then $\triangle ABC \cong \triangle DEF$;

- * (ASA) : If $\angle A \cong \angle D$, $\overline{AB} \cong \overline{DE}$, and $\angle B \cong \angle E$, then $\triangle ABC \cong \triangle DEF$;
 - * (SAA) : If $\overline{AB} \cong \overline{DE}$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$, then $\triangle ABC \cong \triangle DEF$;
 - * (SSS) : If $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{CA} \cong \overline{FD}$, then $\triangle ABC \cong \triangle DEF$.
- The *Exterior Angle Theorem* states that given $\triangle ABC$ and $B - C - D$, then $m(\angle ACD) = m(\angle A) + m(\angle B)$. So for $\triangle ABC$ we have $m(\angle A) + m(\angle B) + m(\angle C) = 180$.
 - Given $\triangle ABC$ and $\triangle DEF$ such that $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$, then $\triangle ABC \sim \triangle DEF$, where “ $\sim$ ” is read “is similar to”. If two of these angle congruences hold, then the third congruence necessarily holds and the triangles are similar; this result is known as the *Angle-Angle Similarity Theorem*. Two triangles are also similar if and only if their corresponding sides are proportional.
 - At times, we shall need to talk about *directed angles* and *directed angle measure*, say from $AB^{\rightarrow}$ to $AC^{\rightarrow}$, with counterclockwise orientation chosen as positive, and clockwise orientation chosen as negative. In general, for real numbers r and s , we agree that $r^{\circ} = s^{\circ} \iff r = s + 360k$ for some integer k .
 - Given line $\mathcal{L}$, the points of the plane are partitioned into three sets, namely the line itself and the two *halfplanes* of the line.
 - Lines $\mathcal{L}_1$ and $\mathcal{L}_2$ are *parallel* if either $\mathcal{L}_1 = \mathcal{L}_2$ or else $\mathcal{L}_1$ and $\mathcal{L}_2$ have no points in common.
 - The locus of all points equidistant from two points A and B is the *perpendicular bisector* of A and B , which is the line through the midpoint of $\overline{AB}$ and perpendicular to $\overline{AB}$.

Exercise 3 Show that the lines

$$(\mathcal{L}) \quad ax + by + c = 0 \quad \text{and} \quad (\mathcal{M}) \quad dx + ey + f = 0$$

are *parallel* if and only if $ae - bd = 0$, and are *perpendicular* if and only if $ad + be = 0$.

Exercise 4 PROVE or DISPROVE : Through any point P off a line $\mathcal{L}$, there passes a unique line parallel to the given line $\mathcal{L}$.

Exercise 5 Show that three points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ and $P_3 = (x_3, y_3)$ are *collinear* if and only if

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 0.$$

1.2 Transformations

One of the most important concepts in geometry is that of a *transformation*.

NOTE : Transformations are a special class of functions. Consider two sets $\mathbf{S}$ and $\mathbf{T}$. A *function* (or *mapping*) α from $\mathbf{S}$ to $\mathbf{T}$ is a rule that associates with each element s of $\mathbf{S}$ a unique element $t = \alpha(s)$ of $\mathbf{T}$; the element $\alpha(s)$ is called the *image* of s under α , and s is a *preimage* of $\alpha(s)$. The set $\mathbf{S}$ is called the *domain* (or *source*) of α , and the set $\mathbf{T}$ is the *codomain* (or *target*) of α . The set of all $\alpha(s)$ with $s \in S$ is called the *image* (or *range*) of α and is denoted by $\alpha(\mathbf{S})$. If any two different elements of the domain have different images under α (that is, if $\alpha(s_1) = \alpha(s_2)$ implies that $s_1 = s_2$), then α is *one-to-one* (or *injective*). If all elements of the codomain are images under α (that is, if $\alpha(\mathbf{S}) = \mathbf{T}$), then α is *onto* (or *surjective*). If a function is injective and surjective, it is said to be *bijective*.

Exercise 6 If there exists a one-to-one mapping $f : \mathbf{A} \rightarrow \mathbf{A}$ which is *not* onto, what can be said about the set $\mathbf{A}$?

When both the domain and codomain of a mapping are “geometrical” the mapping may be referred to as a transformation. We shall find it convenient to use the word *transformation* ONLY IN THE SPECIAL SENSE of a bijective mapping of a set (space) onto itself. We make the following definition.

1.2.1 DEFINITION. A **transformation** on the plane is a bijective mapping of $\mathbb{E}^2$ onto itself.

Transformations will be denoted by lowercase Greek letters.

For a given transformation α , this means that for every point P there is a unique point Q such that $\alpha(P) = Q$ and, conversely, for every point S there is a unique point R such that $\alpha(R) = S$.

NOTE : Not every mapping on $\mathbb{E}^2$ is a transformation. Suppose a mapping α is given by $(x, y) \mapsto (\alpha_1(x, y), \alpha_2(x, y))$. Then α is a bijection (i.e. a transformation) if and only if, given the *equations* (of α)

$$\begin{aligned}x' &= \alpha_1(x, y) \\ y' &= \alpha_2(x, y),\end{aligned}$$

one can solve *uniquely* for (the “old” coordinates) x and y in terms of (the “new” coordinates) x' and y' : $x = \beta_1(x', y')$ and $y = \beta_2(x', y')$.

1.2.2 EXAMPLES. The following mappings on $\mathbb{E}^2$ are transformations:

1. $(x, y) \mapsto (x, y)$ (identity);
2. $(x, y) \mapsto (-x, y)$ (reflection);
3. $(x, y) \mapsto (x - 1, y + 2)$ (translation);
4. $(x, y) \mapsto (-y, x)$ (rotation);
5. $(x, y) \mapsto (2x, 2y)$ (dilation);
6. $(x, y) \mapsto (x + y, y)$ (shear);
7. $(x, y) \mapsto (-x + \frac{y}{2}, x + 2)$ (affinity);
8. $(x, y) \mapsto (x, x^2 + y)$ (generalized shear);
9. $(x, y) \mapsto (x, y^3)$;
10. $(x, y) \mapsto (x + |y|, y)$.

1.2.3 EXAMPLES. The following mappings on $\mathbb{E}^2$ are *not* transformations:

1. $(x, y) \mapsto (x, 0);$
2. $(x, y) \mapsto (xy, xy);$
3. $(x, y) \mapsto (x^2, y);$
4. $(x, y) \mapsto (-x + \frac{y}{2}, 2x - y);$
5. $(x, y) \mapsto (e^x \cos y, e^x \sin y).$

1.2.4 EXAMPLE. Consider the mapping

$$\beta: \mathbb{E}^2 \rightarrow \mathbb{E}^2, \quad (x, y) \mapsto (x', y') = (x^2 - y^2, 2xy).$$

Let us first use *polar coordinates* r, t so that

$$x = r \cos t, \quad y = r \sin t, \quad 0 \leq t \leq 2\pi.$$

By using some trigonometric identities, we can express $\beta((x, y))$ as

$$\beta((r \cos t, r \sin t)) = (r^2 \cos 2t, r^2 \sin 2t), \quad 0 \leq t \leq 2\pi.$$

From this it follows that under β the image curve of the circle of radius r and center at the origin counterclockwise once is the circle of radius r^2 and center at the origin counterclockwise twice. Thus *the effect of β is to wrap the plane $\mathbb{E}^2$ smoothly around itself, leaving the origin fixed*, since $\beta((0, 0)) = (0, 0)$, and therefore β is surjective but not injective.

Exercise 7 Verify that the mapping

$$(x, y) \mapsto \left(x - \frac{2a}{a^2 + b^2}(ax + by + c), y - \frac{2b}{a^2 + b^2}(ax + by + c) \right)$$

is a transformation.

Collineations

1.2.5 DEFINITION. A transformation α with the property that if $\mathcal{L}$ is a line, then $\alpha(\mathcal{L})$ is also a line is called a **collineation**.

NOTE : We take the view that *a line is a set of points* and so $\alpha(\mathcal{L})$ is the set of all points $\alpha(P)$ with point P on line $\mathcal{L}$; that is,

$$\alpha(\mathcal{L}) = \{\alpha(P) \mid P \in \mathcal{L}\} \subset \mathbb{E}^2.$$

Clearly, $\alpha(P) \in \alpha(\mathcal{L}) \iff P \in \mathcal{L}$.

1.2.6 EXAMPLE. The mapping

$$\alpha : \mathbb{E}^2 \rightarrow \mathbb{E}^2, \quad (x, y) \mapsto (x, y^3)$$

is a transformation as $(u, \sqrt[3]{v})$ is the unique point sent to (u, v) for given numbers u and v (given the equations $u = x$ and $v = y^3$, one can solve uniquely for x and y in terms of u and v). However, α is *not* a collineation, since the line with equation $y = x$ is not sent to a line, but rather to the *cubic curve* with equation $y = x^3$.

1.2.7 EXAMPLE. The mapping

$$\beta : \mathbb{E}^2 \rightarrow \mathbb{E}^2, \quad (x, y) \mapsto \left(-x + \frac{y}{2}, x + 2\right)$$

is a collineation. Indeed, from (the equations of β)

$$\begin{aligned} x' &= -x + \frac{y}{2} \\ y' &= x + 2 \end{aligned}$$

we get (uniquely)

$$\begin{aligned} x &= y' - 2 \\ y &= 2x' + 2y' - 4. \end{aligned}$$

Hence β is a transformation.

Now consider the line $\mathcal{L}$ with equation $ax + by + c = 0$, and let $P' = (x', y')$ denote the image of the (arbitrary) point $P = (x, y)$ under (the transformation) β . Recall that

$$P' = (x', y') \in \beta(\mathcal{L}) \iff P = (x, y) \in \mathcal{L}.$$

Then

$$a(y' - 2) + b(2x' + 2y' - 4) + c = 0$$

or, equivalently,

$$(2b)x' + (a + 2b)y' + c - 4b - 2a = 0.$$

(Observe that $(2b)^2 + (a + 2b)^2 \neq 0$ since $a^2 + b^2 \neq 0$.) So the line $\mathcal{L}$ with equation $ax + by + c = 0$ goes to the line with equation $(2b)x + (a + 2b)y + c - 4b - 2a = 0$. Hence β is a collineation.

Exercise 8 PROVE or DISPROVE : Collineations preserve *parallelness* among lines (i.e. the images of two parallel lines under a given collineation are also parallel lines).

1.3 Properties of Transformations

Various sets of transformations correspond to important geometric properties.

We will look at properties of sets of transformations that make them algebraically interesting. Let $\mathfrak{G}$ be a set of transformations.

Sets of transformations will be denoted by uppercase Gothic letters.

1.3.1 DEFINITION. The transformation defined by

$$\iota : \mathbb{E}^2 \rightarrow \mathbb{E}^2, \quad P \mapsto P$$

is called the **identity transformation**.

NOTE : No other transformation is allowed to use the Greek letter *iota*. The identity transformation may seem of little importance by itself, but its presence simplifies investigations about transformations, just as the number 0 simplifies addition of numbers.

If ι is in the set $\mathfrak{G}$, then $\mathfrak{G}$ is said to have the **identity property**.

Recall that α is a transformation if (and only if) for every point P there is a unique point Q such that $\alpha(P) = Q$ and, conversely, for every point S

there is a point R such that $\alpha(R) = S$. From this definition we see that the mapping $\alpha^{-1} : \mathbb{E}^2 \rightarrow \mathbb{E}^2$, defined by

$$\alpha^{-1}(A) = B \iff \alpha(B) = A$$

is a transformation, called the **inverse** of α .

NOTE : We read “ α^{-1} ” as “alpha inverse”. If (the transformation) α is given by

$$(x, y) \mapsto (x', y') = (\alpha_1(x, y), \alpha_2(x, y))$$

with $x = \beta_1(x', y')$ and $y = \beta_2(x', y')$, then (the transformation)

$$\beta : (x, y) \mapsto (\beta_1(x, y), \beta_2(x, y))$$

is the inverse of α ; that is, $\beta = \alpha^{-1}$.

If α^{-1} is also in $\mathfrak{G}$ for every transformation α in our set $\mathfrak{G}$ of transformations, then $\mathfrak{G}$ is said to have the **inverse property**.

Whenever two transformations are brought together they might form new transformations. In fact, one transformation might form new transformations by itself, as we can see by considering $\alpha = \beta$ below.

1.3.2 DEFINITION. Given two transformations α and β , the mapping

$$\beta\alpha : \mathbb{E}^2 \rightarrow \mathbb{E}^2, \quad P \mapsto \beta(\alpha(P))$$

is called the **product** of the transformation β by the transformation α .

NOTE : Transformation α is applied first and then transformation β is applied. We read “ $\beta\alpha$ ” as “the product beta-alpha”.

1.3.3 PROPOSITION. *The product of two transformations is itself a transformation.*

PROOF : Let α and β be two transformations. Since for every point C there is a point B such that $\alpha(B) = C$ and for every point B there is a point A such that $\alpha(A) = B$, then for every point C there is a point A

such that $\beta\alpha(A) = \beta(\alpha(A)) = \beta(B) = C$. So $\beta\alpha$ is an onto mapping. Also, $\beta\alpha$ is one-to-one, as the following argument shows. Suppose $\beta\alpha(P) = \beta\alpha(Q)$. Then $\beta(\alpha(P)) = \beta(\alpha(Q))$ by the definition of $\beta\alpha$. So $\alpha(P) = \alpha(Q)$ since β is one-to-one. Then $P = Q$ as α is one-to-one. Therefore, $\beta\alpha$ is both one-to-one and onto. $\square$

If our set $\mathfrak{G}$ has the property that the product $\beta\alpha$ is in $\mathfrak{G}$ whenever α and β are in $\mathfrak{G}$, then $\mathfrak{G}$ is said to have the **closure property**. Since both $\alpha^{-1}\alpha(P) = P$ and $\alpha\alpha^{-1}(P) = P$ for every point P , we see that

$$\alpha^{-1}\alpha = \alpha\alpha^{-1} = \iota.$$

Hence if $\mathfrak{G}$ is a nonempty set of transformations having both the inverse property and the closure property, then $\mathfrak{G}$ must necessarily have the identity property.

Our set $\mathfrak{G}$ of transformations is said to have the **associativity property**, as any elements α, β, γ in $\mathfrak{G}$ satisfy the *associativity law* :

$$\gamma(\beta\alpha) = (\gamma\beta)\alpha.$$

Indeed, for every point P ,

$$(\gamma(\beta\alpha))(P) = \gamma(\beta\alpha(P)) = \gamma(\beta(\alpha(P))) = (\gamma\beta)(\alpha(P)) = ((\gamma\beta)\alpha)(P).$$

Groups of transformations

The important sets of transformations are those that simultaneously satisfy the closure property, the associativity property, the identity property, and the inverse property. Such a set is called a **group (of transformations)**.

NOTE : We mention all four properties because *it is these four properties that are used for the definition of an abstract group in algebra*. However, when we want to check that a nonempty set $\mathfrak{G}$ of transformations forms a group, we need check *only* the closure property and the inverse property.

1.3.4 PROPOSITION. *The set of all transformations forms a group.*

PROOF : The closure property and the inverse property hold for the set of all transformations. $\square$

Exercise 9 Let α be a collineation. Show that, given a line $\mathcal{L}$, there exists a line $\mathcal{M}$ such that $\alpha(\mathcal{M}) = \mathcal{L}$.

1.3.5 PROPOSITION. *The set of all collineations forms a group.*

PROOF : We suppose α and β are collineations. Suppose $\mathcal{L}$ is a line. Then $\alpha(\mathcal{L})$ is a line since α is a collineation, and $\beta(\alpha(\mathcal{L}))$ is then a line since β is a collineation. Hence, $\beta\alpha(\mathcal{L})$ is a line, and $\beta\alpha$ is a collineation. So the set of collineations satisfies the closure property. There is a line $\mathcal{M}$ such that $\alpha(\mathcal{M}) = \mathcal{L}$. So

$$\alpha^{-1}(\mathcal{L}) = \alpha^{-1}(\alpha(\mathcal{M})) = \alpha^{-1}\alpha(\mathcal{M}) = \iota(\mathcal{M}) = \mathcal{M}.$$

Hence, α^{-1} is a collineation, and the set of all collineations satisfies the inverse property. The set is not empty as the identity is a collineation. Therefore, the set of all collineations forms a group. $\square$

If every element of transformation group $\mathfrak{G}'$ is an element of transformation group $\mathfrak{G}$, then $\mathfrak{G}'$ is a **subgroup** of $\mathfrak{G}$. *All of our groups will be subgroups of the group of all collineations.* These transformation groups will be a very important part of our study of geometry.

NOTE : The word *group* now has a technical meaning and should never be used as a general collective noun in place of the word *set*.

Transformations α and β may or may not satisfy the *commutativity law* : $\alpha\beta = \beta\alpha$. If the commutativity law is always satisfied by the elements from a group, then that group is said to be **commutative** (or **Abelian**). The term *Abelian* is after the Norwegian mathematician N.H. ABEL (1801-1829).

Orders and generators

Given a transformation α , the product $\alpha\alpha\ldots\alpha$ (n times) is denoted by α^n . As expected, we define α^0 to be ι . Also, we write

$$(\alpha^{-1})^n = \alpha^{-n}, \quad n \in \mathbb{Z}.$$

If group $\mathfrak{G}$ has exactly n elements, then $\mathfrak{G}$ is said to be **finite** and have **order** n ; otherwise, $\mathfrak{G}$ is said to be **infinite**. Analogously, if there is a smallest positive integer n such that $\alpha^n = \iota$, then transformation α is said to have **order** n ; otherwise α is said to have **infinite order**.

1.3.6 EXAMPLE. Let ρ be a *rotation* of $\frac{360}{n}$ degrees about the origin with n a positive integer and let

$$\tau : \mathbb{E}^2 \rightarrow \mathbb{E}^2, \quad (x, y) \mapsto (x + 1, y).$$

Then

- ρ has order n ,
- the set $\{\rho, \rho^2, \dots, \rho^n\}$ forms a group,
- τ has infinite order,
- the set $\{\tau^k : k \in \mathbb{Z}\}$ forms an infinite group.

If every element of a group containing α is a power of α , then we say that the group is **cyclic** with **generator** α and denote the group as $\langle \alpha \rangle$.

1.3.7 EXAMPLE. If ρ is a rotation of 36° , then $\langle \rho \rangle$ is a cyclic group of order 10. Note that this same group is generated by β where $\beta = \rho^3$. In fact, we have

$$\langle \rho \rangle = \langle \rho^3 \rangle = \langle \rho^7 \rangle = \langle \rho^9 \rangle.$$

So a cyclic group may have more than one generator.

NOTE : Since the powers of a transformation always commute (i.e. $\alpha^m \alpha^n = \alpha^{m+n} = \alpha^n \alpha^m$ for integers m and n), we see that *a cyclic group is always Abelian*.

If $\mathfrak{G} = \langle \alpha, \beta, \gamma, \dots \rangle$, then every element of group $\mathfrak{G}$ can be written as a product of powers of $\alpha, \beta, \gamma, \dots$ and $\mathfrak{G}$ is said to be **generated** by $\{\alpha, \beta, \gamma, \dots\}$.

Involutions and multiplication tables

Among the particular transformations that will command our attention are the *involutions*.

1.3.8 DEFINITION. A transformation α is an **involution** if $\alpha^2 = \iota$ but $\alpha \neq \iota$.

NOTE : The identity transformation is *not* an involution by definition.

1.3.9 EXAMPLE. The following transformations are involutions :

1. $(x, y) \mapsto (y, x)$;
2. $(x, y) \mapsto (-x + 2a, -y + 2b)$;
3. $(x, y) \mapsto \left(\frac{1}{2}(x + \sqrt{3}y), \frac{1}{2}(\sqrt{3}x - y)\right)$.

1.3.10 PROPOSITION. A nonidentity transformation α is an involution if and only if $\alpha = \alpha^{-1}$.

PROOF : ($\Rightarrow$) Assume the nonidentity transformation α is an involution. Then $\alpha^2 = \iota$. By multiplying both sides by α^{-1} , we get

$$\alpha^{-1}(\alpha\alpha) = \alpha^{-1}\iota \iff (\alpha^{-1}\alpha)\alpha = \alpha^{-1} \iff \iota\alpha = \alpha^{-1} \iff \alpha = \alpha^{-1}.$$

($\Leftarrow$) Conversely, assume the nonidentity transformation α is such that $\alpha = \alpha^{-1}$. Then by multiplying both sides by α , we get

$$\alpha^2 = \alpha\alpha = \alpha\alpha^{-1} = \iota.$$

□

Exercise 10 Determine whether the transformation

$$(x, y) \mapsto \left(x - \frac{2a}{a^2 + b^2}(ax + by + c), y - \frac{2b}{a^2 + b^2}(ax + by + c) \right)$$

is an involution.

A multiplication table for a finite group is often called a **Cayley table** for the group. This is in honour of the English mathematician A. CAYLEY (1821-1895). In a Cayley table, the product $\beta\alpha$ is found in the row headed “ β ” and the column headed “ α ”.

1.3.11 EXAMPLE. Consider the group $\mathfrak{C}_4$ that is generated by a rotation ρ of 90° about the origin. The Cayley table for $\mathfrak{C}_4$ is given below :

$\mathfrak{C}_4$	ι	ρ	ρ^2	ρ^3
ι	ι	ρ	ρ^2	ρ^3
ρ	ρ	ρ^2	ρ^3	ι
ρ^2	ρ^2	ρ^3	ι	ρ
ρ^3	ρ^3	ι	ρ	ρ^2

Clearly, $\mathfrak{C}_4$ is a group of order 4 (it is easy to check the closure property and the inverse property). Group $\mathfrak{C}_4$ is cyclic and is generated by ρ . Since

$$(\rho^3)^2 = \rho^6 = \rho^2, \quad (\rho^3)^3 = \rho^9 = \rho, \quad \text{and} \quad (\rho^3)^4 = \rho^{12} = \iota,$$

then $\mathfrak{C}_4$ is also generated by ρ^3 . So

$$\mathfrak{C}_4 = \langle \rho \rangle = \langle \rho^3 \rangle.$$

Note, also, that group $\mathfrak{C}_4$ contains the one involution ρ^2 .

1.3.12 EXAMPLE. Consider the group $\mathfrak{V}_4 = \{\iota, \sigma_O, \sigma_h, \sigma_v\}$, where

$$\iota((x, y)) = (x, y), \quad \sigma_O((x, y)) = (-x, -y),$$

$$\sigma_h((x, y)) = (x, -y), \quad \sigma_v((x, y)) = (-x, y).$$

The Cayley table for $\mathfrak{V}_4$ can be computed algebraically without any geometric interpretation.

$\mathfrak{V}_4$	ι	σ_h	σ_v	σ_O
ι	ι	σ_h	σ_v	σ_O
σ_h	σ_h	ι	σ_O	σ_v
σ_v	σ_v	σ_O	ι	σ_h
σ_O	σ_O	σ_v	σ_h	ι

Group $\mathfrak{V}_4$ is Abelian but not cyclic. Every element of $\mathfrak{V}_4$ except the identity is an involution.

1.4 Exercises

Exercise 11 Let P, Q , and R be three distinct points. Prove that

$$PQ + QR = PR \iff Q = (1 - t)P + tR \text{ for some } 0 < t < 1.$$

(The line segment $\overline{PR}$ consists of P, R and all points between P and R . Hence

$$\overline{PR} = \{(1 - t)P + tR \mid 0 \leq t \leq 1\}.)$$

Exercise 12 Which of the following mappings defined on the Euclidean plane $\mathbb{E}^2$ are transformations ?

- (a) $(x, y) \mapsto (x^3, y^3)$.
- (b) $(x, y) \mapsto (\cos x, \sin y)$.
- (c) $(x, y) \mapsto (x^3 - x, y)$.
- (d) $(x, y) \mapsto (2x, 3y)$.
- (e) $(x, y) \mapsto (-x, x + 3)$.
- (f) $(x, y) \mapsto (3y, x + 2)$.
- (g) $(x, y) \mapsto (\sqrt[3]{x}, e^y)$.
- (h) $(x, y) \mapsto (-x, -y)$.
- (i) $(x, y) \mapsto (x + 2, y - 3)$.

Exercise 13 Which of the transformations in the exercise above are collineations ? For each collineation, find the image of the line with equation $ax + by + c = 0$.

Exercise 14 Find the image of the line with equation $y = 5x + 7$ under collineation α if $\alpha((x, y))$ is :

- (a) $(-x, y)$.
- (b) $(x, -y)$.
- (c) $(-x, -y)$.
- (d) $(2y - x, x - 2)$.

Exercise 15 TRUE or FALSE ? Suppose α is a transformation on the plane.

- (a) If $\alpha(P) = \alpha(Q)$, then $P = Q$.
- (b) For any point P there is a unique point Q such that $\alpha(P) = Q$.
- (c) For any point P there is a point Q such that $\alpha(P) = Q$.
- (d) For any point P there is a unique point Q such that $\alpha(Q) = P$.
- (e) For any point P there is a point Q such that $\alpha(Q) = P$.
- (f) A collineation is necessarily a transformation.
- (g) A transformation is necessarily a collineation.
- (h) A collineation is a mapping that is one-to-one.
- (i) A collineation is a mapping that is onto.
- (j) A transformation is onto but not necessarily one-to-one.

Exercise 16 Give three examples of transformations on the plane that are not collineations.

Exercise 17 Find the preimage of the line with equation $y = 3x + 2$ under the collineation

$$\alpha : \mathbb{E}^2 \rightarrow \mathbb{E}^2, \quad (x, y) \mapsto (3y, x - y).$$

Exercise 18 If

$$\begin{cases} x' = ax + by + h \\ y' = cx + dy + k \end{cases}$$

are the equations for mapping $\alpha : \mathbb{E}^2 \rightarrow \mathbb{E}^2$, then what are the necessary and sufficient conditions on the coefficients for α to be a *transformation* ? Is such a transformation always a collineation ?

Exercise 19 Let $\mathbf{P} = \{P_1, \dots, P_n\}$ be a finite set of points (in the plane), and let C be its **centre of gravity**, namely

$$C := \frac{1}{n} (P_1 + \dots + P_n).$$

Consider a transformation $\alpha : \mathbb{E}^2 \rightarrow \mathbb{E}^2$ of the form

$$(x, y) \mapsto (ax + by + h, cx + dy + k) \quad \text{with } ad - bc \neq 0$$

and let $P'_i = \alpha(P_i)$, $i = 1, 2, \dots, n$ and $C' = \alpha(C)$. Show that

$$C' = \frac{1}{n} (P'_1 + \dots + P'_n).$$

Exercise 20 Sketch the image of the unit square under the following transformations :

- (a) $(x, y) \mapsto (x, x + y)$.
- (b) $(x, y) \mapsto (y, x)$.
- (c) $(x, y) \mapsto (x, x^2 + y)$.
- (d) $(x, y) \mapsto (-x + \frac{y}{2}, x + 2)$.

Exercise 21 Prove that if α, β , and γ are elements in a group, then

- (a) $\beta\alpha = \gamma\alpha$ implies $\beta = \gamma$;
- (b) $\beta\alpha = \beta\gamma$ implies $\alpha = \gamma$;
- (c) $\beta\alpha = \alpha$ implies $\beta = \iota$;
- (d) $\beta\alpha = \beta$ implies $\alpha = \iota$;
- (e) $\beta\alpha = \iota$ implies $\beta = \alpha^{-1}$ and $\alpha = \beta^{-1}$.

Exercise 22 TRUE or FALSE ?

- (a) If α and β are transformations, then $\alpha = \beta$ if and only if $\alpha(P) = \beta(P)$ for every point P .
- (b) Transformation ι is in every group of transformations.
- (c) If $\alpha\beta = \iota$, then $\alpha = \beta^{-1}$ and $\beta = \alpha^{-1}$ for transformations α and β .
- (d) “ $\alpha\beta$ ” is read “the product beta-alpha”.
- (e) If α and β are both in group $\mathfrak{G}$, then $\alpha\beta = \beta\alpha$.

- (f) $(\alpha\beta)^{-1} = \alpha^{-1}\beta^{-1}$ for transformations α and β .

Exercise 23 PROVE or DISPROVE : There is an infinite cyclic group of rotations.

Exercise 24 TRUE or FALSE ?

- (a) $\langle \iota \rangle$ is a cyclic group of order 1.
- (b) $\langle \gamma \rangle = \langle \gamma^{-1} \rangle$ for any transformation γ .
- (c) An Abelian group is always cyclic, but a cyclic group is not always Abelian.
- (d) If $\langle \alpha \rangle = \langle \beta \rangle$, then $\alpha = \beta$ or $\alpha = \beta^{-1}$.

Exercise 25 Find all a and b such that the transformation

$$(x, y) \mapsto \left(ay, \frac{x}{b} \right)$$

is an involution.

DISCUSSION :

The Euclidean plane can be approached in many ways. One can take the view that plane geometry is about points, lines, circles, and proceed from “self-evident” properties of these figures (axioms) to deduce the less obvious properties as theorems. This was the classical approach to geometry, also known as *synthetic*. It was based on the conviction that geometry describes actual space and, in particular, that the theory of lines and circles describes what one can do with ruler and compass. To develop this theory, EUCLID (c. 300 B.C.) stated certain plausible properties of lines and circles as axioms and derived theorems from them by pure logic. Actually he occasionally made use of unstated axioms; nevertheless his approach is feasible and it was eventually made rigorous by DAVID HILBERT (1862-1943).

EUCLID’s approach has some undeniable advantages. Above all, it presents geometry in a pure and self-contained manner, without use of “non-geometric” concepts. One feels that the “real reason” for geometric theorems are revealed in such a system. Visual intuition not only supplies the axioms, it also prompts the steps in a proof, so that some extremely short and elegant proofs result.

Nevertheless, with the enormous growth of mathematics over the last two centuries, EUCLID’s approach has become isolated and inefficient. It is isolated because Euclidean geometry is no longer *the* geometry of space and the basis for most of

mathematics. Nowadays, numbers and sets are regarded as more fundamental than points and lines. They form a much broader basis, not only for geometry, but for mathematics as a whole. Moreover, geometry can be built more efficiently on this basis because the powerful techniques of algebra and analysis can be brought into play.

The construction of geometry from numbers and sets is implicit in the coordinate geometry of RENÉ DESCARTES (1596-1650), though DESCARTES, in fact, took the classical view that points, lines, and curves had a prior existence, and he regarded coordinates and equations as merely a convenient way to study them. Perhaps the first to grasp the deeper value of the coordinate approach was BERNHARD RIEMANN (1826-1866), who wrote the following : “It is well known that geometry assumes as given not only the concept of space, but also the basic principles of construction in space. It gives only nominal definitions of these things; their determination being in the form of axioms. As a result, the relationships between these assumptions are left in the dark; one does not see whether, or to what extent, connections between them are necessary, or even whether they are a priori possible.”

RIEMANN went on to outline a very general approach to geometry in which “points” in an “ n -dimensional space” are n -tuples of numbers, and all geometric relations are determined by a *metric* on this space, a differentiable function giving the “distance” between two “points”. This *analytic* approach allows a vast range of spaces to be considered simultaneously, and RIEMANN found that their geometric properties were largely controlled by a property of the metric he called its *curvature*.

The concept of curvature illuminates the axioms of Euclidean geometry by showing them to hold only in the presence of *zero* curvature. In particular, the Euclidean plane is a two-dimensional space of zero curvature (though not the only one). It also becomes obvious what the natural alternatives to Euclidean geometry are – those of constant positive and negative curvature – and one can pinpoint precisely where change of curvature causes a change in axioms.

RIEMANN set up analytic machinery to study spaces whose curvature varies from point to point. However, simpler machinery suffices for spaces of constant curvature. The reason is that the geometry of these spaces is reflected in *isometries* (distance-preserving transformations) and isometries turn out to be easily understood. This approach is due to FELIX KLEIN (1849-1925). The concept of isometry actually fills a gap in EUCLID’s approach to geometry, where the idea of “moving” one figure

until it coincides with another is used without being formally recognized. Thus, when geometry is based on coordinates *and* isometries, it is possible to enjoy the benefits of both the analytic and synthetic approaches.

A point is that which has no parts.

EUCLID

A “point” is much more subtle object than naive intuition suggests.

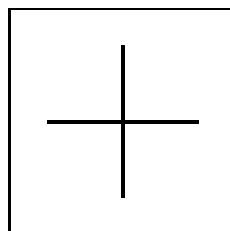
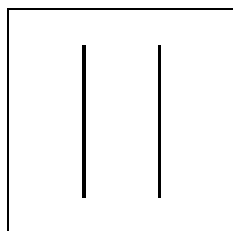
JOHN STILLWELL

Chapter 2

Translations and Halfturns

Topics :

1. TRANSLATIONS
 2. HALFTURNS
-



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2.1 Translations

Let $\mathbb{E}^2$ be the Euclidean plane.

2.1.1 DEFINITION. A **translation** (or **parallel displacement**) is a mapping

$$\tau : \mathbb{E}^2 \rightarrow \mathbb{E}^2, \quad (x, y) \mapsto (x + h, y + k).$$

We use to say that such a translation τ *has equations*

$$\begin{cases} x' = x + h \\ y' = y + k. \end{cases}$$

Given any two of (x, y) , (x', y') , and (h, k) , the third is then uniquely determined by this last set of equations. Hence, *a translation is a transformation*.

NOTE : We shall use the Greek letter *tau* only for translations.

2.1.2 PROPOSITION. *Given points P and Q , there is a unique translation $\tau_{P,Q}$ taking P to Q .*

PROOF : Let $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$. Then there are unique numbers h and k such that

$$x_Q = x_P + h \quad \text{and} \quad y_Q = y_P + k.$$

So the unique translation $\tau_{P,Q}$ that takes P to Q has equations

$$\begin{cases} x' = x + x_Q - x_P \\ y' = y + y_Q - y_P. \end{cases}$$

□

By the proposition above, if $\tau_{P,Q}(R) = S$, then $\tau_{P,Q} = \tau_{R,S}$ for points P, Q, R, S .

NOTE : The identity is a special case of a translation as

$$\iota = \tau_{P,P} \quad \text{for each point } P.$$

2.1.3 COROLLARY. *If $\tau_{P,Q}(R) = R$ for point R , then $P = Q$.*

2.1.4 PROPOSITION. *Suppose A, B, C are noncollinear points. Then $\tau_{A,B} = \tau_{C,D}$ if and only if $\square CABD$ is a parallelogram.*

PROOF : The translation $\tau_{A,B}$ has equations

$$\begin{cases} x' = x + x_B - x_A \\ y' = y + y_B - y_A. \end{cases}$$

Then the following are equivalent :

- (1) $\tau_{A,B} = \tau_{C,D}$.
- (2) $D = \tau_{A,B}(C)$.
- (3) $D = (x_D, y_D) = (x_C + x_B - x_A, y_C + y_B - y_A)$.
- (4) $\frac{1}{2}(A + D) = \frac{1}{2}(B + C)$.
- (5) $\square CABD$ is a parallelogram.

□

Exercise 26 Prove the equivalence (3) $\iff$ (4).

Exercise 27 What happens (in PROPOSITION 2.1.4) if we drop the requirement that the points A, B, C are noncollinear ?

It follows that *a translation moves each point the same distance in the same direction*. For nonidentity translation $\tau_{A,B}$, the distance is given by AB and the direction by (the directed line segment) $\overrightarrow{AB}$.

NOTE : The translation $\tau_{A,B}$ can be *identified* with the (geometric) vector

$$\mathbf{v} = \begin{bmatrix} x_B - x_A \\ y_B - y_A \end{bmatrix}$$

where $A = (x_A, y_A)$ and $B = (x_B, y_B)$. A vector is really the same thing as a translation, although one uses different phraseologies for vectors and translations.

It may be helpful to make this idea more precise. What is a vector ? The school textbooks usually define a vector as a “quantity having magnitude and direction”, such as the *velocity vector* of an object moving through space (in our case, the Euclidean plane). It is helpful to represent a vector as an “arrow” attached to a point of the space. But one is not supposed to think of the vector as being firmly rooted just at one point. For instance, one wants to *add* vectors, and the recipe for doing this is to pick up one vector and move it around without changing its length or direction until its tail lies on the head of the other one.

It is better, then, to think of a vector as a *instruction to move* rather than as an arrow pointing from one fixed point to another. The instruction makes sense wherever you are, even if it may be rather difficult to carry out, whereas the arrow is not much use unless you are already at its origin. The “instruction” idea makes vector addition simple : to add two vectors, you just carry out one instruction after the other. Not every instruction to move is a vector. For an instruction to be a vector, it must specify movement through the same distance and in the same direction for every point.

This idea of an “instruction” is expressed mathematically as a function (or mapping). A *vector* is a mapping $\mathbf{v}$ (on the plane) which associates to each point A a new point $\mathbf{v}(A)$, having the property that for any two points A and B , the midpoint of $\overline{A\mathbf{v}(B)}$ is equal to the midpoint of $\overline{B\mathbf{v}(A)}$. Thus, if $\mathbf{v}$ is a vector and A and B are any two points, then $\square AB\mathbf{v}(B)\mathbf{v}(A)$ is a parallelogram. Given two points P and Q , there is exactly one vector $\mathbf{v}$ such that $\mathbf{v}(P) = Q$. This unique vector is denoted by $\overrightarrow{PQ}$; if $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$, it is convenient to represent $\mathbf{v} = \overrightarrow{PQ}$ by the 2×1 matrix

$$\begin{bmatrix} x_Q - x_P \\ y_Q - y_P \end{bmatrix}.$$

We have yet to show that *a translation is a collineation*. Suppose line $\mathcal{L}$ has equation $ax + by + c = 0$ and nonidentity translation $\tau_{P,Q}$ has equations

$$\begin{cases} x' = x + h \\ y' = y + k. \end{cases}$$

So

$$\tau_{P,Q} = \tau_{O,R}, \quad \text{where } R = (h, k) \quad \text{and} \quad \overleftrightarrow{PQ} \parallel \overleftrightarrow{OR}.$$

Under the equations for $\tau_{P,Q}$, we see that

$$ax + by + c = 0 \iff ax' + by' + (c - ah - bk) = 0.$$

We calculate that $\tau_{P,Q}(\mathcal{L})$ is the line $\mathcal{M}$ with equation

$$ax + by + (c - ah - bk) = 0.$$

We have shown more than the fact that a translation is a collineation. By comparing the equations for lines $\mathcal{L}$ and $\mathcal{M}$, we see that $\mathcal{L}$ and $\mathcal{M}$ are parallel. Thus, *a translation always sends a line to a parallel line*. We make the following definition.

2.1.5 DEFINITION. A collineation α is a **dilatation** if $\mathcal{L} \parallel \alpha(\mathcal{L})$ for every line $\mathcal{L}$.

NOTE : While any collineation sends a *pair* of parallel lines to a pair of parallel lines, a dilatation sends *each* given line to a line parallel to the given line. For example, we shall see that a rotation of 90° is a collineation but not a dilatation.

Let α be a transformation and $\mathbf{S}$ a set of points.

2.1.6 DEFINITION. Transformation α **fixes** set $\mathbf{S}$ if $\alpha(\mathbf{S}) = \mathbf{S}$.

NOTE : In particular, the set $\mathbf{S}$ can be a point or a line.

2.1.7 PROPOSITION. *A translation is a dilatation. If $P \neq Q$, then $\tau_{P,Q}$ fixes no points and fixes exactly those lines that are parallel to $\overleftrightarrow{PQ}$.*

PROOF : Our calculation above has shown that a translation is a dilatation.

Clearly, if $P \neq Q$, then $\tau_{P,Q}$ fixes no points. As above, let $\mathcal{L}$ be the line with equation

$$ax + by + c = 0$$

and $\mathcal{M}$ the line with equation

$$ax + by + (c - ah - bk) = 0.$$

These two lines are the same if and only if $ah + bk = 0$. Since $\overleftrightarrow{OR}$ has equation $kx - hy = 0$, then

$$ah + bk = 0 \iff \mathcal{L} \parallel \overleftrightarrow{OR}.$$

Thus

$$\tau_{P,Q}(\mathcal{L}) = \mathcal{L} \iff \mathcal{L} \parallel \overleftrightarrow{PQ}.$$

□

2.1.8 PROPOSITION. *The translations form a commutative group $\mathfrak{T}$, called the **translation group**.*

PROOF : Translations are collineations.

Let $S = (a, b)$, $T = (c, d)$, and $R = (a + c, b + d)$. Then

$$\tau_{O,T\tau_{O,S}}((x, y)) = \tau_{O,T}((x + a, y + b)) = (x + a + c, y + b + d) = \tau_{O,R}((x, y)).$$

Since

$$\tau_{O,T\tau_{O,S}} = \tau_{O,R}$$

then a product of two translations is a translation.

Also, by taking $R = O$, we see that the inverse of the translation $\tau_{O,S}$ is the translation $\tau_{O,S'}$, where $S' = (-a, -b)$.

Further, since $a + c = c + a$ and $b + d = d + b$, it follows that

$$\tau_{O,T}\tau_{O,S} = \tau_{O,S}\tau_{O,T}.$$

So the translations form a commutative group (of transformations). $\square$

2.1.9 PROPOSITION. *The dilatations form a group $\mathfrak{D}$, called the **dilatation group**.*

PROOF : Dilatations are collineations.

By the symmetry of parallelness for lines (i.e., $\mathcal{L} \parallel \mathcal{L}' \Rightarrow \mathcal{L}' \parallel \mathcal{L}$), the inverse of a dilatation is a dilatation.

By the transitivity of parallelness for lines (i.e., $\mathcal{L} \parallel \mathcal{L}'$ and $\mathcal{L}' \parallel \mathcal{L}'' \Rightarrow \mathcal{L} \parallel \mathcal{L}''$), the product of two dilatations is a dilatation.

So the dilatations form a group (of transformations). $\square$

2.2 Halfturns

A *halfturn* turns out to be an involutory rotation; that is, a rotation of 180° . So, a halfturn is just a special case of a rotation. Although we have not formally introduced rotations yet, we look at this special case now because halfturns are nicely related to translations and have such easy equations. Informally, we observe that if point A is rotated 180° about point P to point A' , then P is the midpoint of A and A' . Hence, we need only the midpoint formulas to obtain the desired equations. From equations

$$\begin{cases} \frac{x + x'}{2} = a \\ \frac{y + y'}{2} = b \end{cases}$$

we can make our definition as follows.

2.2.1 DEFINITION. If $P = (a, b)$, then the **halfturn** σ_P **about point** P is the mapping

$$\sigma_P : \mathbb{E}^2 \rightarrow \mathbb{E}^2, \quad (x, y) \mapsto (-x + 2a, -y + 2b).$$

Such a halfturn σ_P has equations

$$\begin{cases} x' = -x + 2a \\ y' = -y + 2b. \end{cases}$$

NOTE : For the halfturn about the origin we have

$$\sigma_O((x, y)) = (-x, -y).$$

Under transformation σ_O does (x, y) go to $(-x, -y)$ by going directly through O , by rotating counterclockwise about O , by rotating clockwise about O , or by taking some “fanciful path” ? Either the answer is “None of the above” or, perhaps, it would be better to ask whether the question makes sense. Recall that transformations are just one-to-one correspondences among points. There is actually no physical motion being described. (That is done in the study called *differential geometry*.) We might say *we are describing the end position of physical motion*. Since our thinking is often aided by language indicating physical motion, we continue such usage as the customary “ P goes to Q ” in place of the more formal “ P corresponds to Q ”.

What properties of a halfturn follow immediately from the definition of σ_P ? First, for any point A , the midpoint of A and $\sigma_P(A)$ is P . From this simple fact alone, it follows that σ_P is an *involutory transformation*. Also from this simple fact, it follows that σ_P fixes exactly the one point P . It even follows that σ_P fixes line $\mathcal{L}$ if and only if P is on $\mathcal{L}$.

2.2.2 PROPOSITION. *A halfturn is an involutory dilatation. The midpoint of points A and $\sigma_P(A)$ is P . Halfturn σ_P fixes point A if and only if $A = P$. Halfturn σ_P fixes line $\mathcal{L}$ if and only if P is on $\mathcal{L}$.*

PROOF : We shall show that σ_P is a collineation.

Suppose that line $\mathcal{L}$ has equation $ax + by + c = 0$. Let $P = (h, k)$. Then σ_P has equations

$$\begin{cases} x' = -x + 2h \\ y' = -y + 2k. \end{cases}$$

Then

$$ax + by + c = 0 \iff ax' + by' + c - 2(ah + bk + c) = 0.$$

So $\sigma_P(\mathcal{L})$ is the line $\mathcal{M}$ with equation

$$ax + by + c - 2(ah + bk + c) = 0.$$

Therefore, not only σ_P is a collineation, but a dilatation as $\mathcal{L} \parallel \mathcal{M}$.

Finally, $\mathcal{L}$ and $\mathcal{M}$ are the same if and only if $ah + bk + c = 0$, which holds if and only if (h, k) is on $\mathcal{L}$. $\square$

Since a halfturn is an involution, then $\sigma_P\sigma_P = \iota$. What can be said about the product of two halfturns in general ?

Let $P = (a, b)$ and $Q = (c, d)$. Then

$$\begin{aligned} \sigma_Q\sigma_P((x, y)) &= \sigma_Q((-x + 2a, -y + 2b)) \\ &= (-(-x + 2a) + 2c, -(-y + 2b) + 2d) \\ &= (x + 2(c - a), y + 2(d - b)). \end{aligned}$$

Since $\sigma_Q\sigma_P$ has equations

$$\begin{cases} x' = x + 2(c - a) \\ y' = y + 2(d - b) \end{cases}$$

then $\sigma_Q\sigma_P$ is a translation. This proves the important result that *the product of two halfturns is a translation*.

2.2.3 PROPOSITION. *If Q is the midpoint of points P and R , then*

$$\sigma_Q\sigma_P = \tau_{P,R} = \sigma_R\sigma_Q.$$

PROOF : We have

$$\sigma_Q\sigma_P(P) = \sigma_Q(P) = R \quad \text{and} \quad \sigma_R\sigma_Q(P) = \sigma_R(R) = R.$$

Since there is a unique translation taking P to R , then each of $\sigma_Q\sigma_P$ and $\sigma_P\sigma_R$ must be $\tau_{P,R}$. $\square$

NOTE : A product of two halfturns is a translation and, conversely, a translation is a product of two halfturns. Also, notice that $\sigma_Q\sigma_P$ moves each point *twice* the directed distance from P to Q .

We now consider a product of three halfturns. By thinking about the equations, it should almost be obvious that $\sigma_R\sigma_Q\sigma_P$ is itself a halfturn. We shall prove that and a little more.

2.2.4 PROPOSITION. *A product of three halfturns is a halfturn. In particular, if points P, Q, R are not collinear, then $\sigma_R\sigma_Q\sigma_P = \sigma_S$ where $\square PQRS$ is a parallelogram.*

PROOF : Suppose $P = (a, b)$, $Q = (c, d)$, and $R = (e, f)$. Let $S = (a - c + e, b - d + f)$. In case P, Q, R are not collinear, then $\square PQRS$ is a parallelogram. (This is easy to check as opposite sides of the quadrilateral are congruent and parallel.) We calculated $\sigma_Q\sigma_P((x, y))$ above. Whether P, Q, R are collinear or not, we obtain

$$\begin{aligned}\sigma_R\sigma_Q\sigma_P((x, y)) &= (-x + 2(a - c + e), -y + 2(b - d + f)) \\ &= \sigma_S((x, y)).\end{aligned}$$

□

2.2.5 EXAMPLE. Given any three of the not necessarily distinct points A, B, C, D , then the fourth is uniquely determined by the equation $\tau_{A,B} = \sigma_D\sigma_C$.

PROOF : We can solve the equation $\tau_{A,B} = \sigma_D\sigma_C$ for any one of A, B, C, D in terms of the other three. Knowing C, D and one of A or B , we let the other be defined by the equation $\sigma_D\sigma_C(A) = B$ or the equivalent equation $\sigma_C\sigma_D(B) = A$. In either case, product $\sigma_D\sigma_C$ is the unique translation taking A to B , and so $\sigma_D\sigma_C = \tau_{A,B}$. When we know both A and B , we let M be the midpoint of A and B . So $\tau_{A,B} = \sigma_M\sigma_A$. Knowing A, B, D , we have C is the unique solution for Y in the equation $\sigma_D\sigma_M\sigma_A = \sigma_Y$ as then

$\tau_{A,B} = \sigma_M \sigma_A = \sigma_D \sigma_Y$. Knowing A, B, C , we have D is the unique solution for Z in the equation $\sigma_M \sigma_A \sigma_C = \sigma_Z$ as then $\tau_{A,B} = \sigma_M \sigma_A \sigma_Z \sigma_C$. $\square$

NOTE : In general, halfturns *do not* commute. Indeed, if $\sigma_Q \sigma_P = \tau_{P,R}$, then $\tau_{P,R}^{-1} = \sigma_P \sigma_Q$. So

$$\sigma_Q \sigma_P = \sigma_P \sigma_Q \iff P = Q.$$

2.2.6 PROPOSITION. $\sigma_R \sigma_Q \sigma_P = \sigma_P \sigma_Q \sigma_R$ for any points P, Q, R .

PROOF : For any points P, Q, R , there is a point S such that

$$\sigma_R \sigma_Q \sigma_P = \sigma_S = \sigma_S^{-1} = (\sigma_R \sigma_Q \sigma_P)^{-1} = \sigma_P^{-1} \sigma_Q^{-1} \sigma_R^{-1} = \sigma_P \sigma_Q \sigma_R.$$

$\square$

NOTE : The halfturns *do not* form a group by themselves.

2.2.7 PROPOSITION. *The union of the translations and the halfturns forms a group $\mathfrak{H}$.*

PROOF : The product of two halfturns is a translation. Since a translation is a product of two halfturns, then the product in either order of a translation and a halfturn is a halfturn.

Recall that the inverse of a translation is a translation, and that a halfturn is an involutory transformation.

So the union of the translations and the halfturns forms a group. $\square$

NOTE : A product of an even number of halfturns is a product of translations and, hence, is a translation.

A product of an odd number of halfturns is a halfturn followed by a translation and, hence, is a halfturn.

2.3 Exercises

Exercise 28 If τ is the product of halfturns about O and O' , what is the product of halfturns about O' and O ?

Exercise 29 Prove that

$$\tau_{A,B}\sigma_P\tau_{A,B}^{-1} = \sigma_Q, \quad \text{where } Q = \tau_{A,B}(P).$$

Exercise 30 TRUE or FALSE ?

- (a) A product of two involutions is an involution or ι .
- (b) $\mathfrak{D} \subset \mathfrak{H} \subset \mathfrak{T}$.
- (c) If δ is a dilatation and lines $\mathcal{L}$ and $\mathcal{M}$ are parallel, then $\delta(\mathcal{L})$ and $\delta(\mathcal{M})$ are parallel to $\mathcal{L}$.
- (d) Given points A, B, C , there is a D such that $\tau_{A,B} = \tau_{D,C}$.
- (e) Given points A, B, C , there is a D such that $\tau_{A,B} = \sigma_D\sigma_C$.
- (f) If $\tau_{A,B}(C) = D$, then $\tau_{A,B} = \tau_{C,D}$.
- (g) If $\sigma_Q\sigma_P = \tau_{P,R}$, then $\sigma_P\sigma_Q = \tau_{R,P}$.
- (h) $\sigma_A\sigma_B\sigma_C = \sigma_B\sigma_C\sigma_A$ for points A, B, C .
- (i) A translation has equations $x' = x - a$ and $y' = y - b$.
- (j) $\sigma_Q\sigma_P = \tau_{P,Q}^2$ for any points P and Q .

Exercise 31

$$\begin{cases} x' = -x + 3 \\ y' = -y - 8 \end{cases}$$

are the equations for which transformation ?

What are the equations for $\tau_{S,T}^{-1}$ if $S = (a, c)$ and $T = (g, h)$?

Exercise 32 PROVE or DISPROVE : $\sigma_P\tau_{A,B}\sigma_P = \tau_{C,D}$, where $C = \sigma_P(A)$ and $D = \sigma_P(B)$.

Exercise 33 If $P_i = (a_i, b_i)$, $i = 1, 2, 3, 4, 5$, then what are the equations for the product

$$\tau_{P_4,P_5}\tau_{P_3,P_4}\tau_{P_2,P_3}\tau_{P_1,P_2}\tau_{O,P_1} ?$$

Exercise 34 What is the image of the line with equation $y = 5x + 7$ under σ_P , when $P = (-3, 2)$?

Exercise 35 If α is a translation, show that $\alpha\sigma_P$ is the halfturn about the midpoint of points P and $\alpha(P)$. What is $\sigma_P\alpha$?

Exercise 36 Draw line $\mathcal{L}$ with equation $y = 5x + 7$ and point P with coordinates $(2, 3)$. Then draw $\sigma_P(\mathcal{L})$.

Exercise 37 Show that $\tau_{P,Q}$ has infinite order if $P \neq Q$.

Exercise 38 Suppose that $\langle \tau_{P,Q} \rangle$ is a subgroup of $\langle \tau_{R,S} \rangle$. Show there is a positive integer n such that $PQ = nRS$.

Exercise 39 PROVE or DISPROVE : $\langle \tau_{P,Q} \rangle = \langle \tau_{R,S} \rangle$ implies $\tau_{P,Q} = \tau_{R,S}$ or $\tau_{P,Q} = \tau_{S,R}$.

Exercise 40 Consider the points $A = (-1, -1)$, $B = (0, 0)$, $C = (1, 0)$, $D = (1, 1)$, and $E = (0, 1)$. Find points X, Y, Z such that :

- (a) $\sigma_A\sigma_E\sigma_D = \sigma_X$.
- (b) $\sigma_D\tau_{A,C} = \sigma_Y$.
- (c) $\tau_{B,C}\tau_{A,B}\tau_{E,A}(A) = Z$.

DISCUSSION :

In the Euclidean plane $\mathbb{E}^2$, for each line $\mathcal{L}$ and point $P \notin \mathcal{L}$ there is a unique line $\mathcal{L}'$ through P which does not meet $\mathcal{L}$. The line $\mathcal{L}'$ is called the *parallel* to $\mathcal{L}$ through P . Parallels provide us with a global notion of *direction* in the Euclidean plane. Each member of a family of parallel lines has the same direction, measured by the angle a member of the family makes with the x -axis, and parallels are a constant distance apart. A translation slides each member of a family of parallels along itself a constant distance. Consequently, translations always commute.

The situation changes in other spaces (with “non-euclidean” geometries). For example, in the *sphere* $\mathbb{S}^2$ (viewed as a surface of positive constant curvature in Euclidean 3-dimensional space) the “lines” are *great circles* (i.e. intersections of the sphere with planes through the origin), and hence any two of them intersect. Thus, there are *no* parallels, no global notion of direction (which way is north at the north pole ?), and no translations. Each rotation slides just *one* line (great circle) along

itself, together with the curves at constant distance from this line. These “equidistant curves”, however, are not lines.

Another example is the *hyperbolic plane* $\mathbb{H}^2$ (viewed as a surface of negative constant curvature in Euclidean 3-dimensional space, the *pseudosphere*). In this case, there are many lines $\mathcal{L}'$ through a point $P \notin \mathcal{L}$ which do not meet $\mathcal{L}$. (This is typical of the way hyperbolic geometry departs from Euclidean – in the opposite way from spherical geometry.) Translations exist, but each translation slides just one line along itself, together with the curves at constant distance from this line. These “equidistant curves” are also no lines, and translations with different invariant lines do not commute.

The most suggestive and notable achievement of the last [19th] century is the discovery of non-Euclidean geometry.

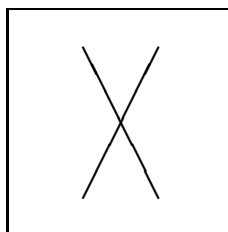
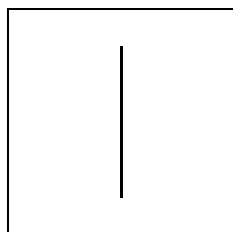
DAVID HILBERT

Chapter 3

Reflections and Rotations

Topics :

1. EQUATIONS FOR A REFLECTION
 2. PROPERTIES OF A REFLECTION
 3. ROTATIONS
-



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3.1 Equations for a Reflection

A *reflection* will be defined as a transformation leaving invariant every point of a fixed line $\mathcal{L}$ and no other points. (An optical reflection along $\mathcal{L}$ in a mirror having both sides silvered, would yield the same result.) We make the following definition.

3.1.1 DEFINITION. **Reflection $\sigma_{\mathcal{L}}$ in line $\mathcal{L}$** is the mapping

$$\sigma_{\mathcal{L}} : \mathbb{E}^2 \rightarrow \mathbb{E}^2, \quad P \mapsto \begin{cases} P, & \text{if point } P \text{ is on } \mathcal{L} \\ Q, & \text{if point } P \text{ is off } \mathcal{L} \text{ and } \mathcal{L} \text{ is the} \\ & \text{perpendicular bisector of } \overline{PQ}. \end{cases}$$

The line $\mathcal{L}$ is usually referred to as the **mirror** of the reflection.

NOTE : We do not use the word *reflection* to denote the image of a point or of a set of points. A reflection is a transformation and never a set of points. Point $\sigma_{\mathcal{L}}(P)$ is the *image* of point P under the reflection $\sigma_{\mathcal{L}}$.

3.1.2 PROPOSITION. *Reflection $\sigma_{\mathcal{L}}$ is an involutory transformation that interchanges the halfplanes of $\mathcal{L}$. Reflection $\sigma_{\mathcal{L}}$ fixes point P if and only if P is on $\mathcal{L}$. Reflection $\sigma_{\mathcal{L}}$ fixes line $\mathcal{M}$ pointwise if and only if $\mathcal{M} = \mathcal{L}$. Reflection $\sigma_{\mathcal{L}}$ fixes line $\mathcal{M}$ if and only if $\mathcal{M} = \mathcal{L}$ or $\mathcal{M} \perp \mathcal{L}$.*

PROOF : It follows immediately from the definition that

$$\sigma_{\mathcal{L}} \neq \iota \quad \text{but} \quad \sigma_{\mathcal{L}}^2 = \iota$$

as the perpendicular bisector of $\overline{PQ}$ is the perpendicular bisector of $\overline{QP}$. Hence, $\sigma_{\mathcal{L}}$ is onto as $\sigma_{\mathcal{L}}(P)$ is the point mapped onto the given point P since $\sigma_{\mathcal{L}}(\sigma_{\mathcal{L}}(P)) = P$ for any point P . Also, $\sigma_{\mathcal{L}}$ is one-to-one as

$$\sigma_{\mathcal{L}}(A) = \sigma_{\mathcal{L}}(B) \quad \text{implies} \quad A = \sigma_{\mathcal{L}}(\sigma_{\mathcal{L}}(A)) = \sigma_{\mathcal{L}}(\sigma_{\mathcal{L}}(B)) = B.$$

Therefore, $\sigma_{\mathcal{L}}$ is an involutory transformation. Then, from the definition of $\sigma_{\mathcal{L}}$, it follows that $\sigma_{\mathcal{L}}$ interchanges the halfplanes of $\mathcal{L}$.

NOTE : In fact, any involutory mapping (on $\mathbb{E}^2$) is a transformation (and hence an involution).

Clearly, $\sigma_{\mathcal{L}}$ fixes point P if and only if P is on $\mathcal{L}$. Not only does $\sigma_{\mathcal{L}}$ fix line $\mathcal{L}$, but $\sigma_{\mathcal{L}}$ fixes every point on $\mathcal{L}$.

NOTE : In general, transformation α is said to **fix pointwise** set $\mathbf{S}$ of points if $\alpha(P) = P$ for every point P in $\mathbf{S}$; that is, if α leaves invariant (unchanged) every point in $\mathbf{S}$. Observe the difference between fixing a set and fixing a set pointwise. Every line perpendicular to $\mathcal{L}$ is fixed by $\sigma_{\mathcal{L}}$, but none of these lines is fixed pointwise as each contains only one fixed point.

Suppose line $\mathcal{M}$ is distinct from $\mathcal{L}$ and is fixed by $\sigma_{\mathcal{L}}$. Let $Q = \sigma_{\mathcal{L}}(P)$ for some point P that is on $\mathcal{M}$ but off $\mathcal{L}$. Then P and Q are both on $\mathcal{M}$ since $\mathcal{M}$ is fixed, and $\mathcal{L}$ is the perpendicular bisector of $\overline{PQ}$. Hence, $\mathcal{L}$ and $\mathcal{M}$ are perpendicular. $\square$

Exercise 41 Show that if the nonidentity mapping $\alpha : \mathbf{A} \rightarrow \mathbf{A}$ is *involutory* (i.e. α^2 is the identity mapping), then it is invertible.

NOTE : We have used the Greek letter *sigma* for both halfturns and reflections; the Greek letter *rho* is left free for use later with rotations. The Greek σ corresponds to the Roman *s* which begins the German word *Spiegelung*, meaning *reflection*. A halfturn is a sort of “reflection in a point”. The similar notation for halfturns and reflections emphasizes the important property they do share, namely that of being involutions :

$$\sigma_{\mathcal{L}} = \sigma_{\mathcal{L}}^{-1}, \quad \sigma_P = \sigma_P^{-1}.$$

What are the equations for a reflection ?

3.1.3 PROPOSITION. *If line $\mathcal{L}$ has equation $ax+by+c=0$, then reflection $\sigma_{\mathcal{L}}$ has equations :*

$$\begin{cases} x' = x - \frac{2a(ax+by+c)}{a^2+b^2} \\ y' = y - \frac{2b(ax+by+c)}{a^2+b^2} . \end{cases}$$

PROOF : Let $P = (x, y)$ and $\sigma_{\mathcal{L}}(P) = (x', y') = Q$. For the moment, suppose that P is off $\mathcal{L}$. Now, the line through points P and Q is perpendicular to line $\mathcal{L}$. This geometric fact is expressed algebraically by the equation

$$b(x' - x) = a(y' - y).$$

Also,

$$\left(\frac{x+x'}{2}, \frac{y+y'}{2} \right) \text{ is the midpoint of } \overline{PQ} \text{ and is on } \mathcal{L}.$$

This geometric fact is expressed algebraically by the equation

$$a \left(\frac{x+x'}{2} \right) + b \left(\frac{y+y'}{2} \right) + c = 0 .$$

Rewriting these two equations as

$$\begin{cases} bx' - ay' = bx - ay \\ ax' + by' = -2c - ax - by \end{cases}$$

we see we have two linear equations in two unknowns x' and y' . Solving these equations for x' and y' (by using Cramer's rule, for instance), we get

$$\begin{cases} x' = \frac{a^2x + b^2x - 2a^2x - 2aby - 2ac}{a^2 + b^2} \\ y' = \frac{a^2y + b^2y - 2b^2y - 2abx - 2bc}{a^2 + b^2} . \end{cases}$$

With these equations in the form

$$\begin{cases} x' = x - \frac{2a(ax+by+c)}{a^2+b^2} \\ y' = y - \frac{2b(ax+by+c)}{a^2+b^2} \end{cases}$$

it is easy to check that the equations also hold when P is on $\mathcal{L}$. This proves the result. $\square$

NOTE : Suppose we had defined a reflection as a transformation having equations given by PROPOSITION 3.1.3. Not only would this have seemed artificial, since these equations are not something you would think of examining in the first place, but just imagine trying to prove PROPOSITION 3.1.2 from these equations. Although this is conceptually easy, the actual computation involves a considerable amount of algebra.

3.2 Properties of a Reflection

We have already mentioned those properties of a reflection that follow immediately from the definition. Another important property is that *a reflection preserves distance*, which means the distance from $\sigma_{\mathcal{L}}(P)$ to $\sigma_{\mathcal{L}}(Q)$ is equal to the distance from P to Q , for all points P and Q . The following definition is fundamental.

3.2.1 DEFINITION. A transformation α is an **isometry** (or **congruent transformation**) if $P'Q' = PQ$ for all points P and Q , where $P' = \alpha(P)$ and $Q' = \alpha(Q)$.

In other words, an isometry is a distance-preserving transformation.

NOTE : (1) In fact, *any distance-preserving mapping is an isometry*. Such a mapping is one-to-one – because points at nonzero distance cannot have images at zero distance – but it is not clear that such a mapping is onto.

(2) The name *isometry* comes from the Greek *isos* (equal) and *metron* (measure). An isometry is also called a **rigid motion**.

The set of all isometries form a group. This group is denoted by **Isom**.

3.2.2 PROPOSITION. *Reflection $\sigma_{\mathcal{L}}$ is an isometry.*

PROOF : We shall consider several cases. Suppose P and Q are two points, $P' = \sigma_{\mathcal{L}}(P)$ and $Q' = \sigma_{\mathcal{L}}(Q)$. We must show $P'Q' = PQ$.

(a) If $\overleftrightarrow{PQ} = \mathcal{L}$ or if $\overleftrightarrow{PQ} \perp \mathcal{L}$, then the desired result follows immediately from the definition of $\sigma_{\mathcal{L}}$.

(b) Also, if $\overleftrightarrow{PQ}$ is parallel to $\mathcal{L}$ but distinct from $\mathcal{L}$, the result follows easily as $\square PQQ'P'$ is a rectangle and so opposite sides $\overline{PQ}$ and $\overline{P'Q'}$ are congruent.

(c) Further, if one of P or Q , say P , is on $\mathcal{L}$ and Q is off $\mathcal{L}$, then $P'Q' = PQ$ follows from the fact that $P' = P$ and that $\mathcal{L}$ is the locus of all points equidistant from Q and Q' .

(d) Finally, suppose P and Q are both off $\mathcal{L}$ and that $\overleftrightarrow{PQ}$ intersects $\mathcal{L}$ at point R , but is not perpendicular to $\mathcal{L}$. So $RP = RP'$ and $RQ = RQ'$. The desired result, $P'Q' = PQ$, then follows provided R, P', Q' are shown to be collinear. $\square$

Exercise 42 Prove the preceding statement.

Exercise 43 Are translations and halfturns isometries? Why? (Hence the group of translations $\mathfrak{T}$ and the group $\mathfrak{H}$ are subgroups of $\mathfrak{Isom}$.)

Now that we know a reflection is an isometry, a long sequence of other properties dependent only on distance will follow.

3.2.3 PROPOSITION. *An isometry is a collineation that preserves betweenness, midpoints, segments, rays, triangles, angles, angle measure, and perpendicularity.*

PROOF : Since these properties are shared by all isometries, we shall consider a general isometry α .

(a) Suppose A, B, C are any three points and let $A' = \alpha(A)$, $B' = \alpha(B)$, $C' = \alpha(C)$. Since α preserves distance, if $AB + BC = AC$ then $A'B' + B'C' = A'C'$ as $A'B' = AB$, $B'C' = BC$, and $A'C' = AC$. Hence, $A - B - C$ implies $A' - B' - C'$; in other words, if B is between A and C , then B' is between A' and C' . We describe this by saying that α preserves betweenness.

(b) The special case $AB = BC$ in the argument above implies $A'B' = B'C'$. In other words, if B is the midpoint of A and C , then B' is the midpoint of A' and C' . Thus we say α *preserves midpoints*.

(c) More generally, since $\overline{AB}$ is the union of A, B , and all points between A and B , then $\alpha(\overline{AB})$ is the union of A', B' , and all points between A' and B' . So $\alpha(\overline{AB}) = \overline{A'B'}$ and we say α *preserves segments*.

(d) Likewise, since α is onto by definition and $AB^\rightarrow$ is the union of $\overline{AB}$ and all points C such that $A - B - C$, then $\alpha(AB^\rightarrow)$ is the union of $\overline{A'B'}$ and all points C' such that $A' - B' - C'$. So $\alpha(AB^\rightarrow) = A'B'^\rightarrow$ and we say α *preserves rays*.

(e) Since $\overleftrightarrow{AB}$ is the union $AB^\rightarrow$ and $BA^\rightarrow$, then $\alpha(\overleftrightarrow{AB})$ is the union of $A'B'^\rightarrow$ and $B'A'^\rightarrow$, which is $\overleftrightarrow{A'B'}$. So α is a transformation that preserves lines ; in other words, α *is a collineation*.

(f) If A, B, C , are not collinear, then $AB + BC > AC$ and so $A'B' + B'C' > A'C'$ and A', B', C' are not collinear. Then, since $\triangle ABC$ is a union of the three segments $\overline{AB}, \overline{BC}, \overline{CA}$, then we conclude that $\alpha(\triangle ABC)$ is just $\triangle A'B'C'$. So an isometry *preserves triangles*.

(g) It follows that α *preserves angles* as $\alpha(\angle ABC) = \angle A'B'C'$.

(h) Not only does α preserve angles, but α also *preserves angle measure*. That is, $m(\angle ABC) = m(\angle A'B'C')$ since $\triangle ABC \cong \triangle A'B'C'$ by *SSS*.

(i) Finally, if $\overrightarrow{BA} \perp \overrightarrow{BC}$ then $\overrightarrow{B'A'} \perp \overrightarrow{B'C'}$ since $m(\angle ABC) = 90$ implies $m(\angle A'B'C') = 90$. So α *preserves perpendicularity*. $\square$

3.3 Rotations

We shall now formally define rotations in the most elementary manner.

3.3.1 DEFINITION. A **rotation about point C through directed angle of r°** is the transformation $\rho_{C,r}$ that fixes C and otherwise sends a point P to the point P' , where $CP' = CP$ and r is the directed angle measure of the directed angle from $\overrightarrow{CP}$ to $\overrightarrow{CP'}$.

We agree that $\rho_{C,0}$ is the identity ι . Rotation $\rho_{C,r}$ is said to have **centre** C and **directed angle** r° .

3.3.2 PROPOSITION. *A rotation is an isometry.*

PROOF : Suppose $\rho_{C,r}$ sends points P and Q to P' and Q' , respectively. If C, P, Q are collinear, then $PQ = P'Q'$ by the definition. If C, P, Q are not collinear, then $\triangle PCQ \cong \triangle P'C'Q'$ by *SAS* and $PQ = P'Q'$. So $\rho_{C,r}$ is a transformation that preserves distance. $\square$

3.3.3 PROPOSITION. *A nonidentity rotation fixes exactly one point, its centre. A rotation with centre C fixes every circle with centre C .*

PROOF : For distinct points C and P , circle C_P is defined to be the circle with centre C and radius CP . So $\overline{CP}$ is a radius of the circle C_P , and point P is on the circle. The result also follows immediately from the definition of a rotation. $\square$

Exercise 44 Show that (for point C and real numbers r and s)

$$\rho_{C,s}\rho_{C,r} = \rho_{C,r+s} \quad \text{and} \quad \rho_{C,r}^{-1} = \rho_{C,-r}.$$

3.3.4 COROLLARY. *The rotations with centre C form a commutative group.*

NOTE : (1) The involutory rotations are the halfturns, and (for any point C)

$$\rho_{C,180} = \sigma_C.$$

(2) Observe that, for example, $\rho_{C,30} = \rho_{C,390} = \rho_{C,-330}$. In general, for real numbers r and s , we have

$$r^\circ = s^\circ \iff r = s + 360k, \quad k \in \mathbb{Z}.$$

For distinct intersecting lines $\mathcal{L}$ and $\mathcal{M}$, there are *two* directed angles from $\mathcal{L}$ to $\mathcal{M}$. Clearly, these will have directed angle measures that differ by a multiple of 180. If r and s are the directed angle measures of the two directed angles from $\mathcal{L}$ to $\mathcal{M}$, then $(2r)^\circ = (2s)^\circ$, since numbers r and s differ by a multiple of 180. So, *if we are talking about the rotation through twice a directed angle from line $\mathcal{L}$ to line $\mathcal{M}$, then it makes no difference which of the two directed angles we choose.*

3.4 Exercises

Exercise 45 Given point P off line $\mathcal{L}$, construct $\rho_{P,60}(\mathcal{L})$.

Exercise 46 TRUE or FALSE ?

- (a) If isometry α interchanges distinct points P and Q , then α fixes the midpoint of P and Q .
- (b) $\sigma_{\mathcal{L}} = \sigma_P^{-1}$ if point P is on line $\mathcal{L}$.
- (c) Reflection $\sigma_{\mathcal{L}}$ fixes the halfplanes of $\mathcal{L}$ but does not fix the halfplanes pointwise.
- (d) Reflection $\sigma_{\mathcal{L}}$ fixes line $\mathcal{M}$ if and only if $\mathcal{L} \perp \mathcal{M}$.
- (e) For line $\mathcal{L}$ and point P , $\sigma_{\mathcal{L}} = \sigma_{\mathcal{L}}^{-1} \neq \iota$ and $\sigma_P = \sigma_P^{-1} \neq \iota$.
- (f) $\rho_{C,r}^{-1} = \rho_{C,-r} = \sigma_C$ for any point C .

Exercise 47 What are the images of $(0, 0)$, $(1, -3)$, $(-2, 1)$, and $(2, 4)$ under the reflection in the line with equation $y = 2x - 5$?

Exercise 48 Describe the product of the reflection in $\overleftrightarrow{OO'}$ and the halfturn about O .

Exercise 49 PROVE or DISPROVE :

- (a) $\sigma_{\mathcal{L}}\sigma_{\mathcal{M}} = \sigma_{\mathcal{M}}\sigma_{\mathcal{L}} \iff \mathcal{L} \perp \mathcal{M}$.
- (b) $\sigma_P\sigma_{\mathcal{L}} = \sigma_{\mathcal{L}}\sigma_P \iff P \in \mathcal{L}$.

Exercise 50 PROVE or DISPROVE : If ρ is a rotation, then the cyclic group $\langle \rho \rangle$ is finite.

DISCUSSION :

An *axiomatic system* provides an explicit foundation for a mathematical subject. Axiomatic systems include several parts : the logical language, rules of proof, undefined terms, axioms, definitions, theorems and proofs of theorems, and models.

Consider EUCLID's definition of a point as "that which has no part". This definition is more a philosophical statement about the nature of a point than a way to

prove statements. EUCLID's definition of a straight line, "a line which lies evenly with the points on itself", is unclear as well as not useful. In essence, points and lines were so basic to EUCLID's work that there is no good way to define them. Mathematicians realized centuries ago the need for undefined terms to establish an unambiguous beginning. (Otherwise, each term would have to be defined with other terms, leading either to a cycle of terms or an infinite sequence of terms. Neither of these options is acceptable for carefully reasoned mathematics.) Of course, we then define all other terms from these initial, undefined terms. However, undefined terms are, by their nature, unrestricted. How can we be sure that two people *mean* the same thing when they use undefined terms? In short, we can't. The axioms of a mathematical system become the "key": they tell us how the undefined terms *behave*. (Axioms describe how to use terms and how they relate to one another, rather than telling us what the terms "really mean".) Indeed, mathematicians permit any *interpretation* of undefined terms, as long as all the axioms hold in that interpretation.

Unlike the Greek understanding of axioms as "self-evident truths", we do not claim the truth of axioms. However, this does not mean that we consider axioms to be false. Rather, we are free to choose axioms to formulate the fundamental relationships we want to investigate. From a logical point of view, the choice of axioms is arbitrary; in actuality, though, mathematicians carefully pick axioms to focus on particular features. *Axiomatic systems allow us to formulate and logically explore abstract relationships, freed from the specificity and imprecision of real situations.* There are two basic types of axiomatic systems. One completely characterizes a particular mathematical system (for example, HILBERT's axioms characterize *Euclidean plane geometry* completely). The second focuses on the common features of a family of *structures* (e.g. groups, vector spaces, or metric spaces); such axiomatic systems unite a wide variety of examples within one powerful theoretical framework.

Mathematical definitions are built from undefined terms and previously defined terms.

In an axiomatic system, a *theorem* is a statement whose proof depends only on previously proven theorems, the axioms, the definitions, and the rules of logic. (This condition ensures that the entire edifice of theorems rests securely on the explicit axioms of the system.) Proofs of theorems in an axiomatic system cannot depend on diagrams, even though diagrams have been part of geometry since the ancient Greeks drew figures in the sand.

Axiomatic systems are a workable compromise between the austere formal languages of mathematical logic and Euclid's work with its many implicit assumptions. *Mathematicians need both the careful reasoning of proofs and the intuitive understanding of content.* Axiomatic systems provide more than a way to give careful proofs. They enable us to understand the relationship of particular concepts, to explore the consequences of assumptions, to contrast different systems, and to unify seemingly disparate situations under one framework. In short, *axiomatic systems are one important way in which mathematicians obtain insight.*

Mathematical models provide an explicit link between intuitions and undefined terms. The usual (Cartesian) model of Euclidean plane geometry is the set $\mathbb{R}^2$, where a *point* is interpreted as an ordered pair of (real) numbers and a *line* is interpreted as the locus of points that satisfy an appropriate (first degree) algebraic equation $ax + by + c = 0$. (In making a model, we are free to interpret the undefined terms in any way we want, provided that all the axioms hold under our interpretation. Note that the axioms are not by themselves true; a context is needed to give meaning to the axioms in order for them to be true or false.) Models do much more than provide concrete examples of axiomatic systems : they lead to important understandings about axiomatic systems. The most important property of an axiomatic system is *consistency*, which says that we cannot prove two statements that contradict each other. An axiomatic system is consistent if and only if it has a model.

I am coming more and more to the conviction that the necessity of our geometry cannot be demonstrated, at least neither by, nor for, the human intellect. [...] geometry should be ranked, not with arithmetic, which is purely aprioristic, but with mechanics.

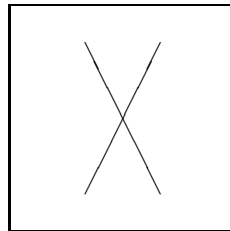
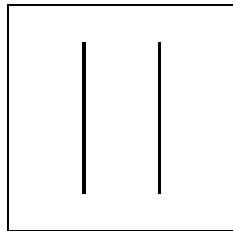
CARL FRIEDRICH GAUSS

Chapter 4

Isometries I

Topics :

1. ISOMETRIES AS PRODUCT OF REFLECTIONS
 2. THE PRODUCT OF TWO REFLECTIONS
 3. FIXED POINTS AND INVOLUTIONS
-



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4.1 Isometries as Product of Reflections

A product of reflections is clearly an isometry. The converse is also true; that is, *every isometry is a product of reflections*. We prove now this fact in seven (small) steps. (Actually, we shall do better than that by showing the product has *at most* three factors, not necessarily distinct.)

Looking at the fixed points of isometries turns out to be very rewarding in general.

4.1.1 PROPOSITION. *If an isometry fixes two distinct points on a line, then the isometry fixes that line pointwise.*

PROOF : Knowing point P is on the line through distinct points A and B and knowing the nonzero distance AP , we do not know which of the two possible points is P . However, if we also know the distance BP , then P is uniquely determined. It follows that an isometry fixing both A and B must also fix the point P , since an isometry is a collineation that preserves distance. In other words, an isometry fixing distinct points A and B must fix every point on the line through A and B .

□

4.1.2 PROPOSITION. *If an isometry fixes three noncollinear points, then the isometry must be the identity.*

PROOF : Suppose that an isometry fixes each of three noncollinear points A, B, C . Then the isometry must fix every point on $\triangle ABC$ as the isometry fixes every point on any one of the lines $\overleftrightarrow{AB}, \overleftrightarrow{BC}, \overleftrightarrow{CA}$. Every point Q in the plane lies on a line that intersects $\triangle ABC$ in two distinct points. Hence the point Q is on a line containing two fixed points and, therefore, must also be fixed. So an isometry that fixes three noncollinear points must fix every point Q in the plane.

□

4.1.3 PROPOSITION. *If α and β are isometries such that*

$$\alpha(P) = \beta(P), \quad \alpha(Q) = \beta(Q), \quad \text{and} \quad \alpha(R) = \beta(R)$$

for three noncollinear points P, Q, R , then $\alpha = \beta$.

PROOF : Multiplying each of the given equations by β^{-1} on the left, we see that $\beta^{-1}\alpha$ fixes each of the noncollinear points P, Q, R . Then $\beta^{-1}\alpha = \iota$ by PROPOSITION 4.1.2. Multiplying this last equation by β on the left, we have $\alpha = \beta$. $\square$

4.1.4 PROPOSITION. *An isometry that fixes two points is a reflection or the identity.*

PROOF : Suppose isometry α fixes distinct points P and Q on line $\mathcal{L}$. We know two possibilities for α , namely ι and $\sigma_{\mathcal{L}}$. We shall show these are the only two possibilities by supposing $\alpha \neq \iota$ and proving $\alpha = \sigma_{\mathcal{L}}$. If $\alpha \neq \iota$, then there is a point R not fixed by α . So R is off $\mathcal{L}$, and P, Q, R are three noncollinear points. Let $R' = \alpha(R)$. So $PR = PR'$ and $QR = QR'$, as α is an isometry. Therefore, $\mathcal{L}$ is the perpendicular bisector of $\overline{RR'}$ as each of P and Q is in the locus of all points equidistant from R and R' . Hence,

$$\alpha(R) = R' = \sigma_{\mathcal{L}}(R) \quad \text{as well as} \quad \alpha(P) = P = \sigma_{\mathcal{L}}(P) \quad \text{and} \quad \alpha(Q) = Q = \sigma_{\mathcal{L}}(Q).$$

By PROPOSITION 4.1.3 we have $\alpha = \sigma_{\mathcal{L}}$. $\square$

4.1.5 PROPOSITION. *An isometry that fixes exactly one point is a product of two reflections.*

PROOF : Suppose isometry α fixes exactly one point C . Let P be a point different from C , let $\alpha(P) = P'$, and let $\mathcal{L}$ be the perpendicular bisector of $\overline{PP'}$. Since $CP = CP'$ as α is an isometry, then C is on $\mathcal{L}$. So $\sigma_{\mathcal{L}}(C) = C$ and $\sigma_{\mathcal{L}}(P') = P$. Then

$$\sigma_{\mathcal{L}}\alpha(C) = \sigma_{\mathcal{L}}(C) = C \quad \text{and} \quad \sigma_{\mathcal{L}}\alpha(P) = \sigma_{\mathcal{L}}(P') = P.$$

By PROPOSITION 4.1.4

$$\sigma_{\mathcal{L}}\alpha = \iota \quad \text{or} \quad \sigma_{\mathcal{L}}\alpha = \sigma_{\mathcal{M}}, \quad \text{where } \mathcal{M} = \overleftrightarrow{CP}.$$

However, $\sigma_{\mathcal{L}}\alpha \neq \iota$ as otherwise α is $\sigma_{\mathcal{L}}$ and fixes more points than C . Thus $\sigma_{\mathcal{L}}\alpha = \sigma_{\mathcal{M}}$ for some line $\mathcal{M}$. Multiplying this equation by $\sigma_{\mathcal{L}}$ on the left, we have $\alpha = \sigma_{\mathcal{L}}\sigma_{\mathcal{M}}$. $\square$

4.1.6 PROPOSITION. *An isometry that fixes a point is a product of at most two reflections.*

PROOF : Since $\iota = \sigma_{\mathcal{L}}\sigma_{\mathcal{L}}$ for any line $\mathcal{L}$, the result follows as a corollary of PROPOSITION 4.1.5. $\square$

We are now prepared to prove the main result.

4.1.7 THEOREM. *Every isometry is a product of at most three reflections.* (We count the number of factors even though the factors themselves may not be distinct.)

PROOF : The identity is a product of two reflections. Suppose nonidentity isometry α sends point P to different point Q . Let $\mathcal{L}$ be the perpendicular bisector of $\overline{PQ}$. Then $\sigma_{\mathcal{L}}\alpha$ fixes point P . We have just seen that $\sigma_{\mathcal{L}}\alpha$ must be a product β of at most two reflections. Hence $\alpha = \sigma_{\mathcal{L}}\beta$ and α is a product of at most three reflections. $\square$

Congruence

Suppose $\triangle PQR \cong \triangle ABC$. We know there is *at most* one isometry α such that

$$\alpha(P) = A, \alpha(Q) = B, \quad \text{and} \quad \alpha(R) = C.$$

The question is whether there exists *at least* one such isometry α . It is possible to construct effectively such an isometry (as a product of at most three reflections).

4.1.8 PROPOSITION. *If $\triangle PQR \cong \triangle ABC$, then there is a unique isometry α such that*

$$\alpha(P) = A, \quad \alpha(Q) = B, \quad \text{and} \quad \alpha(R) = C.$$

PROOF : Suppose $\triangle PQR \cong \triangle ABC$. So $AB = PQ$, $AC = PR$, and $BC = QR$. If $P \neq A$, then let $\alpha_1 = \sigma_{\mathcal{L}}$, where $\mathcal{L}$ is the perpendicular bisector of $\overline{PA}$. If $P = A$, then let $\alpha_1 = \iota$. In either case, then $\alpha_1(P) = A$. Let $\alpha_1(Q) = Q_1$ and $\alpha_1(R) = R_1$. If $Q_1 \neq B$, then let $\alpha_2 = \sigma_{\mathcal{M}}$, where $\mathcal{M}$ is the perpendicular bisector of $\overline{Q_1B}$. In this case, point A is on $\mathcal{M}$ as $AB = PQ = AQ_1$. If $Q_1 = B$, then let $\alpha_2 = \iota$. In either case, we have $\alpha_2(A) = A$ and $\alpha_2(Q_1) = B$. Let $\alpha_2(R_1) = R_2$. If $R_2 \neq C$, then let $\alpha_3 = \sigma_{\mathcal{N}}$, where $\mathcal{N}$ is the perpendicular bisector of $\overline{R_2C}$. In this case, $\mathcal{N} = \overleftrightarrow{AB}$ as $AC = PR = AR_1 = AR_2$ and $BC = QR = Q_1R_1 = BR_2$. If $R_2 = C$, then let $\alpha_3 = \iota$. In any case, we have $\alpha_3(A) = A$, $\alpha_3(B) = B$, and $\alpha_3(R_2) = C$. Let $\alpha = \alpha_3\alpha_2\alpha_1$. Then

$$\begin{aligned} \alpha(P) &= \alpha_3\alpha_2\alpha_1(P) = \alpha_3\alpha_2(A) = \alpha_3(A) = A \\ \alpha(Q) &= \alpha_3\alpha_2\alpha_1(Q) = \alpha_3\alpha_2(Q_1) = \alpha_3(B) = B \\ \alpha(R) &= \alpha_3\alpha_2\alpha_1(R) = \alpha_3\alpha_2(R_1) = \alpha_3(R_2) = C \end{aligned}$$

as desired. □

4.1.9 COROLLARY. *Two segments, two angles, or two triangles are, respectively, congruent if and only if there is an isometry taking one to other.*

NOTE : In elementary plane geometry there are three different relations indicated by the same words “is congruent to”, one for segments, one for angles, and a third for triangles. All three can be combined under a generalized definition that applies to arbitrary sets of points as follows. If $\mathbf{S}_1$ and $\mathbf{S}_2$ are sets of points, then $\mathbf{S}_1$ and $\mathbf{S}_2$ are said to be **congruent** if there is an isometry α such that

$$\alpha(\mathbf{S}_1) = \mathbf{S}_2.$$

Exercise 51 Give a reasonable definition for $\square ABCD \cong \square PQRS$.

4.2 The Product of Two Reflections

Every isometry is a product of at most three reflections (see THEOREM 4.1.7). So each isometry is of the form

$$\sigma_{\mathcal{L}}, \quad \sigma_{\mathcal{M}}\sigma_{\mathcal{L}}, \quad \text{or} \quad \sigma_{\mathcal{N}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}.$$

We shall examine now the case $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$. Since a reflection is an involution, we know that $\sigma_{\mathcal{L}}\sigma_{\mathcal{L}} = \iota$ for any line $\mathcal{L}$.

Thus we are concerned with the product of two reflections in distinct lines $\mathcal{L}$ and $\mathcal{M}$. There are two cases : either $\mathcal{L}$ and $\mathcal{M}$ are parallel lines or else $\mathcal{L}$ and $\mathcal{M}$ intersect at a unique point.

Case 1 : $\mathcal{L}$ and $\mathcal{M}$ are (distinct) parallel lines

4.2.1 PROPOSITION. *If lines $\mathcal{L}$ and $\mathcal{M}$ are parallel, then $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$ is the translation through twice the directed distance from $\mathcal{L}$ to $\mathcal{M}$.*

PROOF : Let $\mathcal{L}$ and $\mathcal{M}$ be distinct parallel lines. Suppose $\overleftrightarrow{LM}$ is a common perpendicular to $\mathcal{L}$ and $\mathcal{M}$ with L on $\mathcal{L}$ and M on $\mathcal{M}$. The directed distance from $\mathcal{L}$ to $\mathcal{M}$ is the directed distance from L to M . (We are going to use PROPOSITION 4.1.3.) With K a point on $\mathcal{L}$ distinct from L , let $L' = \sigma_{\mathcal{M}}(L)$ and $K' = \tau_{L,L'}(K)$. Then (by PROPOSITION 2.1.2 and PROPOSITION 2.1.4) we have $\tau_{K,K'} = \tau_{L,L'}$ and $\square LKK'L'$ is a rectangle with $\mathcal{M}$ the common perpendicular bisector of $\overline{LL'}$ and of $\overline{KK'}$. So

$$\sigma_{\mathcal{M}}(K) = K'.$$

Now, let $J = \sigma_{\mathcal{L}}(M)$. Then, since L is the midpoint of $\overline{JM}$ and M is the midpoint of $\overline{LL'}$, we have

$$\tau_{J,M} = \tau_{L,L'}$$

where $\tau_{L,L'}$ is the translation through *twice* the directed distance from $\mathcal{L}$ to

$\mathcal{M}$. Hence

$$\begin{aligned}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}(J) &= \sigma_{\mathcal{M}}(M) = M = \tau_{L,L'}(J) \\ \sigma_{\mathcal{M}}\sigma_{\mathcal{L}}(K) &= \sigma_{\mathcal{M}}(K) = K' = \tau_{L,L'}(K) \\ \sigma_{\mathcal{M}}\sigma_{\mathcal{L}}(L) &= \sigma_{\mathcal{M}}(L) = L' = \tau_{L,L'}(L).\end{aligned}$$

Since an isometry is determined by any three noncollinear points (see PROPOSITION 4.1.2), the equations above give the desired result

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \tau_{L,L'} = \tau_{L,M}^2.$$

□

4.2.2 PROPOSITION. *If line $\mathcal{A}$ is perpendicular to line $\mathcal{L}$ at L and to line $\mathcal{M}$ at M , then*

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \tau_{L,M}^2 = \sigma_M\sigma_L.$$

PROOF : In the proof above we have $\tau_{L,L'} = \sigma_M\sigma_L$ (PROPOSITION 2.2.3). □

Is every translation a product of two reflections (in parallel lines) ? The answer is “Yes”. The following result holds.

4.2.3 THEOREM. *Every translation is a product of two reflections in parallel lines, and, conversely, a product of two reflections in parallel lines is a translation.*

PROOF : Given nonidentity translation $\tau_{L,N}$, then $\tau_{L,N} = \sigma_M\sigma_L$, where M is the midpoint of $\overleftrightarrow{LN}$. With $\mathcal{L}$ the perpendicular to $\overleftrightarrow{LM}$ at L and $\mathcal{M}$ the perpendicular to $\overleftrightarrow{LM}$ at M , we have

$$\sigma_M\sigma_L = \sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$$

by PROPOSITION 4.2.2. So

$$\tau_{L,N} = \sigma_{\mathcal{M}}\sigma_{\mathcal{L}} \quad \text{with} \quad \mathcal{L} \parallel \mathcal{M}.$$

□

4.2.4 PROPOSITION. *If lines $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are perpendicular to line $\mathcal{A}$, then there are unique lines $\mathcal{P}$ and $\mathcal{Q}$ such that*

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}} = \sigma_{\mathcal{Q}}\sigma_{\mathcal{N}}.$$

Further, the lines $\mathcal{P}$ and $\mathcal{Q}$ are perpendicular to $\mathcal{A}$.

PROOF : The equations

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}} = \sigma_{\mathcal{Q}}\sigma_{\mathcal{N}}$$

have unique solutions for lines $\mathcal{P}$ and $\mathcal{Q}$, given lines $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are parallel. To show this, let line $\mathcal{A}$ be perpendicular to lines $\mathcal{L}, \mathcal{M}, \mathcal{N}$ at points L, M, N , respectively. Let P and Q be the unique points on $\mathcal{A}$ such that

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}} = \sigma_{\mathcal{Q}}\sigma_{\mathcal{N}}.$$

Let line $\mathcal{P}$ be perpendicular to $\mathcal{A}$ at P , and let line $\mathcal{Q}$ be perpendicular to $\mathcal{A}$ at Q . Then

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}} \quad \text{and} \quad \sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{Q}}\sigma_{\mathcal{N}} = \sigma_{\mathcal{Q}}\sigma_{\mathcal{N}}.$$

The uniqueness of these lines $\mathcal{P}$ and $\mathcal{Q}$ that satisfy the equations follows from the *cancellation laws*. (For example, $\sigma_{\mathcal{N}}\sigma_{\mathcal{P}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}'}$ implies $\sigma_{\mathcal{P}} = \sigma_{\mathcal{P}'}$, which implies $\mathcal{P} = \mathcal{P}'$.) $\square$

NOTE : In the proposition above, $\mathcal{P}$ is just the unique line such that directed distance from $\mathcal{P}$ to $\mathcal{N}$ equals the directed distance from $\mathcal{L}$ to $\mathcal{M}$ and that $\mathcal{Q}$ is just the unique line such that the directed distance from $\mathcal{N}$ to $\mathcal{Q}$ equals the directed distance from $\mathcal{L}$ to $\mathcal{M}$.

4.2.5 COROLLARY. *If $P \neq Q$, then $\tau_{P,Q}$ may be expressed as $\sigma_{\mathcal{B}}\sigma_{\mathcal{A}}$, where either one of $\mathcal{A}$ or $\mathcal{B}$ is an arbitrarily chosen line perpendicular to $\overleftrightarrow{PQ}$ and the other is then a uniquely determined line perpendicular to $\overleftrightarrow{PQ}$.*

Observe that

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}} \quad \text{and} \quad \sigma_{\mathcal{N}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{P}}$$

are equivalent equations.

4.2.6 COROLLARY. *If lines $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are perpendicular to line $\mathcal{A}$, then $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$ is a reflection in a line perpendicular to $\mathcal{A}$.*

Case 2 : $\mathcal{L}$ and $\mathcal{M}$ are (distinct) intersecting lines

$\mathcal{L}$ and $\mathcal{M}$ are lines intersecting at a point C . We shall follow much the same path as we did for parallel lines.

4.2.7 PROPOSITION. *If lines $\mathcal{L}$ and $\mathcal{M}$ intersect at point C and the directed angle measure of a directed angle from $\mathcal{L}$ to $\mathcal{M}$ is $\frac{r}{2}$, then $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \rho_{C,r}$.*

PROOF : We first show that $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$ is a rotation about C by using the fact that three noncollinear points determine an isometry. Suppose $\frac{r}{2}$ is the directed angle measure of one of the two directed angles from $\mathcal{L}$ to $\mathcal{M}$. We may as well suppose $-90 < \frac{r}{2} \leq 90$. (Note that the notation suggests correctly that we are going to encounter *twice* the directed angle from $\mathcal{L}$ to $\mathcal{M}$ in our conclusion.) Let L be a point on $\mathcal{L}$ different from C . Let point M be the intersection of line $\mathcal{M}$ and circle C_L such that the directed angle measure from $\overrightarrow{CL}$ to $\overrightarrow{CM}$ is $\frac{r}{2}$. We have $\mathcal{L} = \overleftrightarrow{CL}$ and $\mathcal{M} = \overleftrightarrow{CM}$. Let $L' = \rho_{C,r}(L)$. Then L' is on circle C_L , and $\mathcal{M}$ is the perpendicular bisector of $\overline{LL'}$. So $L' = \sigma_{\mathcal{M}}(L)$. Let $J = \sigma_{\mathcal{L}}(M)$. Then $\mathcal{L}$ is the perpendicular bisector of $\overline{JM}$. So J is on circle C_L , and the directed angle measure from $\overrightarrow{CJ}$ to $\overrightarrow{CM}$ is r . Hence, $M = \rho_{C,r}(J)$. Therefore,

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}(C) = \sigma_{\mathcal{M}}(C) = C = \rho_{C,r}(C)$$

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}(J) = \sigma_{\mathcal{M}}(M) = M = \rho_{C,r}(J)$$

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}(L) = \sigma_{\mathcal{M}}(L) = L' = \rho_{C,r}(L).$$

Since points C, J, L are not collinear, we conclude

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \rho_{C,r}.$$

So $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$ is the rotation about C through *twice* a directed angle from $\mathcal{L}$ to $\mathcal{M}$. $\square$

The following result (analogue of THEOREM 4.2.3) holds.

4.2.8 THEOREM. *Every rotation is a product of two reflections in intersecting lines, and, conversely, a product of two reflections in intersecting lines is a rotation.*

PROOF : Suppose $\rho_{C,r}$ is given. Let $\mathcal{L}$ be any line through C , and let $\mathcal{M}$ be the line through C such that a directed angle from $\mathcal{L}$ to $\mathcal{M}$ has directed angle measure $\frac{r}{2}$. Then $\rho_{C,r} = \sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$, and this completes the proof. $\square$

4.2.9 PROPOSITION. *If lines $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are concurrent at point C , then there are unique lines $\mathcal{P}$ and $\mathcal{Q}$ such that*

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}} = \sigma_{\mathcal{Q}}\sigma_{\mathcal{N}}.$$

Further, the lines $\mathcal{P}$ and $\mathcal{Q}$ are concurrent at C .

PROOF : Given rays $CL^{\rightarrow}, CM^{\rightarrow}$, and $CN^{\rightarrow}$, there are unique rays $CP^{\rightarrow}$ and $CQ^{\rightarrow}$ such that the directed angle from $CL^{\rightarrow}$ to $CM^{\rightarrow}$, the directed angle from $CP^{\rightarrow}$ to $CN^{\rightarrow}$, and the directed angle from $CN^{\rightarrow}$ to $CQ^{\rightarrow}$, all have the same directed angle measure. With $\mathcal{N} = \overleftrightarrow{CN}$, $\mathcal{P} = \overleftrightarrow{CP}$, and $\mathcal{Q} = \overleftrightarrow{CQ}$, we have solutions $\mathcal{P}$ and $\mathcal{Q}$ to the equations

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}} = \sigma_{\mathcal{Q}}\sigma_{\mathcal{N}}.$$

when $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are given lines concurrent at C . The uniqueness of such lines $\mathcal{P}$ and $\mathcal{Q}$ follows from the cancellation laws. $\square$

4.2.10 COROLLARY. *Rotation $\rho_{C,r}$ may be expressed as $\sigma_{\mathcal{B}}\sigma_{\mathcal{A}}$, where either one of $\mathcal{A}$ or $\mathcal{B}$ is an arbitrarily chosen line through C and the other is then a uniquely determined line through C .*

4.2.11 COROLLARY. *Halfturn σ_P is the product (in either order) of the two reflections in any two lines perpendicular at P .*

4.2.12 COROLLARY. *If lines $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are concurrent at point C , then $\sigma_N \sigma_M \sigma_L$ is a reflection in a line through C .*

4.2.13 THEOREM. *A product of two reflections is a translation or a rotation; only the identity is both a translation and a rotation.*

PROOF : Clearly,

$$\sigma_{\mathcal{L}} \sigma_{\mathcal{L}} = \tau_{P,P} = \rho_{P,0} = \iota$$

for any line $\mathcal{L}$ and any point P . Also, a rotation has a fixed point while a nonidentity translation does not. From these observations and the fact that lines $\mathcal{L}$ and $\mathcal{M}$ must be parallel or intersect, we have the result. $\square$

4.3 Fixed Points and Involutions

We have not considered products of three reflections, except in the very special cases where the reflections are in lines that are parallel or in lines that are concurrent. Therefore, it would be fairly surprising if we could at this stage *classify all the isometries that have fixed points* and *classify all the isometries that are involutions*. Such is the case, however.

4.3.1 PROPOSITION. *An isometry that fixes exactly one point is a nonidentity rotation. An isometry that fixes a point is a rotation or a reflection.*

PROOF : An isometry with a fixed point is a product of at most two reflections (PROPOSITION 4.1.6). Of course, the identity and a reflection have fixed points. Otherwise, an isometry with a fixed point must be a translation or a rotation (THEOREM 4.2.13). Since a nonidentity translation has no fixed points and a nonidentity rotation has exactly one fixed point, the desired result follows. $\square$

The involutions come next.

4.3.2 PROPOSITION. *The involutory isometries are the reflections and the halfturns.*

PROOF : Suppose α is an involutory isometry. Since α is not the identity, there are points P and Q such that $\alpha(P) = Q \neq P$. Since $P = \alpha^2(P) = \alpha(Q)$, then α interchanges distinct points P and Q . Hence (PROPOSITION 3.2.3), α must fix the midpoint of $\overline{PQ}$. Therefore, α must be a rotation or a reflection by PROPOSITION 4.3.1. Since the involutory rotations are halfturns (see **Exercise 44**), we obtain the desired result. $\square$

Exercise 52 Do involutory isometries form a group ?

Although we know that halfturn σ_P fixes line $\mathcal{L}$ if and only if point P is on line $\mathcal{L}$ (see PROPOSITION 2.2.2), we have not considered the fixed lines of an arbitrary rotation. We do so now.

4.3.3 PROPOSITION. *A nonidentity rotation that fixes a line is a halfturn.*

PROOF : Suppose nonidentity rotation $\rho_{C,r}$ fixes line $\mathcal{L}$. Let $\mathcal{M}$ be the line through C that is perpendicular to $\mathcal{L}$. Then (COROLLARY 4.2.10), there is a line $\mathcal{N}$ through C and different from $\mathcal{M}$ such that $\rho_{C,r} = \sigma_{\mathcal{N}}\sigma_{\mathcal{M}}$. Since $\mathcal{L}$ and $\mathcal{M}$ are perpendicular, then (PROPOSITION 3.1.2) we have

$$\mathcal{L} = \rho_{C,r}(\mathcal{L}) = \sigma_{\mathcal{N}}\sigma_{\mathcal{M}}(\mathcal{L}) = \sigma_{\mathcal{N}}(\mathcal{L}).$$

So $\sigma_{\mathcal{N}}$ fixes line $\mathcal{L}$. Then $\mathcal{N} = \mathcal{L}$ or $\mathcal{N} \perp \mathcal{L}$. Lines $\mathcal{M}$ and $\mathcal{N}$ cannot be two intersecting lines and both perpendicular to $\mathcal{L}$. Hence, $\mathcal{N} = \mathcal{L}$. So $\mathcal{M}$ and $\mathcal{N}$ are perpendicular at C and $\rho_{C,r}$ is the halfturn σ_C . $\square$

4.4 Exercises

Exercise 53 Given $\triangle ABC \cong \triangle DEF$, where $A = (0, 0)$, $B = (5, 0)$, $C = (0, 10)$, $D = (4, 2)$, $E = (1, -2)$, and $F = (12, -4)$, find equations of lines such that the product of reflections in these lines takes $\triangle ABC$ to $\triangle DEF$.

Exercise 54 Suppose lines $\mathcal{L}, \mathcal{M}, \mathcal{N}$ have, respectively, equations $x = 2$, $y = 3$, and $y = 5$. Find the equations for $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$ and $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}$.

Exercise 55 PROVE or DISPROVE : Every isometry is either a product of five reflections or a product of six reflections.

Exercise 56 PROVE or DISPROVE : The images of a triangle under two distinct isometries cannot be identical.

Exercise 57 TRUE or FALSE ?

- (a) $(\sigma_{\mathcal{Z}}\sigma_{\mathcal{Y}}\sigma_{\mathcal{X}} \cdots \sigma_{\mathcal{C}}\sigma_{\mathcal{B}}\sigma_{\mathcal{A}})^{-1} = \sigma_{\mathcal{A}}\sigma_{\mathcal{B}}\sigma_{\mathcal{C}} \cdots \sigma_{\mathcal{X}}\sigma_{\mathcal{Y}}\sigma_{\mathcal{Z}}$ for all lines $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$.
- (b) If $A_B = C_D$, then $AD = BC$.
- (c) A product of four reflections is an isometry.
- (d) The set of all rotations *generates* a commutative group.
- (e) The set of all reflections *generates* **Isom**.
- (f) If A and B are two distinct points, $PA = PB$ and $QA = QB$, then $P = Q$.
- (g) An isometry that fixes a point is an involution.
- (h) If isometry α fixes points A, B , and C , then $\alpha = \iota$.
- (i) If α and β are isometries and $\alpha^2 = \beta^2$, then $\alpha = \beta$ or $\alpha = \beta^{-1}$.

Exercise 58 Prove the if $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}$ fixes point P and $\mathcal{M} \neq \mathcal{N}$, then P is on both $\mathcal{M}$ and $\mathcal{N}$.

Exercise 59 PROVE or DISPROVE : If α is an involution, then $\beta\alpha\beta^{-1}$ is an involution for any transformation β .

Exercise 60 If $\mathcal{M} \parallel \mathcal{N}$, find points M and N such that

$$\sigma_{\mathcal{N}}\sigma_{\mathcal{M}} = \sigma_N\sigma_M.$$

Exercise 61 What are the equations for $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}$ if line $\mathcal{M}$ has equation $y = -2x + 3$ and line $\mathcal{N}$ has equation $y = -2x + 8$?

Exercise 62 Show that $\sigma_{\mathcal{L}}\rho_{C,r}\sigma_{\mathcal{L}} = \rho_{C,-r}$ if point C is on line $\mathcal{L}$.

Exercise 63 TRUE or FALSE ?

- (a) If a directed angle from line $\mathcal{L}$ to line $\mathcal{M}$ is 240° , then $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$ is a rotation of 120° .
- (b) $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \tau_{L,M}^2 = \sigma_M\sigma_L$ if point L is on line $\mathcal{L}$ and point M is on line $\mathcal{M}$.
- (c) An isometry has a unique fixed point if and only if the isometry is a nonidentity rotation.
- (d) An isometry that is its own inverse must be a halfturn, a reflection, or the identity.
- (e) If $L' = \sigma_{\mathcal{M}}(L)$ and $K' = \tau_{L,L'}(K)$, then $\mathcal{M}$ is the perpendicular bisector of $\overline{KK'}$.
- (f) Given points L, M, N , there is a point P such that $\sigma_M\sigma_L = \sigma_N\sigma_P$.
- (g) Given lines $\mathcal{L}, \mathcal{M}, \mathcal{N}$, there is a line $\mathcal{P}$ such that $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{P}}$.
- (h) If lines $\mathcal{L}$ and $\mathcal{M}$ intersect at point C and a directed angle from $\mathcal{L}$ to $\mathcal{M}$ is r° , then $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \rho_{C,2r}$.
- (i) An isometry that fixes a point must be a rotation, a reflection, or the identity.
- (j) Isometry $\alpha\beta\alpha^{-1}$ is an involution for any isometry α if and only if isometry β is an involution.

Exercise 64 Given nonparallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, show there is a rotation ρ such that $\rho(\overleftrightarrow{AB}) = \overleftrightarrow{CD}$.

Exercise 65 PROVE or DISPROVE : Every translation is a product of two noninvolutory rotations.

Exercise 66 PROVE or DISPROVE : If $P \neq Q$, then there is a unique translation taking point P to point Q but there are an infinite number of rotations that take P to Q .

Exercise 67 What lines are fixed by rotation $\rho_{C,r}$?

Exercise 68 If $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}((x, y)) = (x + 6, y - 3)$, find equations for lines $\mathcal{M}$ and $\mathcal{N}$.

Exercise 69 If $\sigma_{\mathcal{C}}\sigma_{\mathcal{B}}\sigma_{\mathcal{A}}$ is a reflection, show that lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are either concurrent or parallel to each other.

Exercise 70 Show that $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{L}}\sigma_{\mathcal{M}}\sigma_{\mathcal{N}}$ whenever lines $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ are concurrent or have a common perpendicular.

DISCUSSION :

The regular polyhedra are five figures from the classical geometry of $\mathbb{E}^3$ (the so-called “solid geometry” encountered in high school) : the tetrahedron, cube, octohedron, dodecahedron, and icosahedron. The faces of the tetrahedron, octahedron and icosahedron are equilateral triangles, those of the cube are squares, and those of the dodecahedron are regular pentagons. It is easy to show that these there are the only possible *convex* polyhedra whose faces are all regular polygons of the same type, because at least three faces must meet at each vertex and, hence, the polygons must have angles $< \frac{2\pi}{3}$.

Corresponding to each regular polyhedron $\mathbf{P}$ we get a *regular tessellation* of the sphere $\mathbb{S}^2$ by placing $\mathbf{P}$ so that its center is at the origin O and projecting the edges of $\mathbf{P}$ from O onto $\mathbb{S}^2$. Each regular polygonal face of $\mathbf{P}$ projects to a regular spherical polygon on $\mathbb{S}^2$.

Each regular polyhedron $\mathbf{P}$ has a *symmetry group* which is a finite group of rotations of $\mathbb{S}^2$. If we imagine a solid $\mathbf{P}$ occupying a $\mathbf{P}$ -shaped “hole” in $\mathbb{E}^3$, then the rotations in the symmetry group are those which turn $\mathbf{P}$ to a position which fits the hole. There are only three such symmetry groups, called the *polyhedral groups*, because the octahedron and cube have the same group, as do the icosahedron and dodecahedron.

In addition to the polyhedral groups, there are two infinite families of finite groups of rotations of $\mathbb{S}^2$. The first consists of *cyclic groups* $\mathfrak{C}_n$, each generated by a rotation through $\frac{2\pi}{n}$. The second consists of *dihedral groups* $\mathfrak{D}_n$. $\mathfrak{D}_n$ can be regarded as the symmetry group of a degenerate polyhedron – the “dihedron” – with two regular n -gonal faces.

The fact that there are only five regular polyhedra has the (not quite obvious) consequence that *the only finite groups of rotations of $\mathbb{S}^2$ are $\mathfrak{C}_n$, $\mathfrak{D}_n$, and the three polyhedral groups*. An even more remarkable consequence was proved by FELIX KLEIN

(1849-1925): *Any finite group of linear fractional transformations is isomorphic to one of $\mathfrak{C}_n$, $\mathfrak{D}_n$ or the three polyhedral groups.*

The psychological aspects of true geometric intuition will perhaps never be cleared up. At one time it implied primarily the power of visualization in three-dimensional space. Now that higher-dimensional spaces have mostly driven out the more elementary problems, visualization can at best be partial or symbolic. Some degree of tactile imagination seems also to be involved.

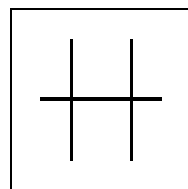
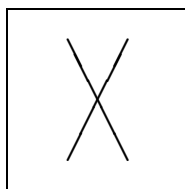
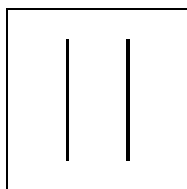
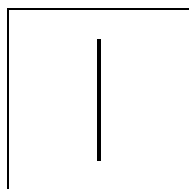
ANDRÉ WEIL

Chapter 5

Isometries II

Topics :

1. EVEN AND ODD ISOMETRIES
 2. CLASSIFICATION OF ISOMETRIES
 3. EQUATIONS FOR ISOMETRIES
-



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5.1 Even and Odd Isometries

A product of two reflections is a translation or a rotation. By considering the fixed points of each, we see that neither a translation nor a rotation can be equal to a reflection. Thus, for lines $\mathcal{L}, \mathcal{M}, \mathcal{N}$

$$\sigma_{\mathcal{N}}\sigma_{\mathcal{M}} \neq \sigma_{\mathcal{L}}.$$

When a given isometry is expressed as a product of reflections, the number of reflections is not invariant. Although the product of two reflections cannot be a reflection, we know that in some cases a product of three reflections is a reflection. (We shall see this is possible only because both 3 and 1 are odd integers.) We make the following definitions.

5.1.1 DEFINITION. An isometry that is a product of an even number of reflections is said to be **even**.

5.1.2 DEFINITION. An isometry that is a product of an odd number of reflections is said to be **odd**.

NOTE : It is intuitively clear that the product of an even number of reflections preserves the *sense* of a clockwise oriented circle in the plane, whereas the product of an odd number of reflections reverses it. We say that even isometries are *orientation-preserving* and that odd isometries are *orientation-reversing* isometries.

We shall refer to the property of an isometry of being even or odd as the *parity*. But is this concept “well-defined” ? Observe that, since an isometry is a product of reflections, *an isometry is even or odd*. Of course, no integer can be both even and odd, but is it not conceivable some product of ten reflections could equal to some product of seven reflections ? We shall show this is impossible.

Exercise 71 Show that if P is a point and $\mathcal{A}$ and $\mathcal{B}$ are lines, then there are lines $\mathcal{C}$ and $\mathcal{D}$ with $\mathcal{C}$ passing through P such that $\sigma_{\mathcal{B}}\sigma_{\mathcal{A}} = \sigma_{\mathcal{D}}\sigma_{\mathcal{C}}$.

Based on this simple fact, we can now prove the following

5.1.3 PROPOSITION. *A product of four reflections is a product of two reflections.*

PROOF : Suppose product $\sigma_S\sigma_R\sigma_Q\sigma_P$ is given. We want to show this product is equal to a product of two reflections. Let P a point on line $\mathcal{P}$. There are lines $\mathcal{Q}'$ and $\mathcal{R}'$ such that $\sigma_R\sigma_Q = \sigma_{\mathcal{R}'}\sigma_{\mathcal{Q}'}$ with P on $\mathcal{Q}'$. Also, there are lines $\mathcal{R}''$ and $\mathcal{M}$ such that $\sigma_S\sigma_{\mathcal{R}'} = \sigma_{\mathcal{M}}\sigma_{\mathcal{R}''}$ with P on $\mathcal{R}''$. Since $\mathcal{P}$, $\mathcal{Q}'$, $\mathcal{R}''$ are concurrent at P , then there is a line $\mathcal{L}$ such that $\sigma_{\mathcal{R}''}\sigma_{\mathcal{Q}'}\sigma_P = \sigma_{\mathcal{L}}$. Therefore,

$$\sigma_S\sigma_R\sigma_Q\sigma_P = \sigma_S\sigma_{\mathcal{R}'}\sigma_{\mathcal{Q}'}\sigma_P = \sigma_{\mathcal{M}}\sigma_{\mathcal{R}''}\sigma_{\mathcal{Q}'}\sigma_P = \sigma_{\mathcal{M}}\sigma_{\mathcal{L}}.$$

□

NOTE : Not only are there lines such that the given product of four reflections is equal to $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$, but our proof even tells us how to find such lines.

5.1.4 PROPOSITION. *An even isometry is a product of two reflections. An odd isometry is a reflection or a product of three reflections. No isometry is both even and odd.*

PROOF : Given a long product of reflections, we can use PROPOSITION 5.1.3 repeatedly to replace the first four reflections by two reflections until we have obtained a product with less than four reflections. By repeated application of the result to an even isometry, we can reduce the even isometry to a product of two reflections. Also, by repeated application of the result to an odd isometry, we can reduce the odd isometry to a product of three reflections or to a reflection. Therefore, to show an isometry cannot be both even and odd, we need to show only that a product of two reflections cannot equal a reflection or a product of three reflections. Assume there are lines $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$ such that $\sigma_R\sigma_Q\sigma_P = \sigma_S\sigma_T$. Then, we have shown above that there are lines $\mathcal{L}$ and $\mathcal{M}$ such that

$$\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = \sigma_S\sigma_R\sigma_Q\sigma_P = \sigma_S\sigma_S\sigma_T = \sigma_T.$$

We have a contradiction since $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$ is a translation or a rotation and cannot be equal to reflection $\sigma_{\mathcal{T}}$. A product of two reflections is never equal to a reflection or a product of three reflections. $\square$

5.1.5 PROPOSITION. *An even involutory isometry is a halfturn; an odd involutory isometry is a reflection.*

PROOF : The even isometries are the translations and the rotations. Since the involutory isometries are the halfturns and the reflections, the result follows. $\square$

5.1.6 PROPOSITION. *The even isometries form a group $\mathbf{Isom}^+$.*

PROOF : An isometry and its inverse have the same parity, since the inverse of a product of reflections is the product of the reflections in reverse order. So the set $\mathbf{Isom}^+$ of all even isometries has the inverse property. Further, the set $\mathbf{Isom}^+$ has the closure property since the sum of two even integers is even. So the even isometries form a group. $\square$

NOTE : $\mathbf{Isom}^+$ will always denote the group of even isometries. So $\mathbf{Isom}^+$ consists of the translations and the rotations.

Some “technical” results

Exercise 72 Suppose that α and β are two isometries. Prove that

- (a) $\alpha\beta\alpha^{-1}$ is an involution $\iff \beta$ is an involution.
- (b) $\alpha\beta\alpha^{-1}$ and β must have the same parity.

NOTE : In general, $\alpha\beta\alpha^{-1}$ is called the **conjugate** of β by α .

5.1.7 PROPOSITION. *If P is a point, $\mathcal{L}$ is a line, and α is an isometry, then*

$$\alpha\sigma_{\mathcal{L}}\alpha^{-1} = \sigma_{\alpha(\mathcal{L})} \quad \text{and} \quad \alpha\sigma_P\alpha^{-1} = \sigma_{\alpha(P)}.$$

PROOF : Since σ_P is an even involutory isometry, so is $\alpha\sigma_P\alpha^{-1}$ (**Exercise 72**). By PROPOSITION 5.1.5, $\alpha\sigma_P\alpha^{-1}$ must be a halfturn. Since halfturn

$\alpha\sigma_P\alpha^{-1}$ fixes point $\alpha(P)$, then $\alpha\sigma_P\alpha^{-1}$ must be the halfturn about $\alpha(P)$; that is,

$$\alpha\sigma_P\alpha^{-1} = \sigma_{\alpha(P)}.$$

In similar fashion, since $\alpha\sigma_{\mathcal{L}}\alpha^{-1}$ is an odd involutory isometry, then $\alpha\sigma_{\mathcal{L}}\alpha^{-1}$ is a reflection. Hence, since $\alpha\sigma_{\mathcal{L}}\alpha^{-1}$ clearly fixes every point $\alpha(P)$ on line $\alpha(\mathcal{L})$, then $\alpha\sigma_{\mathcal{L}}\alpha^{-1}$ must be the reflection in the line $\alpha(\mathcal{L})$. That is,

$$\alpha\sigma_{\mathcal{L}}\alpha^{-1} = \sigma_{\alpha(\mathcal{L})}.$$

□

5.1.8 PROPOSITION. *If α is an isometry, then*

$$\alpha\tau_{A,B}\alpha^{-1} = \tau_{\alpha(A),\alpha(B)} \quad \text{and} \quad \alpha\rho_{C,r}\alpha^{-1} = \rho_{\alpha(C),\pm r}$$

where the positive sign applies when α is even and the negative sign applies when α is odd.

PROOF : If M is the midpoint of $\overline{AB}$, then point $\alpha(M)$ is the midpoint of $\overline{\alpha(A)\alpha(B)}$. Also,

$$\tau_{A,B} = \sigma_M\sigma_A \quad \text{and} \quad \tau_{\alpha(A),\alpha(B)} = \sigma_{\alpha(M)}\sigma_{\alpha(A)}.$$

Now

$$\begin{aligned} \alpha\tau_{A,B}\alpha^{-1} &= \alpha\sigma_M\sigma_A\alpha^{-1} = (\alpha\sigma_M\alpha^{-1})(\alpha\sigma_A\alpha^{-1}) = \sigma_{\alpha(M)}\sigma_{\alpha(A)} \\ &= \tau_{\alpha(A),\alpha(B)}. \end{aligned}$$

That is,

$$\alpha\tau_{A,B}\alpha^{-1} = \tau_{\alpha(A),\alpha(B)}.$$

Finding the conjugate of a rotation is slightly more complicated.

We first examine the conjugate of $\rho_{C,r}$ by $\sigma_{\mathcal{L}}$. Let $\mathcal{M}$ be the line through C that is perpendicular to $\mathcal{L}$. Then there exists a line $\mathcal{N}$ through C such that

$$\rho_{C,r} = \sigma_{\mathcal{N}}\sigma_{\mathcal{M}}.$$

Now $\sigma_{\mathcal{L}}(\mathcal{M}) = \mathcal{M}$ and $\sigma_{\mathcal{L}}(\mathcal{N})$ intersect at $\sigma_{\mathcal{L}}(C)$, and a directed angle from $\sigma_{\mathcal{L}}(\mathcal{M})$ to $\sigma_{\mathcal{L}}(\mathcal{N})$ is the negative of a directed angle from $\mathcal{M}$ to $\mathcal{N}$. (This explains the negative sign on the far right in the following calculation.) We have

$$\begin{aligned}\sigma_{\mathcal{L}}\rho_{C,r}\sigma_{\mathcal{L}}^{-1} &= \sigma_{\mathcal{L}}\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}^{-1} = (\sigma_{\mathcal{L}}\sigma_{\mathcal{N}}\sigma_{\mathcal{L}}^{-1})(\sigma_{\mathcal{L}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}^{-1}) = \sigma_{\sigma_{\mathcal{L}}(\mathcal{N})}\sigma_{\sigma_{\mathcal{L}}(\mathcal{M})} \\ &= \rho_{\sigma_{\mathcal{L}}(C),-r}.\end{aligned}$$

If $\alpha = \sigma_{\mathcal{T}}\sigma_{\mathcal{S}}$, then

$$\alpha\rho_{C,r}\alpha^{-1} = \sigma_{\mathcal{T}}(\sigma_{\mathcal{S}}\rho_{C,r}\sigma_{\mathcal{S}}^{-1})\sigma_{\mathcal{T}}^{-1} = \rho_{\alpha(C),r}.$$

If $\alpha = \sigma_{\mathcal{T}}\sigma_{\mathcal{S}}\sigma_{\mathcal{R}}$, then the sign in front of r is back to a negative sign again. $\square$

Commuting isometries

By taking $\alpha = \rho_{D,s}$ in PROPOSITION 5.1.8, we can show that *nonidentity rotation $\rho_{D,s}$ does not commute with nonidentity rotation $\rho_{C,r}$ unless $D = C$* . We leave this as an exercise.

Exercise 73 Prove that nonidentity rotations with different centres do not commute.

We are also in a position to answer the question “When do reflections commute?”

5.1.9 PROPOSITION. $\sigma_{\mathcal{M}}\sigma_{\mathcal{N}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{M}}$ if and only if $\mathcal{M} = \mathcal{N}$ or $\mathcal{M} \perp \mathcal{N}$.

PROOF : For lines $\mathcal{M}$ and $\mathcal{N}$ the following five statements are seen to be equivalent:

- (1) $\sigma_{\mathcal{M}}\sigma_{\mathcal{N}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{M}}$.
- (2) $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}\sigma_{\mathcal{N}} = \sigma_{\mathcal{M}}$.
- (3) $\sigma_{\sigma_{\mathcal{N}}(\mathcal{M})} = \sigma_{\mathcal{M}}$.

$$(4) \quad \sigma_{\mathcal{N}}(\mathcal{M}) = \mathcal{M}.$$

$$(5) \quad \mathcal{M} = \mathcal{N} \text{ or } \mathcal{M} \perp \mathcal{N}.$$

Comparing (1) and (2), we have the answer to our question. $\square$

We now consider products of even isometries. We already know that

- The product of two translations is a translation (PROPOSITION 2.1.8).
- The product of two rotations *can be* a translation in some cases; for example, $\sigma_B \sigma_A = \tau_{A,B}^2$ (PROPOSITION 2.2.3).
- $\rho_{C,s} \rho_{C,r} = \rho_{C,r+s}$ (**Exercise 44**).

5.1.10 THEOREM. (THE ANGLE-ADDITION THEOREM) *A rotation of r° followed by a rotation of s° is a rotation of $(r+s)^\circ$ unless $(r+s)^\circ = 0^\circ$, in which case the product is a translation.*

PROOF : Let's consider the product

$$\rho_{B,s} \rho_{A,r}$$

of two nonidentity rotations with different centres. With $\mathcal{C} = \overleftrightarrow{AB}$, there is a line $\mathcal{A}$ through A and a line $\mathcal{B}$ through B such that

$$\rho_{A,r} = \sigma_{\mathcal{C}} \sigma_{\mathcal{A}} \quad \text{and} \quad \rho_{B,s} = \sigma_{\mathcal{B}} \sigma_{\mathcal{C}}.$$

So

$$\rho_{B,s} \rho_{A,r} = \sigma_{\mathcal{B}} \sigma_{\mathcal{C}} \sigma_{\mathcal{C}} \sigma_{\mathcal{A}} = \sigma_{\mathcal{B}} \sigma_{\mathcal{A}}.$$

When $(r+s)^\circ = 0^\circ$, then the lines $\mathcal{A}$ and $\mathcal{B}$ are parallel and our product is a translation. On the other hand, when $(r+s)^\circ \neq 0^\circ$, then the lines $\mathcal{A}$ and $\mathcal{B}$ intersect at some point C and our product is a rotation. We can see (by The Exterior Angle Theorem) that one directed angle from $\mathcal{A}$ to $\mathcal{B}$ is $(\frac{r}{2} + \frac{s}{2})^\circ$. Hence, our product $\sigma_{\mathcal{B}} \sigma_{\mathcal{A}}$ is a rotation about C through an angle of $(r+s)^\circ$. That is,

$$\rho_{B,s} \rho_{A,r} = \rho_{C,r+s}.$$

$\square$

NOTE : The ANGLE-ADDITION THEOREM can also be proved by using the equations for the even isometries that will be developed later.

Now, what is the product of a translation and a nonidentity rotation ?

Exercise 74 Prove that

- (a) A translation followed by a nonidentity rotation of r° is a rotation of r° .
- (b) A nonidentity rotation of r° followed by a translation is a rotation of r° .

5.2 Classification of Plane Isometries

We have classified all the even isometries as translations or rotations. An odd isometry is a reflection or a product of three reflections. Only those odd isometries $\sigma_C\sigma_B\sigma_A$, where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are neither concurrent nor have a common perpendicular remain to be considered.

Glide reflections

We begin with the special case where $\mathcal{A}$ and $\mathcal{B}$ are perpendicular to $\mathcal{C}$. Then $\sigma_B\sigma_A$ is a translation and σ_C is, of course, a reflection. We make the following definition.

5.2.1 DEFINITION. If $\mathcal{A}$ and $\mathcal{B}$ are distinct lines perpendicular to line $\mathcal{C}$, then $\sigma_C\sigma_B\sigma_A$ is called a **glide reflection** with **axis** $\mathcal{C}$.

We might as well call line $\mathcal{M}$ the **axis** of $\sigma_{\mathcal{M}}$ as the reflection and the glide reflection then share the property that the midpoint of any point and its image under the isometry lies on the axis.

5.2.2 PROPOSITION. *A glide reflection fixes no points. A glide reflection fixes exactly one line, its axis. The midpoint of any point and its image under a glide reflection lies on the axis of the glide reflection.*

PROOF : Suppose P is any point. Let line $\mathcal{L}$ be the perpendicular from P

to $\mathcal{C}$. Then there is a line $\mathcal{M}$ perpendicular to $\mathcal{C}$ such that $\sigma_{\mathcal{B}}\sigma_{\mathcal{A}} = \sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$. If M is the intersection of $\mathcal{M}$ and $\mathcal{C}$, then P and M are distinct points such that

$$\sigma_{\mathcal{C}}\sigma_{\mathcal{B}}\sigma_{\mathcal{A}}(P) = \sigma_{\mathcal{C}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}(P) = \sigma_{\mathcal{C}}\sigma_{\mathcal{M}}(P) = \sigma_M(P) \neq P.$$

Since $\sigma_{\mathcal{C}}\sigma_{\mathcal{B}}\sigma_{\mathcal{A}}(P) = \sigma_M(P)$ and M is the midpoint of distinct points P and $\sigma_M(P)$, we have shown that glide reflection $\sigma_{\mathcal{C}}\sigma_{\mathcal{B}}\sigma_{\mathcal{A}}$ fixes no point but the midpoint of any point P and its image $\sigma_{\mathcal{C}}\sigma_{\mathcal{B}}\sigma_{\mathcal{A}}(P)$ lies on the axis of the glide reflection. So a glide reflection interchanges the halfplanes of its axis. Hence, any line fixed by the glide reflection must intersect the axis at least twice. That is, the glide reflection can fix no line except its axis. The axis of a glide reflection is the unique line fixed by the glide reflection. $\square$

5.2.3 PROPOSITION. *A glide reflection is the composite of a reflection in some line $\mathcal{A}$ followed by a halfturn about some point off $\mathcal{A}$. A glide reflection is the composite of a halfturn about some point A followed by a reflection in some line off A . Conversely, if point P is off line $\mathcal{L}$, then $\sigma_P\sigma_{\mathcal{L}}$ and $\sigma_{\mathcal{L}}\sigma_P$ are glide reflections with axis the perpendicular from P to $\mathcal{L}$.*

PROOF : If γ is a glide reflection, then there are distinct lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ such that $\gamma = \sigma_{\mathcal{C}}\sigma_{\mathcal{B}}\sigma_{\mathcal{A}}$, where $\mathcal{A}$ and $\mathcal{B}$ are perpendicular to $\mathcal{C}$, say at points A and B , respectively. Now

$$\sigma_A = \sigma_{\mathcal{A}}\sigma_{\mathcal{C}} = \sigma_{\mathcal{C}}\sigma_{\mathcal{A}} \quad \text{and} \quad \sigma_B = \sigma_{\mathcal{B}}\sigma_{\mathcal{C}} = \sigma_{\mathcal{C}}\sigma_{\mathcal{B}}.$$

Hence

$$\begin{aligned} \gamma = \sigma_{\mathcal{C}}(\sigma_{\mathcal{B}}\sigma_{\mathcal{A}}) &= (\sigma_{\mathcal{C}}\sigma_{\mathcal{B}})\sigma_{\mathcal{A}} = \sigma_{\mathcal{B}}(\sigma_{\mathcal{C}}\sigma_{\mathcal{A}}) = (\sigma_{\mathcal{B}}\sigma_{\mathcal{A}})\sigma_{\mathcal{C}} \\ &= \sigma_B\sigma_{\mathcal{A}} = \sigma_{\mathcal{B}}\sigma_A. \end{aligned}$$

The first line of these equations tells us that γ is the product of the glide $\sigma_{\mathcal{B}}\sigma_{\mathcal{A}}$ and the reflection $\sigma_{\mathcal{C}}$ in either order. More important, the second line tells us that γ is a product $\sigma_B\sigma_{\mathcal{A}}$ with B off $\mathcal{A}$ and a product $\sigma_{\mathcal{B}}\sigma_A$ with A off $\mathcal{B}$.

We want to show, conversely, that such a product is a glide reflection. Suppose point P is off line $\mathcal{L}$. Let $\mathcal{P}$ be the perpendicular from P to $\mathcal{L}$ and let $\mathcal{M}$ be the perpendicular at P to $\mathcal{P}$. Lines $\mathcal{L}$ and $\mathcal{M}$ are distinct since P is off $\mathcal{L}$. Furthermore,

$$\sigma_P \sigma_{\mathcal{L}} = \sigma_{\mathcal{P}} \sigma_{\mathcal{M}} \sigma_{\mathcal{L}} \quad \text{and} \quad \sigma_{\mathcal{L}} \sigma_P = \sigma_{\mathcal{L}} \sigma_{\mathcal{P}} \sigma_{\mathcal{M}} = \sigma_{\mathcal{P}} \sigma_{\mathcal{L}} \sigma_{\mathcal{M}}.$$

Therefore, the products $\sigma_P \sigma_{\mathcal{L}}$ and $\sigma_{\mathcal{L}} \sigma_P$ are glide reflections by the definition of a glide reflection. $\square$

5.2.4 COROLLARY. *The set of all glide reflections has the inverse property.*

NOTE : The set of all glide reflections does *not* have the closure property because the product of two glide reflections (= odd isometries) must be an even isometry.

5.2.5 PROPOSITION. *Lines $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ are neither concurrent nor have a common perpendicular if and only if $\sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}$ is a glide reflection.*

PROOF : ($\Leftarrow$) If $\sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}$ is a glide reflection, then $\sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}$ is not a reflection and the lines $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ cannot be either concurrent or parallel.

($\Rightarrow$) Suppose $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ are any lines that are neither concurrent nor have a common perpendicular. We wish to prove that $\sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}$ is a glide reflection.

First, we consider the case lines $\mathcal{P}$ and $\mathcal{Q}$ intersect at some point Q . Then Q is off $\mathcal{R}$ as the lines are not concurrent. Let P be the foot of the perpendicular from Q to $\mathcal{R}$, and let $\mathcal{M}$ be the line through P and Q . There is a line $\mathcal{L}$ through Q such that $\sigma_{\mathcal{Q}} \sigma_{\mathcal{P}} = \sigma_{\mathcal{M}} \sigma_{\mathcal{L}}$. Since $\mathcal{P} \neq \mathcal{Q}$, then $\mathcal{L} \neq \mathcal{M}$ and P is off $\mathcal{L}$. Hence,

$$\sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}} = \sigma_{\mathcal{R}} \sigma_{\mathcal{M}} \sigma_{\mathcal{L}} = \sigma_P \sigma_{\mathcal{L}}$$

with P off $\mathcal{L}$. Therefore, $\sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}$ is a glide reflection by PROPOSITION 5.2.3.

There remains the case $\mathcal{P} \parallel \mathcal{Q}$. In this case, lines $\mathcal{R}$ and $\mathcal{Q}$ must intersect as otherwise $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ have a common perpendicular. Then, by what we just

proved, there is some point P off some line $\mathcal{L}$ such that $\sigma_P\sigma_Q\sigma_R = \sigma_P\sigma_L$. Hence,

$$\sigma_R\sigma_Q\sigma_P = (\sigma_P\sigma_Q\sigma_R)^{-1} = (\sigma_P\sigma_L)^{-1} = \sigma_L\sigma_P$$

with point P off line $\mathcal{L}$. Therefore, again we have $\sigma_R\sigma_Q\sigma_P$ is a glide reflection. $\square$

An immediate corollary of this result is that *a product of three reflections is a reflection or a glide reflection*. Thus, we have a classification of odd isometries.

5.2.6 PROPOSITION. *An odd isometry is either a reflection or a glide reflection.*

We finally have

5.2.7 THEOREM. (THE CLASSIFICATION THEOREM FOR PLANE ISOMETRIES)
Each nonidentity isometry is exactly one of the following : translation, rotation, reflection or a glide reflection.

Exercise 75 Prove that

- (a) A translation that fixes line $\mathcal{C}$ commutes with a glide reflection with axis $\mathcal{C}$.
- (b) The square of a glide reflection is a nonidentity translation.

5.2.8 PROPOSITION. *If γ is a glide reflection with axis $\mathcal{C}$ and α is an isometry, then $\alpha\gamma\alpha^{-1}$ is a glide reflection with axis $\alpha(\mathcal{C})$.*

PROOF : We have $\gamma^2 \neq \iota$. So, a glide reflection is *not* an involution. Since $\alpha\gamma\alpha^{-1}$ is an odd isometry that fixes line $\alpha(\mathcal{C})$ but is not an involution, then $\alpha\gamma\alpha^{-1}$ has to be a glide reflection with axis $\alpha(\mathcal{C})$. $\square$

5.3 Equations for Isometries

The equations for a general translation were incorporated in the definition of a translation. Equations for a reflection were determined in PROPOSITION 3.1.3. We now turn to rotations.

5.3.1 PROPOSITION. *Rotation $\rho_{O,r}$ about the origin has equations*

$$\begin{cases} x' = (\cos r)x - (\sin r)y \\ y' = (\sin r)x + (\cos r)y. \end{cases}$$

PROOF : Let

$$\rho_{O,r} = \sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$$

where $\mathcal{L}$ is the x -axis. Then one directed angle from $\mathcal{L}$ to $\mathcal{M}$ has directed measure $\frac{r}{2}$. From the definition of trigonometric functions we know that $(\cos \frac{r}{2}, \sin \frac{r}{2})$ is a point on $\mathcal{M}$. So line $\mathcal{M}$ has equation

$$\left(\sin \frac{r}{2}\right)x - \left(\cos \frac{r}{2}\right)y = 0.$$

Hence $\sigma_{\mathcal{M}}$ has equations :

$$\begin{aligned} x' &= x - \frac{2 \sin \frac{r}{2} ((\sin \frac{r}{2})x - (\cos \frac{r}{2})y)}{\sin^2 \frac{r}{2} + \cos^2 \frac{r}{2}} \\ &= \left(1 - 2 \sin^2 \frac{r}{2}\right)x + \left(2 \sin \frac{r}{2} \cos \frac{r}{2}\right)y \\ &= (\cos r)x + (\sin r)y, \\ y' &= y + \frac{2 \cos \frac{r}{2} ((\sin \frac{r}{2})x - (\cos \frac{r}{2})y)}{\sin^2 \frac{r}{2} + \cos^2 \frac{r}{2}} \\ &= \left(2 \sin \frac{r}{2} \cos \frac{r}{2}\right)x - \left(1 - 2 \cos^2 \frac{r}{2}\right)y \\ &= (\sin r)x - (\cos r)y. \end{aligned}$$

Since $\sigma_{\mathcal{L}}$ has equations

$$\begin{cases} x' = x \\ y' = -y \end{cases}$$

then the rotation $\rho_{O,r} = \sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$ has the equations

$$\begin{cases} x' = (\cos r)x - (\sin r)y \\ y' = (\sin r)x + (\cos r)y. \end{cases}$$

□

5.3.2 PROPOSITION. *The general equations for an even isometry are*

$$\begin{cases} x' = ax - by + h \\ y' = bx + ay + k \end{cases} \quad \text{with } a^2 + b^2 = 1$$

and, conversely, such equations are those of an even isometry.

PROOF : Let $C = (u, v)$. Since

$$\rho_{C,r} = \tau_{O,C}\rho_{O,r}\tau_{C,O} \quad (\text{by PROPOSITION 5.1.8})$$

the equations for rotation $\rho_{C,r}$ about the point $C = (u, v)$ are easily obtained by composing three sets of equations. The rotation has equations

$$\begin{cases} x' = (\cos r)(x - u) - (\sin r)(y - v) + u \\ y' = (\sin r)(x - u) + (\cos r)(y - v) + v. \end{cases}$$

These equations for the rotation $\rho_{C,r}$ have the form

$$\begin{cases} x' = (\cos r)x - (\sin r)y + h \\ y' = (\sin r)x + (\cos r)y + k \end{cases}$$

which, conversely, are the equations of a rotation unless $r^\circ = 0^\circ$. Indeed, given h, k , and r , there are unique solutions for u and v given by

$$\begin{aligned} h &= u(1 - \cos r) + v \sin r \\ k &= u(-\sin r) + v(1 - \cos r) \end{aligned}$$

unless $r^\circ = 0^\circ$. In case $r^\circ = 0^\circ$, the equations above are those of a general translation.

Since the even isometries are the translations and the rotations, setting

$$a = \cos r \quad \text{and} \quad b = \sin r$$

we have the general equations for an even isometry :

$$\begin{cases} x' = ax - by + h \\ y' = bx + ay + k \end{cases} \quad \text{with} \quad a^2 + b^2 = 1.$$

□

5.3.3 PROPOSITION. *The general equations for an isometry (on the plane) are*

$$\begin{cases} x' = ax - by + h \\ y' = \pm(bx + ay) + k \end{cases} \quad \text{with} \quad a^2 + b^2 = 1$$

and, conversely, such equations are those of an isometry.

PROOF : If α is an odd isometry and $\mathcal{L}$ any line, then α is the product of even isometry $\sigma_{\mathcal{L}}\alpha$ followed by $\sigma_{\mathcal{L}}$. Taking $\mathcal{L}$ as the x -axis, we have *any odd isometry is the product of an even isometry followed by the reflection in the x -axis*. This observation, together with PROPOSITION 5.3.2, gives the desired result, where the positive sign applies when isometry is even and negative sign applies when isometry is odd. □

5.4 Exercises

Exercise 76 TRUE or FALSE ?

- (a) An even isometry that fixes two points is the identity.
- (b) The set of rotations *generates* $\mathbf{Isom}^+$.
- (c) An odd isometry is a product of three reflections.

- (d) An even isometry is a product of four reflections.
- (e) If $\rho_{\alpha(C),r} = \rho_{C,r}$ for isometry α , then α fixes C .
- (f) $\rho_{B,r}\rho_{A,-r}$ is the translation that takes A to $\rho_{B,r}(A)$.

Exercise 77 PROVE or DISPROVE : Given $\tau_{A,B}$ and nonidentity rotation $\rho_{C,r}$, there is a rotation $\rho_{D,s}$ such that $\tau_{A,B} = \rho_{D,s}\rho_{C,r}$.

Exercise 78 Show that if ρ_1 , ρ_2 , $\rho_2\rho_1$, and $\rho_2^{-1}\rho_1$ are rotations, then the centres of ρ_1 , $\rho_2\rho_1$, and $\rho_2^{-1}\rho_1$ are collinear.

Exercise 79 Show that translation τ commutes with σ_C if and only if τ fixes C . Also, that τ commutes with a glide reflection with axis C if and only if τ fixes C .

Exercise 80 TRUE or FALSE ?

- (a) Every isometry is a product of two involutions.
- (b) An isometry that does not fix a point is a glide reflection.
- (c) If $\gamma = \sigma_L\sigma_P$, then γ is a glide reflection with axis the line through P that is perpendicular to L .
- (d) If γ is a glide reflection with axis C and P is a point on C , then there are unique lines L and M such that $\gamma = \sigma_M\sigma_P = \sigma_P\sigma_L$.
- (e) If $\sigma_C\sigma_B\sigma_A$ fixes line L , then $\sigma_C\sigma_B\sigma_A$ is a glide reflection with axis L .
- (f) If $\sigma_C\sigma_B\sigma_A$ fixes line L , then $\sigma_C\sigma_B\sigma_A$ is a glide reflection with axis L .

Exercise 81 PROVE or DISPROVE : If point M is on the axis of glide reflection γ , then there is a point P such that M is the midpoint of P and $\gamma(P)$.

Exercise 82 PROVE or DISPROVE : Every glide reflection is a product of three reflections in the three lines containing the sides of some triangle.

Exercise 83 Which isometries are dilatations ?

Exercise 84 Prove that if τ is a translation, then there is a glide reflection γ such that $\tau = \gamma^2$.

Exercise 85 What are the equations for each of the rotations $\rho_{O,90}$, $\rho_{O,180}$, and $\rho_{O,270}$?

Exercise 86 If

$$\begin{cases} x' = ax + by + h \\ y' = bx - ay + k \end{cases} \quad \text{with } a^2 + b^2 = 1$$

are the equations for isometry α , show that α is a reflection if and only if $ah + bk + h = 0$ and $ak - bh - k = 0$.

Exercise 87 TRUE or FALSE ?

- (a) $x' = -x + 6$ and $y' = -y - 7$ are equations for a rotation.
- (b) $x' = px - qy + r$ and $y' = qx + py + s$ are equations for an even isometry.
- (c) $x' = -px - qy - r$ and $y' = qx - py - s$ are equations for an even isometry if $p^2 + q^2 = 1$.
- (d) $x' = -ax + by + h$ and $y' = bx + ay + k$ are equations for an odd isometry if $a^2 + b^2 = 1$.
- (e) $x' = \pm ax - by + h$ and $y' = \pm bx + ay + k$ are equations for an isometry if $a^2 + b^2 = 1$.
- (f) If $\mathcal{M}$ is any line, then every odd isometry is the product of $\sigma_{\mathcal{M}}$ followed by an even isometry.
- (g) If $\mathcal{M}$ is any line, then every odd isometry is the product of an even isometry followed by $\sigma_{\mathcal{M}}$.

Exercise 88 If

$$\begin{cases} x' = -\frac{\sqrt{3}}{2}x - \frac{1}{2}y + 1 \\ y' = \frac{1}{2}x - \frac{\sqrt{3}}{2}y - \frac{1}{2} \end{cases}$$

are equations for $\rho_{P,r}$, then find P and r .

Exercise 89 If

$$\begin{cases} x' = (\cos r)x - (\sin r)y + h \\ y' = (\sin r)x + (\cos r)y + k \end{cases}$$

are equations for nonidentity rotation $\rho_{C,r}$, then find C .

Exercise 90 If

$$\begin{cases} x' = ax + by + h \\ y' = bx - ay + k \end{cases} \quad \text{with } a^2 + b^2 = 1.$$

are equations for $\sigma_{\mathcal{L}}$, then find $\mathcal{L}$.

DISCUSSION :

The idea of doing geometry in terms of numbers and equations caught on after the publication of DESCARTES' *La Géométrie* in 1637. However, the idea that numbers and equations *are* geometric objects arose much later. In fact, the idea had no solid foundation until 1858, when the set $\mathbb{R}$ of real numbers was first given a clear definition, by RICHARD DEDEKIND (1831-1916). DEDEKIND's definition explains in particular the *continuity* of $\mathbb{R}$ which enables it to serve as a model for the line.

Once one has this model for the line it is relatively straightforward to model the plane by $\mathbb{R}^2$ and to verify EUCLID's axioms. This was first done in detail by DAVID HILBERT (1862-1943), thus subordinating geometry to the number concept after 2000 years of independence. It should be mentioned, however, that any construction of $\mathbb{R}$ from the natural numbers $0, 1, 2, \dots$ involves infinite sets. Thus, a "point" is a much subtle object than naïve intuition suggests.

The idea of interpreting points as numbers has been unexpectedly fruitful. Of course, we expect $\mathbb{R}$ to behave like a line because $\mathbb{R}$ was constructed with that purpose in mind, and it is no surprise that $+$ and $\times$ have a geometric meaning on the line ($+$ as a translation, $\times$ as a dilation). It may also not be a surprise that $+$ has a meaning in $\mathbb{R}^2$ (translation = vector addition) and so does multiplication by a real number (dilation again). But it is surely an unexpected bonus when multiplication by a *complex* number turns out to be geometrically meaningful.

After all, this multiplication is forced on us by algebra – by the demand that $i^2 = -1$ and that the field laws hold – yet when $a + ib \in \mathbb{C}$ is interpreted as $(a, b) \in \mathbb{R}^2$, multiplication by a complex number is simply the product of a dilation and a rotation. In particular, we have the miraculous fact that multiplication by e^{ir} is rotation through r . And this is just the beginning of the interplay between complex numbers and angles, leading to many applications of complex numbers, and particularly *complex functions*, in geometry.

In terms of the correspondence between vectors, points, and complex numbers we can set up a “dictionary” between geometry and complex numbers, as follows :

Vector, $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ (or point, $P = (x, y)$)	Complex number, $z = x + iy$
Length of a vector, $\ \vec{v}\ $	Modulus, $ z $
Distance between two points, $P_1 P_2$	Modulus of the difference, $ z_1 - z_2 $
Dot product, $\vec{v}_1 \bullet \vec{v}_2$	Real part of product, $\operatorname{Re}(\bar{z}_1 z_2)$
Collinear points, $P_1 - P_2 - P_3$	Vanishing imaginary part, $\frac{z_2 - z_1}{z_3 - z_1} \in \mathbb{R}$
Oriented angle between $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}$	Argument, $\arg(z) \in (-\pi, \pi]$
Orientation, $\vec{v} \mapsto \vec{v}^\perp$	Multiplication by i , $z \mapsto iz$
Translation, $\vec{v} \mapsto \vec{v} + \vec{a}$	Addition, $z \mapsto z + w$
Rotation, $\vec{v} \mapsto \rho_r(\vec{v})$	Multiplication by e^{ir} , $z \mapsto e^{ir} z$
Reflection in x -axis,	Complex conjugation, $z \mapsto \bar{z}$.

Using this dictionary we could translate all that we have done so far into the language of complex numbers.

The meaning of the word geometry changes with time and with the speaker.

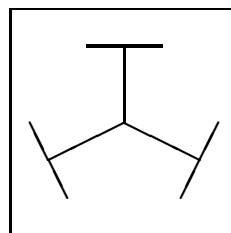
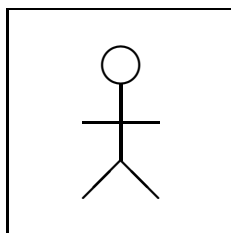
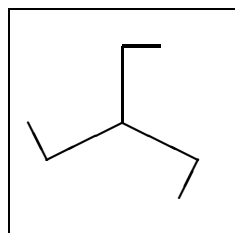
SHIING-SHEN CHERN

Chapter 6

Symmetry

Topics :

1. SYMMETRY AND GROUPS
 2. THE CYCLIC AND DIHEDRAL GROUPS
 3. FINITE SYMMETRY GROUPS
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6.1 Symmetry and Groups

There is an abundant supply of objects (bodies, organisms, structures, etc.) with *symmetry* in nature. Figures with *symmetry* appear throughout the visual arts. There are also many scientific applications of symmetry (for instance the classification of crystals and quasicrystals in chemistry). Theoretical physics makes heavy use of symmetry. But *what is symmetry* ?

When we say that a geometric figure (shape) is “symmetrical” we mean that we can apply certain isometries, called *symmetry operations*, which leave the whole figure unchanged while permuting its parts.

6.1.1 EXAMPLE. The capital letters **E** and **A** have *bilateral* (or *mirror*) *symmetry*, the mirror being horizontal for the former, vertical for the latter. (Bilateral symmetry is the symmetry of left and right, which is so noticeable in the structure of higher animals, especially the human body.)

6.1.2 EXAMPLE. The capital letter **N** is left unchanged by a halfturn, which may be regarded as the result of reflecting horizontally and then vertically, or vice versa. (Alternatively, one may prefer to view the halfturn as a rotation about the “centre” through an angle of 180° .) We can say that the capital letter **N** has *rotational symmetry*.

6.1.3 EXAMPLE. Another basic kind of symmetry is *translational symmetry*. Several combinations of these so-called basic symmetries may occur (for instance, bilateral and rotational symmetry, glide symmetry, translational and rotational symmetry, two independent translational symmetries, etc.)

Exercise 91 Find simple geometric figures (patterns) exhibiting each of the foregoing kinds of symmetry.

NOTE : In counting the symmetry operations of a figure, it is usual to include the identity transformation; any figure has this trivial symmetry.

We make the following definitions. Let **S** be a set of points (in $\mathbb{E}^2$).

6.1.4 DEFINITION. Line $\mathcal{L}$ is a **line of symmetry** (or **symmetry axis**) for $\mathbf{S}$ if

$$\sigma_{\mathcal{L}}(\mathbf{S}) = \mathbf{S}.$$

6.1.5 DEFINITION. Point P is a **point of symmetry** (or **symmetry centre**) for $\mathbf{S}$ if

$$\sigma_P(\mathbf{S}) = \mathbf{S}.$$

Exercise 92 Can a figure have

- (a) *exactly* two lines of symmetry ?
- (b) *exactly* two points of symmetry ?

Exercise 93 Why can't a (capital) letter of the alphabet (*written in most symmetric form*) have two points of symmetry ?

6.1.6 DEFINITION. Isometry α is a **symmetry** for $\mathbf{S}$ if

$$\alpha(\mathbf{S}) = \mathbf{S}.$$

6.1.7 EXAMPLE. Find the symmetries of a rectangle $\mathbf{R} = \square ABCD$ that is not a square.

SOLUTION : Without loss of generality, we may assume that

$$A = (h, k), B = (-h, k), C = (-h, -k), \quad \text{and} \quad D = (h, -k); \quad h, k > 0, h \neq k.$$

Evidently, the x - and y -axes are lines of symmetry for the rectangle, and the origin is a point of symmetry for the rectangle. Denoting the reflection in the x -axis by σ_x and the reflection in the y -axis by σ_y , we have that $\sigma_x, \sigma_y, \sigma_O$, and ι are symmetries for $\mathbf{R}$. Note that ι is a symmetry for *any* set of points. Since the image of the rectangle is known once it is known which of A, B, C, D is the image of A , then *these four transformations are the only possible symmetries for $\mathbf{R}$.*

NOTE : The (four) symmetries for a rectangle that is not a square form a group. Traditionally, this group is denoted by $\mathfrak{V}_4$ and is known as **Klein's four-group** (*Vierergruppe* in German).

6.1.8 PROPOSITION. *The set of all symmetries of a set of points forms a group.*

PROOF : Let $\mathbf{S}$ be any set of points. The set of symmetries for $\mathbf{S}$ is not empty as ι is a symmetry for $\mathbf{S}$.

Suppose α and β are symmetries for $\mathbf{S}$. Then

$$\beta\alpha(\mathbf{S}) = \beta(\alpha(\mathbf{S})) = \beta(\mathbf{S}) = \mathbf{S}.$$

So the set of symmetries has the closure property.

If α is a symmetry for $\mathbf{S}$, then α and α^{-1} are transformations and

$$\alpha^{-1}(\mathbf{S}) = \alpha^{-1}(\alpha(\mathbf{S})) = \iota(\mathbf{S}) = \mathbf{S}.$$

So the set of symmetries also has the inverse property.

Hence, the set of all symmetries of the set $\mathbf{S}$ forms a group. $\square$

The group of all symmetries of the set (figure) $\mathbf{S}$ is denoted by $\mathfrak{Sym}(\mathbf{S})$ and is called the **symmetry group** of $\mathbf{S}$.

What happens if the set of points is taken to be the set of *all* points, that is the plane $\mathbb{E}^2$? In this special case, the symmetries are exactly the same thing as the isometries. So

$$\mathfrak{Isom} = \mathfrak{Sym}(\mathbb{E}^2).$$

In other words, the group $\mathfrak{Isom}$ is the symmetry group of the Euclidean plane.

6.1.9 COROLLARY. *The set of all isometries forms a group.*

Examples of symmetry groups

Symmetry groups can be complicated. However, the *discrete* ones can be completely classified and listed (at least for Euclidean geometry).

6.1.10 EXAMPLE. The symmetry group of the capital letter **E** (or **A**) is the so-called **dihedral group** of order 2, generated by a single reflection and denoted by $\mathfrak{D}_1$.

NOTE : The Greek origin of the word *dihedral* is almost equivalent to the Latin origin of *bilateral*.

Exercise 94 What is the *symmetry group* of

- (a) a *scalene* triangle ?
- (b) an *isosceles* triangle that is not equilateral ?

6.1.11 EXAMPLE. The symmetry group of the capital letter **N** is likewise of order 2, but in this case the generator is a halfturn and we speak of the **cyclic group** $\mathfrak{C}_2$.

NOTE : The two groups $\mathfrak{D}_1$ and $\mathfrak{C}_2$ are “abstractly identical” (or *isomorphic*).

Exercise 95 What is the *symmetry group* of

- (a) a parallelogram that is not a rhombus ?
- (b) a parallelogram that is neither a rectangle nor a rhombus ?

6.2 The Cyclic and Dihedral Groups

Let $\mathfrak{G}$ be a group of isometries (i.e. a subgroup of $\mathfrak{Isom}$). Recall that $\mathfrak{G}$ is said to be **finite** if it consists of a finite number of elements (transformations); otherwise, $\mathfrak{G}$ is said to be **infinite**. The **order** of a (finite) group is the number of elements it contains.

The cyclic groups

Let $\alpha \in \mathfrak{G}$. If every element of $\mathfrak{G}$ is a power of α , then we say that $\mathfrak{G}$ is **cyclic** with *generator* α and denoted by $\langle \alpha \rangle$.

NOTE : The group $\langle \alpha \rangle$ is the smallest subgroup of $\mathfrak{G}$ containing the element (transformation) α . Two possibilities may arise :

- (a) All the powers α^k are different. In this case the group $\langle \alpha \rangle$ is infinite and is referred to as an *infinite cyclic group*.
- (b) Among the powers of α there are some that coincide. Then there is a positive power of α which is equal to the identity transformation ι . Denote by n the smallest positive exponent satisfying $\alpha^n = \iota$. In this case the group generated by α is

$$\langle \alpha \rangle = \{\iota, \alpha, \alpha^2, \dots, \alpha^{n-1}\}.$$

Such a (finite) group is a *cyclic group* of order n .

Cyclic groups are Abelian (i.e. commutative).

Let $n \geq 1$ be a positive integer and fix an arbitrary point C in the plane. (Without any loss of generality, we may assume that C is the origin.)

6.2.1 DEFINITION. The **cyclic group** $\mathfrak{C}_n$ is the (finite) group generated by the rotation $\rho = \rho_{C, \frac{360}{n}}$.

This group contains exactly n rotations (about the same centre C). The angles of rotation are multiples of $\frac{360}{n}$. We have

$$\begin{aligned} \mathfrak{C}_n &= \langle \rho \rangle \\ &= \{\iota, \rho, \rho^2, \dots, \rho^{n-1}\} \\ &= \left\{ \iota, \rho_{C, \frac{360}{n}}, \rho_{C, \frac{360 \cdot 2}{n}}, \dots, \rho_{C, \frac{360 \cdot (n-1)}{n}} \right\}. \end{aligned}$$

6.2.2 EXAMPLE. The cyclic group $\mathfrak{C}_1$ is the *trivial group* $\{\iota\}$.

6.2.3 EXAMPLE. The cyclic group $\mathfrak{C}_2$ has two elements : the identity transformation ι and the halfturn σ_C . This is the symmetry group of the capital letter **N**.

6.2.4 EXAMPLE. The *swastika* is symmetrical by rotation through any number of right angles; it admits four distinct symmetry operations : rotations through 1, 2, 3, or 4 right angles. The last is the identity. The first and the third are inverses of each other, since their product is the identity.

The symmetry group of the swastika is $\mathfrak{C}_4$, the cyclic group of order 4, generated by a rotation ρ of 90° (or *quarterturn*).

Exercise 96 Find a figure with symmetry group the cyclic group $\mathfrak{C}_3$.

NOTE : For any positive integer $n \geq 2$, there is polygon having symmetry group $\mathfrak{C}_n$.

The dihedral groups

Again, let $n \geq 1$ be a positive integer and C a fixed point. We are going to extend the cycle groups $\mathfrak{C}_n$ by incorporating appropriate reflections (i.e. bilateral symmetries).

6.2.5 DEFINITION. The **dihedral group** $\mathfrak{D}_n$ is the (finite) group containing the elements (rotations) of $\mathfrak{C}_n$ together with reflections in the n lines through C which divide the plane into $2n$ congruent angular regions.

This group has order $2n$ (i.e. it contains exactly $2n$ elements) : n rotations (about C) and n reflections in lines (passing through C). The angles between the axes of the reflections are multiples of $\frac{180}{n}$.

6.2.6 EXAMPLE. The dihedral group $\mathfrak{D}_1$ has two elements : the identity transformation ι and the reflection $\sigma_{\mathcal{L}}$ in a line $\mathcal{L}$ passing through C (the so-called symmetry axis). This is the symmetry group of the capital letter **A**.

6.2.7 EXAMPLE. The capital letter **H** admits both reflections and rotations as symmetry operations. It has a horizontal mirror (like **E**) and a vertical mirror (like **A**), as well as a center of rotational symmetry (like **N**) where the mirrors intersect. Thus it has four symmetry operations : the identity

ι , the horizontal reflection σ_h , the vertical reflection σ_v , and the halfturn $\sigma_h\sigma_v = \sigma_v\sigma_h$.

The symmetry group of the capital letter **H** is $\mathfrak{D}_2$, the **dihedral** group of order 4, generated by the two reflections σ_h and σ_v . Group $\mathfrak{D}_2$ is the familiar group $\mathfrak{V}_4$ (Klein's four-group).

Exercise 97 Compute the symmetry group of a rectangle that is not a square.

NOTE : Although $\mathfrak{C}_4$ and $\mathfrak{D}_2$ have both order 4, they are *not* isomorphic : they have a different structure, different CAYLEY tables. To see this, it suffices to observe that $\mathfrak{C}_4$ contains two elements of order 4, whereas all the elements of $\mathfrak{D}_2$ (except the identity) are of order 2.

6.2.8 EXAMPLE. Compute the symmetry group of a square.

SOLUTION : We suppose the square is centered at the origin and that one vertex lies on the positive x -axis. We see that the square is fixed by ρ and σ , where

$$\rho = \rho_{O,90} \quad \text{and} \quad \sigma = \sigma_h.$$

Observe that

$$\rho^4 = \sigma^2 = \iota.$$

Since the symmetries of the square form a group, then the square must be fixed by the four distinct rotations $\rho, \rho^2, \rho^3, \rho^4$ and by the four distinct isometries $\rho\sigma, \rho^2\sigma, \rho^3\sigma, \rho^4\sigma$. Let V_1 and V_2 be adjacent vertices of the square. Under a symmetry, V_1 must go to any one of the four vertices, but then V_2 must go to one of the two vertices adjacent to that one and the images of all remaining vertices are then determined. So *there are at most eight symmetries for the square*. We have listed eight distinct symmetries above. Therefore, there are exactly eight symmetries and we have listed all of them. Isometries ρ and σ generate the entire group.

The symmetry group of the square is

$$\mathfrak{D}_4 = \langle \rho, \sigma \rangle = \{\iota, \rho, \rho^2, \rho^3, \sigma, \rho\sigma, \rho^2\sigma, \rho^3\sigma\}$$

the **dihedral** group of order 8. Observe that

$$\sigma\rho = \rho^3\sigma, \quad \sigma\rho^2 = \rho^2\sigma, \quad \sigma\rho^3 = \rho\sigma.$$

The Cayley table for $\mathfrak{D}_4$ is given below.

$\mathfrak{D}_4$	ι	ρ	ρ^2	ρ^3	σ	$\rho\sigma$	$\rho^2\sigma$	$\rho^3\sigma$
ι	ι	ρ	ρ^2	ρ^3	σ	$\rho\sigma$	$\rho^2\sigma$	$\rho^3\sigma$
ρ	ρ	ρ^2	ρ^3	ι	$\rho\sigma$	$\rho^2\sigma$	$\rho^3\sigma$	σ
ρ^2	ρ^2	ρ^3	ι	ρ	$\rho^2\sigma$	$\rho^3\sigma$	σ	$\rho\sigma$
ρ^3	ρ^3	ι	ρ	ρ^2	$\rho^3\sigma$	σ	$\rho\sigma$	$\rho^2\sigma$
σ	σ	$\rho^3\sigma$	$\rho^2\sigma$	$\rho\sigma$	ι	ρ^3	ρ^2	ρ
$\rho\sigma$	$\rho\sigma$	σ	$\rho^3\sigma$	$\rho^2\sigma$	ρ	ι	ρ^3	ρ^2
$\rho^2\sigma$	$\rho^2\sigma$	$\rho\sigma$	σ	$\rho^3\sigma$	ρ^2	ρ	ι	ρ^3
$\rho^3\sigma$	$\rho^3\sigma$	$\rho^2\sigma$	$\rho\sigma$	σ	ρ^3	ρ^2	ρ	ι

6.2.9 EXAMPLE. Let $n \geq 3$ and consider a regular n -sided polygon centered at the origin. Suppose that one vertex lies on the positive x -axis.

The n -sided polygon is fixed by ρ and σ , where

$$\rho = \rho_{O, \frac{360}{n}} \quad \text{and} \quad \sigma = \sigma_h.$$

(σ_h is the reflection in the x -axis.)

Observe that

$$\rho^n = \sigma^2 = \iota.$$

Since the symmetries of the polygon form a group, then the polygon must be fixed by the n distinct rotations

$$\rho, \rho^2, \dots, \rho^{n-1}$$

and by the n distinct odd isometries

$$\sigma, \rho\sigma, \rho^2\sigma, \dots, \rho^{n-1}\sigma.$$

The symmetry group of the n -sided polygon must have at least these $2n$ symmetries. Let V_1 and V_2 be adjacent vertices of the polygon. Under a symmetry, V_1 must go to any one of the n vertices, but then V_2 must go to one of the two vertices adjacent to that one and the images of all remaining vertices are then determined. So *there are at most $2n$ symmetries for the n -sided polygon*. Therefore, there are exactly $2n$ symmetries and we have listed all of them. Isometries ρ and σ generate the entire group.

The symmetry group of the n -sided polygon is

$$\mathfrak{D}_n = \langle \rho, \sigma \rangle = \{\iota, \rho, \rho^2, \dots, \rho^{n-1}, \sigma, \rho\sigma, \rho^2\sigma, \dots, \rho^{n-1}\sigma\},$$

the **dihedral** group of order $2n$.

To compute the entire Cayley table, all that is needed are the equations

$$\sigma\rho^k = \rho^{-k}\sigma \quad \text{and} \quad \rho^n = \sigma^2 = \iota.$$

NOTE : The groups $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are, respectively, symmetry groups of an isosceles triangle that is not equilateral and of a rectangle that is not a square. Hence, or any positive integer $n \geq 1$, there is polygon having symmetry group $\mathfrak{D}_n$.

6.3 Finite Symmetry Groups

We want to investigate the possible finite symmetry groups of figures (in the Euclidean plane $\mathbb{E}^2$). So we are led to the study of finite subgroups $\mathfrak{G}$ of the group **Isom** of isometries (on $\mathbb{E}^2$).

The key observation which allows us to describe *all* finite symmetry groups is the following result.

6.3.1 PROPOSITION. *Let $\mathfrak{G}$ be a finite group of isometries. Then there is a point C in the plane which is left fixed by every element of $\mathfrak{G}$.*

PROOF : Let P be any point in the plane, and let $\mathbf{P}$ be the set of points which are images of P under the various elements (isometries) of $\mathfrak{G}$. So each element $P' \in \mathbf{P}$ has the form $P' = \alpha(P)$ for some $\alpha \in \mathfrak{G}$.

Any element of the group $\mathfrak{G}$ will permute $\mathbf{P}$. (In other words, if $P' \in \mathbf{P}$ and $\alpha \in \mathfrak{G}$, then $\alpha(P') \in \mathbf{P}$.)

We list the elements of $\mathbf{P}$ arbitrarily, writing

$$\mathbf{P} = \{P_1, P_2, \dots, P_n\}.$$

The fixed point we are looking for is the *centre of gravity* of $\mathbf{P}$, namely

$$C = \frac{1}{n}(P_1 + P_2 + \dots + P_n).$$

Any element of $\mathfrak{G}$ permutes the set $\{P_1, P_2, \dots, P_n\}$, hence it sends the centre of gravity to itself. $\square$

Let $\mathfrak{G}$ be a finite symmetry group (hence a finite subgroup of $\mathfrak{Isom}$). Then there is a point C fixed by every element (isometry) of $\mathfrak{G}$, and we may adjust coordinates so that this point is the origin. Also, it follows that $\mathfrak{G}$ cannot contain a nonidentity translation or a glide reflection.

So the group $\mathfrak{G}$ contains only rotations (about the same point) or reflections.

NOTE : The group generated by a nonidentity translation is an infinite subgroup of $\mathfrak{Isom}$. Hence any subgroup of $\mathfrak{Isom}$ which contains either rotations about two different points or a glide reflection is infinite. Indeed, if $\rho_{C,r}$ and $\rho_{D,s}$ are two nonidentity rotations about different centres, then

$$\rho_{C,r}^{-1} \rho_{D,s}^{-1} \rho_{C,r} \rho_{D,s}$$

is a nonidentity translation. Also, the square of any glide reflection is a nonidentity translation.

LEONARDO DA VINCI (1452-1519), who wanted to determine the possible ways to attach chapels and niches to a central building without destroying the symmetry of the nucleus, realized that *all designs (in the plane) with finitely many symmetries have either rotational symmetries and bilateral symmetries or just rotational symmetries*. In other words, the following result holds.

6.3.2 THEOREM. (LEONARDO'S THEOREM) *A finite symmetry group is either a cyclic group $\mathfrak{C}_n$ or a dihedral group $\mathfrak{D}_n$.*

PROOF : We shall consider the case $\mathfrak{G}$ contains only rotations and the case $\mathfrak{G}$ contains at least one reflection separately.

Suppose that (the finite group of isometries) $\mathfrak{G}$ contains only rotations. One possibility is that $\mathfrak{G}$ is the *trivial group* $\mathfrak{C}_1 = \{\iota\}$. Otherwise, we suppose $\mathfrak{G}$ contains a nonidentity rotation $\rho = \rho_{C,r}$. Then all the other elements in $\mathfrak{G}$ are rotations about the same centre C .

We note that

$$\rho_{C,-s} \in \mathfrak{G} \iff \rho_{C,s} \in \mathfrak{G}$$

and that all the elements in $\mathfrak{G}$ can be written in the form $\rho_{C,s}$, where $0 \leq s < 360$.

Let $\rho = \rho_{C,s}$, where s has the *minimum* positive value.

If $\rho_{C,t} \in \mathfrak{G}$ with $t > 0$, then $t - ks$ cannot be positive and less than s for any integer k by the minimality of s . So

- $t = ks$ for some integer k
- $\rho_{C,t} = \rho^k$.

In other words, the elements of $\mathfrak{G}$ are precisely the powers of ρ . We conclude that, in this case, $\mathfrak{G}$ is a cyclic group $\mathfrak{C}_n$ for some positive integer n .

Suppose now that (the finite group of isometries) $\mathfrak{G}$ contains at least one reflection. Since the identity transformation ι is an even isometry, since an isometry and its inverse have the same parity, and since the product of two even isometries is an even isometry, it follows that *the subset of all even isometries in $\mathfrak{G}$ forms a finite subgroup $\mathfrak{G}^+$ of $\mathfrak{G}$* . By the foregoing argument, we see that

$$\mathfrak{G}^+ = \mathfrak{C}_n = \{\iota, \rho, \rho^2, \dots, \rho^{n-1}\}.$$

So the even isometries in $\mathfrak{G}$ are the n rotations $\iota = \rho^n, \rho, \rho^2, \dots, \rho^{n-1}$.

Suppose $\mathfrak{G}$ has m reflections. If σ is a reflection in $\mathfrak{G}$, then the n odd isometries

$$\sigma, \rho\sigma, \rho^2\sigma, \dots, \rho^{n-1}\sigma$$

are in $\mathfrak{G}$. So $n \leq m$.

However, the m odd isometries multiplied (on the right) by σ give m distinct even isometries. So $m \leq n$.

Hence $m = n$ and $\mathfrak{G}$ contains the $2n$ elements generated by rotation ρ and reflection σ . We conclude that, in this case, $\mathfrak{G}$ is a dihedral group $\mathfrak{D}_n$ for some positive integer n . $\square$

Recall that an n -sided polygon (regular or not) has at most $2n$ symmetries. Since the symmetry group of a polygon must then be a finite group (of isometries), LEONARDO'S THEOREM has the following immediate corollary.

6.3.3 COROLLARY. *The symmetry group for a polygon is either a cyclic group or a dihedral group.*

6.4 Exercises

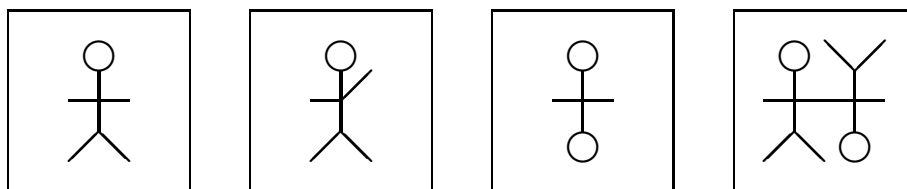
Exercise 98 What is the *symmetry group* of a rhombus that is not a square ?

Exercise 99 TRUE or FALSE ?

- (a) If P is a point of symmetry for set $\mathbf{S}$ of points, then P is in $\mathbf{S}$.
- (b) If $\mathcal{L}$ and $\mathcal{M}$ are perpendicular lines, then $\mathcal{L}$ is a line of symmetry for $\mathcal{M}$.
- (c) A regular pentagon has a point of symmetry.
- (d) The symmetry group of a rectangle has four elements.

Exercise 100 Compute the *symmetry group* of an equilateral triangle.

Exercise 101 Determine the *symmetry groups* of each of the following figures.



Exercise 102 What is the *symmetry group* of the graph of each of the following equations?

- (a) $y = x^2$.
- (b) $y = x^3$.
- (c) $3x^2 + 4y^2 = 12$.
- (d) $xy = 1$.

Exercise 103 Arrange the capital letters *written in most symmetric form* into equivalent classes where two letters are in the same class if and only if the two letters have the same symmetries when superimposed in standard orientation.

Exercise 104

- (a) Prove that every cyclic group $\mathfrak{C}_n$ is commutative.
- (b) Verify that the dihedral groups $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are commutative.
- (c) Prove that the groups $\mathfrak{D}_n$, $n \geq 3$ are *not* commutative.

Exercise 105 Find polygons having *symmetry groups* $\mathfrak{C}_3$ and $\mathfrak{C}_4$, respectively.

DISCUSSION :

Symmetry appeals to artist and scientist alike; it is intimately associated with an innate human appreciation of pattern. Symmetry is bound up in many of the deepest patterns of Nature, and nowadays it is fundamental to our scientific understanding of the Universe. Conservation principles, such as those for energy and momentum, express a symmetry that (we believe) is possessed by the entire space-time continuum : the laws of physics are the same everywhere. The quantum mechanics of fundamental particles is couched in the mathematical language of symmetries. The symmetries of crystals not only classify their shapes, but determine many of their properties. Many natural forms, from starfish to raindrops, from viruses to galaxies, have striking symmetries.

It took humanity roughly two and a half thousand years to attain a precise formulation of the concept of symmetry, counting from the time when the Greek geometers made the first serious mathematical discoveries about that concept, notably the proof that *there exist exactly five regular solids*. (The five regular solids are the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron.) Only after that lengthy period of gestation was the concept of symmetry something that scientists and mathematicians could *use* rather than just admire.

The understanding that symmetries are best viewed as transformations arose when mathematicians realized that the set of symmetries of an object is not just an arbitrary collection of transformations, but has a beautiful internal structure. The fact that the symmetries of an object form a *group* is a significant one. However, it's such a simple and "obvious" fact that for ages nobody even noticed it; and even when they did, it took mathematicians a while to appreciate just how significant this simple observation really is. It leads to a natural and elegant "algebra" of symmetry, known as *Group Theory*.

In 1952 the distinguished mathematician HERMANN WEYL (1885-1955), who was about to retire from the Institute for Advanced Studies at Princeton, gave a series of public lectures on mathematics. His topic, and the title of the book that grew from his talks, was *Symmetry*. It remains one of the classic popularizations of the subject. Some of the WEYL's greatest achievements had been in the deep mathematical setting that underlies the study of symmetry, and his lectures were strongly influenced by his mathematical tastes; but WEYL talked with authority about art and philosophy as well as mathematics and science. You will find in the book discussions of the cyclic groups, dihedral groups, as well as wallpaper groups. Most important, you will find a fascinating treatment in words and pictures of how these purely mathematical abstractions relate to the physical universe and works of art throughout the ages.

Symmetry is a vast subject, significant in art and nature. Mathematics lies at its root, and it would be hard to find a better one on which to demonstrate the working of the mathematical intellect.

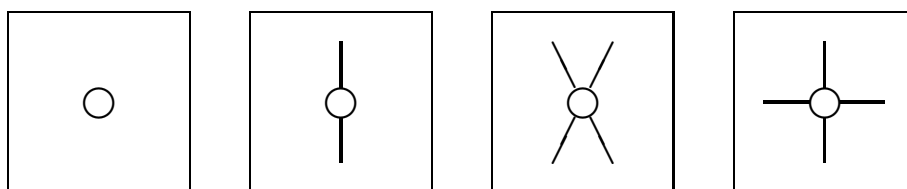
HERMANN WEYL

Chapter 7

Similarities

Topics :

1. CLASSIFICATION OF SIMILARITIES
 2. EQUATIONS FOR SIMILARITIES
-



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7.1 Classification of Similarities

The image of a triangle as seen through a “magnifying glass” is *similar* to the original triangle. For instance, the transformation $(x, y) \mapsto (2x, 2y)$ is a “magnifying glass” for the Euclidean plane, multiplying all distances by 2. (We shall call such a mapping a *stretch*.)

Some definitions

We make the following definition.

7.1.1 DEFINITION. If C is a point and $r > 0$, then a **stretch** (or **homothety**) **of ratio r about C** is the transformation that fixes C and otherwise sends point P to point P' , where P' is the unique point on $\overrightarrow{CP}$ such that $CP' = rCP$ (or, alternatively, where P' is the unique point on $\overleftrightarrow{CP}$ such that $\underline{CP'} = r\underline{CP}$).

We say that the point C is the **centre** and the (positive) factor r is the **magnification ratio** of the stretch. A stretch is also called a **homothetic transformation**.

NOTE : We allow the identity transformation to be a stretch (of ratio 1 and *any* **centre**). Observe, however, that we allow magnification ratios $r \leq 1$, which is in slight conflict with the everyday meaning of the word “magnification”.

Exercise 106 Verify that the set of all stretches with a given centre C forms a commutative group.

There is nothing to stop us from allowing a *negative* ratio in the definition of a stretch. In this case, point P is taken to a point P' lying on $\overleftrightarrow{CP}$ but on the other side of C from where P is located; that is, $\underline{CP'} = r\underline{CP}$. Thus such a transformation is the product (in either order) of a stretch about C and a halfturn about the centre. This motivates the following definition.

7.1.2 DEFINITION. A **dilation about point C** is a stretch about C or else a stretch about C followed by a halfturn about C .

Other transformations can be obtained by composing a stretch with any other transformation (e.g. an isometry). Two such special combinations will be given a name.

7.1.3 DEFINITION. A **stretch reflection** is a nonidentity stretch about some point C followed by the reflection in some line through C .

7.1.4 DEFINITION. A **stretch rotation** is a nonidentity stretch about some point C followed by a nonidentity rotation about C .

Any of the above transformations are *shape-preserving* : *they increase or decrease all lengths in the same ratio but leave shapes unchanged*. We make the following definition.

7.1.5 DEFINITION. If $r > 0$, then a **similarity** (or **similitude**) of ratio r is a transformation α such that

$$P'Q' = rPQ \quad \text{for all points } P \text{ and } Q, \quad \text{where } P' = \alpha(P) \text{ and } Q' = \alpha(Q).$$

Since a similarity is a transformation that multiplies all distances by some positive number, then the image of a triangle under a similarity is a triangle.

Exercise 107 Show that collinear points are mapped onto collinear points by a similarity transformation.

Thus *a similarity is a collineation*. The following proposition is easy to prove (and we shall leave it as an exercise).

7.1.6 PROPOSITION. *The following results hold.*

- (a) *An isometry is a similarity.*
- (b) *A similarity with two fixed points is an isometry.*

- (c) *A similarity with three noncollinear fixed points is the identity.*
- (d) *A similarity is a collineation that preserves betweenness, midpoints, segments, rays, triangles, angles, angle measure, and perpendicularity.*
- (e) *The product of a similarity of ratio r and a similarity of ratio s is a similarity of ratio rs .*
- (f) *The similarities form a group **Sim** that contains the group of **Isom** of all isometries.*

Exercise 108 Prove the preceding proposition.

7.1.7 PROPOSITION. *If $\triangle ABC \sim \triangle A'B'C'$, then there is a unique similarity α such that*

$$\alpha(A) = A', \quad \alpha(B) = B', \quad \text{and} \quad \alpha(C) = C'.$$

PROOF : Suppose $\triangle ABC \sim \triangle A'B'C'$. Let δ be the stretch about A such that $\delta(B) = E$ with $AE = A'B'$. With $F = \delta(C)$, then $\triangle AEF \cong \triangle A'B'C'$ by *ASA*. Since there is an isometry β such that $\beta(A) = A'$, $\beta(E) = B'$, and $\beta(F) = C'$, then $\beta\delta$ is a similarity taking A, B, C to A', B', C' , respectively. If α is a similarity taking A, B, C to A', B', C' , respectively, then $\alpha^{-1}(\beta\delta)$ fixes three noncollinear points and must be the identity. Therefore, $\alpha = \beta\delta$.

□

NOTE : Generalizing from triangles to arbitrary sets of points, we say that (the sets of points) $\mathbf{S}_1$ and $\mathbf{S}_2$ are **similar** provided there is a similarity α such that $\alpha(\mathbf{S}_1) = \mathbf{S}_2$.

What are the dilatations ?

Recall that a dilatation is a collineation α such that $\mathcal{L} \parallel \alpha(\mathcal{L})$ for every line $\mathcal{L}$ and that the group **H** generated by the halfturns is contained in the group **D** of all dilatations.

7.1.8 PROPOSITION. *A dilation is a dilatation and a similarity.*

PROOF : Let α be a dilation. First suppose α is a stretch of ratio r about point C . Transformation α fixes the lines through C . Suppose P, Q, R are three collinear points on a line off C and have images P', Q', R' , respectively, under α . Since

$$CP' = rCP, \quad CQ' = rCQ, \quad \text{and} \quad CR' = rCR$$

it follows (from the theory of similar triangles) that $\overleftrightarrow{P'Q'} \parallel \overleftrightarrow{PQ}$, that points P', Q', R' are collinear, and that $P'Q' = rPQ$. Hence, a stretch is a dilatation and a similarity. Since a halfturn is a dilatation and a similarity, then the product of a stretch and a halfturn is both a dilatation and a similarity. $\square$

7.1.9 PROPOSITION. *If $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$, then there is a unique dilatation δ such that*

$$\delta(A) = A' \quad \text{and} \quad \delta(B) = B'.$$

PROOF : Suppose $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$ and there is a dilatation δ such that $\delta(A) = A'$ and $\delta(B) = B'$. If point P is off $\overleftrightarrow{AB}$, then $\delta(P)$ is uniquely determined as the intersection of the line through A' that is parallel to $\overleftrightarrow{AP}$ and the line through B' that is parallel to $\overleftrightarrow{BP}$. Then, if Q is on $\overleftrightarrow{AB}$, point $\delta(Q)$ is uniquely determined as the intersection of $\overleftrightarrow{A'B'}$ and the line through $\delta(P)$ that is parallel to $\overleftrightarrow{PQ}$. Since the image of each point is uniquely determined by the images of A and B , then there is at most one dilatation δ taking A to A' and B to B' . On the other hand, $\tau_{A,A'}$ followed by the dilation about A' that takes $\tau_{A,A'}(B)$ to B' is a dilatation taking A to A' and B to B' . $\square$

7.1.10 PROPOSITION. *If point A is not fixed by dilatation δ , then line $\overleftrightarrow{AA'}$ is fixed by δ , where $A' = \delta(A)$.*

PROOF : If dilatation δ does not fix point A and if $A' = \delta(A)$, then $\delta(\overleftrightarrow{AA'})$ must be the line through $\delta(A)$ that is parallel to $\overleftrightarrow{AA'}$. $\square$

We can now answer the question “What are the dilatations ?”

7.1.11 PROPOSITION. *A dilatation is a translation or a dilation.*

PROOF : A nonidentity dilatation α must have some nonfixed line $\mathcal{L}$. So $\mathcal{L}$ and $\alpha(\mathcal{L})$ are distinct parallel lines. Any two points A and B on line $\mathcal{L}$ are such that neither $\alpha(A)$ nor $\alpha(B)$ is on $\mathcal{L}$. Let $A' = \alpha(A)$ and $B' = \alpha(B)$. Now $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are distinct parallel lines. If $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'}$, then $\square AA'B'B$ is a parallelogram, $\tau_{A,A'}(B) = B'$, and (PROPOSITION 7.1.9) dilatation α must be the translation $\tau_{A,A'}$. However, suppose $\overleftrightarrow{AA'} \nparallel \overleftrightarrow{BB'}$. Then the lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ are fixed (PROPOSITION 7.1.10) and must intersect at some fixed point C . Since $\overleftrightarrow{AB}$ is not fixed, then C is off both parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ with C, A', A collinear and C, B', B collinear. So $CA'/CA = CB'/CB$. Then there is a dilation δ about C such that $\delta(A) = A'$ and $\delta(B) = B'$. (If point C is between points A and A' , then δ is a stretch followed by σ_C ; otherwise, δ is simply a stretch about C .) By the uniqueness of a dilatation taking A to A' and B to B' , the dilatation α must be the translation $\tau_{A,A'}$ or else the dilation δ . $\square$

The classification theorem

7.1.12 PROPOSITION. *If α is a similarity and P is any point, then $\alpha = \beta\delta$, where δ is a stretch about P and β is an isometry.*

PROOF : A similarity is just a stretch about some point P followed by an isometry. Actually, the point P can be arbitrarily chosen as follows. If α is a similarity of ratio r , let δ be the stretch of ratio r about P . Then δ^{-1} is a stretch of ratio $1/r$. So $\alpha\delta^{-1}$ is an isometry and $\alpha = (\alpha\delta^{-1})\delta$. $\square$

This important result gives us a feeling for the nature of the similarities. We need only one more result on similarities before the CLASSIFICATION THEOREM. However, the proof uses a lemma about *directed distance*.

We suppose the lines in the plane are *directed* (in an arbitrary fashion) and $\underline{AB}$ denotes the **directed distance** from A to B on line $\overleftrightarrow{AB}$. For any

points A and B , we have

$$\underline{AB} = -\underline{BA} \quad \text{and so} \quad \underline{AA} = 0.$$

NOTE : For distinct points A, B, C on line $\mathcal{L}$ the number $\underline{AC}/\underline{CB}$ is independent of the choice of positive direction on line $\mathcal{L}$, as changing the positive direction would change the sign of both numerator and denominator and leave the value of the fraction itself unchanged.

Exercise 109 Show that the function

$$f : \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R} \setminus \{-1\}, \quad x \mapsto \frac{x}{1-x}$$

is a *bijection*.

7.1.13 LEMMA. Given the line $\overleftrightarrow{AB}$, the function

$$\widehat{f} : \overleftrightarrow{AB} \setminus \{B\} \rightarrow \mathbb{R} \setminus \{-1\}, \quad X \mapsto \underline{AX}/\underline{XB}$$

is a *bijection*.

PROOF : There is a one-to-one correspondence between points $X \in \overleftrightarrow{AB}$ and real numbers $x \in \mathbb{R}$ given by (the equation)

$$\underline{AX} = x \underline{AB}.$$

Hence we can identify any point $X \in \overleftrightarrow{AB}$, different from B , with its *intrinsic coordinate* $x \in \mathbb{R} \setminus \{1\}$. Then

$$\underline{XB} = \underline{XA} + \underline{AB} = (1-x) \underline{AB}$$

and so

$$\begin{aligned} \underline{AX}/\underline{XB} &= \frac{x}{1-x} \\ &= f(x). \end{aligned}$$

It follows that the function

$$X \mapsto \underline{AX}/\underline{XB}$$

is a *bijection*. □

7.1.14 COROLLARY. If point $P \in \overleftrightarrow{AB}$, different from B , then

$$\underline{AP}/\underline{PB} \neq -1.$$

7.1.15 COROLLARY. If $t \neq -1$, then there exists a unique point $P \in \overleftrightarrow{AB}$, different from B , such that

$$\underline{AP}/\underline{PB} = t.$$

7.1.16 COROLLARY. Point $P \in \overleftrightarrow{AB}$ is between A and B if and only if $\underline{AP}/\underline{PB}$ is positive.

7.1.17 PROPOSITION. A similarity without a fixed point is an isometry.

PROOF : The lemma above will now be used to prove that a similarity that is not an isometry must have a fixed point. Suppose α is a similarity that is not an isometry. We may suppose α is not a dilatation. (Why ?) So there is a line $\mathcal{L}$ such that $\mathcal{L}' \nparallel \mathcal{L}$ where $\mathcal{L}' = \alpha(\mathcal{L})$. Let $\mathcal{L}$ intersect $\mathcal{L}'$ at point A . With $A' = \alpha(A)$, then A' is on $\mathcal{L}'$. We suppose $A' \neq A$. Let $\mathcal{M}$ be the line through A' that is parallel to $\mathcal{L}$. With $\mathcal{M}' = \alpha(\mathcal{M})$, then $\mathcal{M}' \parallel \mathcal{L}'$. Let $\mathcal{M}'$ intersect $\mathcal{M}$ at point B . With $B' = \alpha(B)$, then B' is on $\mathcal{M}'$ and distinct from A' . We suppose $B' \neq B$. So

$$\mathcal{L}' = \overleftrightarrow{AA'}, \quad \mathcal{M}' = \overleftrightarrow{BB'}, \quad \text{and} \quad \overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'}.$$

Now $\overleftrightarrow{AB} \nparallel \overleftrightarrow{A'B'}$ as otherwise $A'B' = AB$ and α is an isometry. So $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ intersect at some point P off both parallel lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ with P, A, B collinear and P, A', B' collinear. So $AP/PB = A'P/PB'$. If α has ratio r and $P' = \alpha(P)$, then

$$AP/PB = rAP/rPB = A'P'/P'B'.$$

Hence, $A'P'/P'B' = A'P/PB'$. Point P is between A' and B' if and only if P is between A and B since $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'}$, but P is between A and B if and only if P' is between A' and B' . Hence, P is between A' and B' if

and only if P' is between A' and B' . Therefore, by LEMMA 7.1.13 (and its corollaries),

$$\underline{A'P/PB'} = \underline{A'P'/P'B'} \quad \text{and} \quad P = P'.$$

So $\alpha(P) = P$, as desired. $\square$

7.1.18 THEOREM. (THE CLASSIFICATION THEOREM FOR PLANE SIMILARITIES)

Each nonidentity similarity is exactly one of the following : isometry, stretch, stretch rotation or a stretch reflection.

PROOF : In order to classify the similarities, suppose α is a similarity that is not an isometry. Then α has some fixed point C . So $\alpha = \beta\delta$ where δ is a stretch about C and where β is an isometry. Since $\beta(C) = \alpha\delta^{-1}(C) = C$, then β must be one of the identity ι , a rotation ρ about C , or a reflection σ_C with C on $\mathcal{C}$. Hence, α is one of δ , $\rho\delta$, or $\sigma_C\delta$. We have proved the major part of the result. There remains only the task of verifying the “exactly” in the statement of the classification theorem; this is left as an exercise. $\square$

Exercise 110 Finish the proof of the Classification Theorem (for similarities).

7.2 Equations for Similarities

The following technical result is easy to prove.

7.2.1 PROPOSITION. *Suppose $\alpha \in \mathbf{Sim}$. Then*

$$(a) \quad \alpha\gamma\alpha^{-1} \in \mathbf{Isom} \text{ if } \gamma \in \mathbf{Isom}.$$

$$(b) \quad \alpha\delta\alpha^{-1} \in \mathbf{D} \text{ if } \delta \in \mathbf{D}.$$

$$(c) \quad \alpha\eta\alpha^{-1} \in \mathbf{H} \text{ if } \eta \in \mathbf{H}.$$

$$(d) \quad \alpha\tau\alpha^{-1} \in \mathbf{T} \text{ if } \tau \in \mathbf{T}.$$

$$(e) \quad \alpha\sigma_P\alpha^{-1} = \sigma_{\alpha(P)}.$$

$$(f) \quad \alpha\sigma_{\mathcal{L}}\alpha^{-1} = \sigma_{\alpha(\mathcal{L})}.$$

Exercise 111 Prove the preceding proposition.

In order to look at the dilatations a little more closely, a notation for the dilation is introduced as follows. If $a > 0$, then $\delta_{P,a}$ is the stretch about P of (magnification) ratio a and dilation $\delta_{P,-a}$ is defined by

$$\delta_{P,-a} := \sigma_P \delta_{P,a}.$$

Multiplying both sides of this last equation by σ_P on the left, we have $\sigma_P \delta_{P,-a} = \delta_{P,a}$. So

$$\delta_{P,-r} = \sigma_P \delta_{P,r}, \quad r \neq 0.$$

The number r is called the **dilation ratio** of dilation $\delta_{P,r}$. There are two special cases where a dilation is also an isometry :

$$\delta_{P,1} = \iota \quad \text{and} \quad \delta_{P,-1} = \sigma_P.$$

Clearly, the ratio of $\delta_{P,r}$ is the absolute value $|r|$ of the dilation ratio r . For example, $\delta_{P,-3}$ has ratio $+3$ but dilation ratio -3 .

7.2.2 PROPOSITION. *If P is a point, then*

$$\delta_{P,-r} = \sigma_P \delta_{P,r}, \quad \delta_{P,1} = \iota, \quad \delta_{P,-1} = \sigma_P, \quad \delta_{P,s} \delta_{P,r} = \delta_{P,rs} \quad (r, s \neq 0).$$

If $\delta_{P,r}$ is a dilation and α is a similarity, then

$$\alpha \delta_{P,r} \alpha^{-1} = \delta_{\alpha(P),r}.$$

PROOF : From the special case

$$\sigma_P = \delta_{P,r} \sigma_P \delta_{P,r}^{-1} \quad (\text{see PROPOSITION 7.2.1})$$

it follows

$$\sigma_P \delta_{P,r} = \delta_{P,r} \sigma_P$$

and then

$$\delta_{P,s} \delta_{P,r} = \delta_{P,rs} \quad (r, s \neq 0).$$

Thus,

$$\delta_{P,r}^{-1} = \delta_{P,1/r}, \quad r \neq 0.$$

If α is any similarity, then $\alpha\delta_{P,r}\alpha^{-1}$ is a dilatation (PROPOSITION 7.2.1) fixing point $\alpha(P)$ and has ratio $|r|$. Hence,

$$\alpha\delta_{P,r}\alpha^{-1} = \delta_{\alpha(P),s}, \quad \text{where } s = \pm r.$$

The question is “Is r the dilation ratio of $\alpha\delta_{P,r}\alpha^{-1}$?” With $P' = \alpha(P)$ and $Q' = \alpha(Q)$ for $Q \neq P$, that the answer is “Yes” follows from the equivalence of each of the following :

- (1) $r > 0$.
- (2) $\delta_{P,r}$ is a stretch.
- (3) $\delta_{P,r}(Q)$ is on $PQ^{\rightarrow}$.
- (4) $\alpha\delta_{P,r}(Q)$ is on $P'Q'^{\rightarrow}$.
- (5) $\alpha\delta_{P,r}\alpha^{-1}(\alpha(Q))$ is on $P'Q'^{\rightarrow}$.
- (6) $\delta_{P,s}(Q')$ is on $P'Q'^{\rightarrow}$.
- (7) $\delta_{P,s}$ is a stretch.
- (8) $s > 0$.

Since $s = \pm r$ and both r and s have both the same sign, then $r = s$, as desired. $\square$

NOTE : If $r \neq 1$, then the nonidentity dilation $\delta_{P,r}$ is said to have **centre** P .

Further results

Further results on the dilatations are more easily obtained by using coordinates.

7.2.3 PROPOSITION. *If $P = (u, v)$, then (the dilation) $\delta_{P,r}$ has equations*

$$\begin{cases} x' = rx + (1-r)u \\ y' = ry + (1-r)v. \end{cases}$$

PROOF : Let $O = (0, 0)$ be the origin of the plane. We clearly have

$$\delta_{O,r}((x, y)) = (rx, ry), \quad r > 0$$

and this same equation must hold for negative r since $\sigma_O((x, y)) = (-x, -y)$.

So $\delta_{O,r}$ has equations

$$\begin{cases} x' = rx \\ y' = ry \end{cases}$$

in any case. Now, suppose $P = (u, v)$ and $\delta_{P,r}((x, y)) = (x', y')$. Then, from the equations

$$\delta_{P,r} = \tau_{O,P} \delta_{O,r} \tau_{O,P}^{-1} = \tau_{O,P} \delta_{O,r} \tau_{P,O}$$

we have

$$\delta_{P,r}((x, y)) = (r(x - u) + u, r(y - v) + v) = (x', y').$$

Indeed,

$$(x, y) \mapsto (x - u, y - v) \mapsto (r(x - u), r(y - v)) \mapsto (r(x - u) + u, r(y - v) + v).$$

Hence $\delta_{P,r}$ has equations

$$\begin{cases} x' = rx + (1-r)u \\ y' = ry + (1-r)v. \end{cases}$$

□

This simple result has some interesting corollaries.

7.2.4 COROLLARY. *Given $\delta_{A,1/r}$ and $\delta_{B,r}$, then for some point C*

$$\delta_{B,r}\delta_{A,1/r} = \tau_{A,C}.$$

7.2.5 COROLLARY. *Given $\delta_{A,r}$ and $\delta_{B,s}$ with $rs \neq 1$, then for some point C*

$$\delta_{B,s}\delta_{A,r} = \delta_{C,rs}.$$

7.2.6 COROLLARY. *Given $\tau_{A,B}$ and $\delta_{A,r}$ with $r \neq 1$, then for some point C*

$$\tau_{A,B}\delta_{A,r} = \delta_{B,r}\tau_{A,B} = \delta_{C,r}.$$

Exercise 112 In each case, work out an explicit expression for the point C (in terms of A, B, r , and s , as may be the case).

NOTE : Although the coordinate proofs for the corollaries above are easy to give and the content of the equations themselves is easy to understand, the visualization is very hard, if not, in some sense, virtually impossible.

7.2.7 PROPOSITION. *A similarity (of ratio r) has equations of the form*

$$\begin{cases} x' = ax - by + h \\ y' = \pm(bx + ay) + k \end{cases} \quad \text{with } r^2 = a^2 + b^2 \neq 0$$

and, conversely, equations of this form are those of a similarity.

PROOF : A similarity is a stretch about the origin O followed by an isometry (PROPOSITION 7.1.12). From this fact and the equations for an isometry given by PROPOSITION 5.3.3, it follows that a similarity has equations of the form

$$\begin{cases} x' = (r \cos q)x - (r \sin q)y + h \\ y' = \pm((r \sin q)x + (r \cos q)y) + k \end{cases}$$

and, conversely, equations of this form are those of a similarity. With

$$a = r \cos q \quad \text{and} \quad b = r \sin q$$

we get the desired result. $\square$

7.2.8 DEFINITION. A similarity α that is a stretch about some point P followed by an even isometry is said to be **direct**.

7.2.9 DEFINITION. A similarity α that is a stretch about some point P followed by an odd isometry is said to be **opposite**.

From the equations for isometries and similarities it is evident that whether a similarity is direct or opposite is independent of the point P above.

NOTE : In the equations in PROPOSITION 7.2.7, the positive sign applies to direct similarities and the negative sign applies to opposite similarities.

We have

7.2.10 PROPOSITION. *Every similarity is either direct or opposite, but not both. The direct similarities form a group. The product of two opposite similarities is direct. The product of a direct similarity and an opposite similarity is an opposite similarity.*

7.3 Exercises

Exercise 113 For what point P does a dilation about P have equations

$$\begin{cases} x' = -2x + 3 \\ y' = -2y - 4 \end{cases} ?$$

Exercise 114 What are the fixed points and fixed lines of a stretch reflection ?
What are the fixed points and fixed lines of a stretch rotation ?

Exercise 115 TRUE or FALSE ?

- (a) A similarity that is not an isometry has a fixed point, and a dilatation that is not a translation has a fixed point.
- (b) The group of all dilatations is *generated* by the dilations.

- (c) $\sigma_P \delta_{P,r} = \delta_{P,r} \sigma_P$ for any point P and nonzero number r .
- (d) $\delta_{A,r}(B)$ is on $AB^\rightarrow$ if $A \neq B$.
- (e) There is a unique point Q on $\overleftrightarrow{AB}$ such that $AQ/QB = 7$.
- (f) $\alpha \tau_{A,B} \alpha^{-1} = \tau_{\alpha(A), \alpha(B)}$ for any similarity α and points A and B .
- (g) A dilatation is a similarity.

Exercise 116 PROVE or DISPROVE : If α is a transformation and δ is a dilation, then $\alpha \delta \alpha^{-1}$ is a dilatation.

Exercise 117 PROVE or DISPROVE : If $r > 0$, then a mapping α such that $P'Q' = rPQ$ for all points P and Q with $P' = \alpha(P)$ and $Q' = \alpha(Q)$ is a similarity.

Exercise 118 Complete each of the following :

- (a) If $\delta_{P,3}((x, y)) = (3x + 7, 3y - 5)$, then $P = \dots$
- (b) If $x' = 3x + 5y + 2$ and $y' = tx - 3y$ are the equations of a similarity, then $t = \dots$
- (c) If $\sigma_P \delta_{P,15} = \delta_{P,x}$, then $x = \dots$
- (d) If $\delta_{C,r} \tau_{A,B} = \tau_{P,Q} \delta_{C,r}$, then $P = \dots$ and $Q = \dots$
- (e) If $\delta_{B,s} \delta_{A,t} = \delta_{T,t} \delta_{B,s}$, then $T = \dots$
- (f) If $\rho_{A,r} \delta_{A,s} = \delta_{A,s} \rho_{A,x}$, then $x = \dots$
- (g) If $\tau_{A,B}^{-1} = \tau_{A,C}$, then $C = \dots$

Exercise 119 PROVE or DISPROVE : Nonidentity dilatations α and β commute if and only if α and β are translations.

Exercise 120 If $\alpha((1, 2)) = (0, 0)$ and $\alpha((3, 4)) = (3, 4)$, then what is the ratio of similarity α ?

Exercise 121 If $\alpha((0, 0)) = (1, 0)$, $\alpha((1, 0)) = (2, 2)$, and $\alpha((2, 2)) = (-1, 6)$ for similarity α , then find $\alpha((-1, 6))$.

Exercise 122 Show that an involutory similarity is a reflection or a halfturn.

Exercise 123 PROVE or DISPROVE : There are exactly two dilatations taking circle A_B to circle C_D .

Exercise 124 Show that a nonidentity dilation with centre P commutes with $\sigma_{\mathcal{L}}$ if and only if P is on $\mathcal{L}$.

Exercise 125 Show that

- (a) nonidentity dilations $\delta_{A,a}$ and $\delta_{B,b}$ commute if and only if $A = B$.
- (b) dilatations $\delta_{A,a}$ and $\tau_{A,B}$ never commute if $A \neq B$ and $a \neq 1$.

DISCUSSION :

The Euclidean plane admits transformations (called *similarities*) which multiply all distances by a constant factor $r \neq 0$. The typical similarity is (the *stretch*) $(x, y) \mapsto (rx, ry)$. Figures related by a similarity are said to be “of the same shape” or “similar”. In particular, all triangles with the same angles are similar, as are all squares. The existence of squares of different sizes means, for instance, that n^2 unit squares fill a square of side n . This leads to the property of Euclidean area that multiplying the lengths in a figure by r multiplies its area by r^2 .

The Euclidean plane is unique in having this simple dependence of area on length because the sphere and the hyperbolic plane *do not* admit similarities (except with $r = 1$). There the relationships between length and area are more complicated – involving circular and hyperbolic functions, respectively – but the relationship between angles and area is delightfully simple.

This is a benefit of having the triangle’s angle sum unequal to π . One then has a nontrivial angular excess function for triangles $\triangle$:

$$excess(\triangle) = \text{angle sum}(\triangle) - \pi,$$

which is proportional to area because it is *additive*. That is, if triangle $\triangle$ splits into triangles $\triangle_1$ and $\triangle_2$, then

$$excess(\triangle) = excess(\triangle_1) + excess(\triangle_2).$$

Euclidean geometry misses out this property because it is too simple – its angular excess function is zero. It can be shown that any continuous, nonzero, additive function must be proportional to area.

The identification of angular excesses with area clearly shows why similar triangles of different sizes cannot exist in the sphere and hyperbolic plane. Triangles with the same angles have the same area, and hence one cannot be larger than the other.

A branch of mathematics is called geometry, because the name seems good on emotional and traditional grounds to a sufficiently large number of competent people.

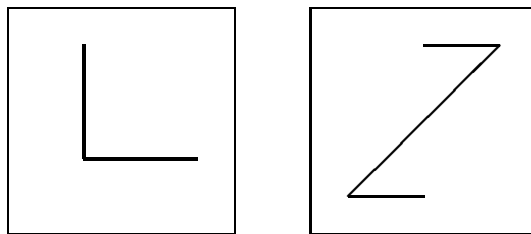
O. VEBLER AND J.H.C. WHITEHEAD

Chapter 8

Affine Transformations

Topics :

1. COLLINEATIONS
 2. AFFINE LINEAR TRANSFORMATIONS
-



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8.1 Collineations

We now turn to transformations that were first introduced by LEONHARD EULER (1707-1783).

Affine transformations (as collineations)

8.1.1 DEFINITION. An **affine transformation** (or **affinity**) is a collineation that preserves parallelness among lines.

So, if $\mathcal{L}$ and $\mathcal{M}$ are parallel lines and α is an affine transformation, then lines $\alpha(\mathcal{L})$ and $\alpha(\mathcal{M})$ are parallel. It is easy to prove the following result.

8.1.2 PROPOSITION. *A collineation is an affine transformation and, conversely, an affine transformation is a collineation.*

PROOF : An affine transformation is by definition a collineation. If β is any collineation and $\mathcal{L}$ and $\mathcal{M}$ are distinct parallel lines, then $\beta(\mathcal{L})$ and $\beta(\mathcal{M})$ cannot contain a common point $\beta(P)$, as point P would then have to be on both $\mathcal{L}$ and $\mathcal{M}$. Therefore, every collineation is an affine transformation. $\square$

NOTE : Affine transformations and collineations are *exactly* the same thing for the Euclidean plane. The choice between the terms *affine transformation* and *collineation* is sometimes arbitrary and sometimes indicates a choice of emphasis on parallelness of lines or on collinearity of points. Loosely speaking, *affine geometry* is what remains after surrendering the ability to measure length (isometries) and surrendering the ability to measure angles (similarities), but maintaining the incidence structure of lines and points (collineations).

8.1.3 EXAMPLE. Similarities preserve parallelness and hence are affine transformations. In particular, isometries are also affine transformations.

8.1.4 EXAMPLE. The mapping

$$\alpha : \mathbb{E}^2 \rightarrow \mathbb{E}^2, \quad (x, y) \mapsto (2x, y)$$

is an affine transformation that is not a similarity.

NOTE : The word *symmetry* brings to mind such general ideas as balance, agreement, order, and harmony. We have been exceedingly conservative in our use of the word *symmetry*; for us, symmetries are restricted to isometries. With a broader mathematical usage of the term, we would certainly be saying that the similarities are the symmetries of similarity geometry and that the collineations are the symmetries of affine geometry. In the most broad usage, *the group of all transformations on a structure that preserves the essence of that structure constitutes the symmetries (also called the automorphisms) of the structure.*

A collineation preserves collinearity of points. We wish to show that, conversely, a transformation such that the image of every three collinear points are themselves collinear must be a collineation.

8.1.5 PROPOSITION. *A transformation such that the images of every three collinear points are themselves collinear is an affine transformation.*

PROOF : We suppose α is a transformation that preserves collinearity and aim to show $\alpha(\mathcal{L})$ is a line whenever $\mathcal{L}$ is a line. Let A and B be distinct points on $\mathcal{L}$, and let $\mathcal{M}$ be the line through $\alpha(A)$ and $\alpha(B)$. By the definition of α , all the points of $\alpha(\mathcal{L})$ are on $\mathcal{M}$. However, are all the points of $\mathcal{M}$ on $\alpha(\mathcal{L})$? Suppose C' is a point on $\mathcal{M}$ distinct from $\alpha(A)$ and $\alpha(B)$, and let C be the point such that $\alpha(C) = C'$. To show C must be on $\mathcal{L}$, we assume C is off $\mathcal{L}$ and then obtain a contradiction. Now the image of all the points of $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{AC}$ are on $\mathcal{M}$ since collinearity is preserved under α . However, any point P in the plane is on a line containing two distinct points of $\triangle ABC$. Since the images of these two points lie on $\mathcal{M}$, then the image of P lies on $\mathcal{M}$. Therefore, the image of every point lies on $\mathcal{M}$, contradicting the fact that α is an onto mapping. Hence, C must lie on $\mathcal{L}$, $\mathcal{M} = \alpha(\mathcal{L})$, and α is a collineation, as desired. $\square$

Are the affine transformations the same as those transformations for which the images of any three noncollinear points are themselves noncollinear ? The answer is “Yes”.

8.1.6 PROPOSITION. *A transformation is an affine transformation if and only if the images of any three noncollinear points are themselves noncollinear.*

PROOF : Suppose α is an affine transformation. Then α^{-1} is an affine transformation and can't take three noncollinear points to three collinear points. Therefore, affine transformation α must take any three noncollinear points to three noncollinear points.

Conversely, suppose β is a transformation such that the images of any three noncollinear points are themselves noncollinear. Assume β is *not* an affine transformation. Then β^{-1} is not an affine transformation. By the contrapositive of the preceding result, then there are three collinear points whose images under β^{-1} are not collinear. Hence, since β is the inverse of β^{-1} , then there are three noncollinear points whose images under β are collinear, contradiction. Therefore, β is an affine transformation. $\square$

An affine transformation preserves betweenness

The result above does *not* state that the image of a triangle under an affine transformation is necessarily a triangle, but states only that the images of the vertices of a triangle are themselves vertices of a triangle. We do not know the image of a segment is necessarily a segment. More fundamental, we do not know that an affine transformation necessarily preserves betweenness. It will take some effort to prove this. We begin by showing that *midpoint is actually an affine concept*; that is, an affine transformation carries the midpoint of two given points to the midpoint of their images.

8.1.7 PROPOSITION. *If α is an affine transformation and M is the midpoint of points A and B , then $\alpha(M)$ is the midpoint of $\alpha(A)$ and $\alpha(B)$.*

PROOF : Suppose A and B are distinct points and α is an affine transformation. Let P be any point off $\overleftrightarrow{AB}$. Let Q be the intersection of the line through A that is parallel to $\overleftrightarrow{PB}$ and the line through B that is parallel to $\overleftrightarrow{PA}$. So $\square APBQ$ is a parallelogram. Let $A' = \alpha(A)$, $B' = \alpha(B)$, $P' = \alpha(P)$,

and $Q' = \alpha(Q)$. Since two parallel lines go to two parallel lines under α , then $\square A'P'B'Q'$ is a parallelogram. (We are *not* claiming that $\alpha(\square APBQ) = \square A'P'B'Q'$ but only that A', P', B', Q' are vertices in order of a parallelogram.) Further, $\overleftrightarrow{M}$, the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{PQ}$, must go to M' , the intersection of $\overleftrightarrow{A'B'}$ and $\overleftrightarrow{P'Q'}$. However, since the diagonals of a parallelogram bisect each other, then M is the midpoint of A and B while M' is the midpoint of A' and B' . Hence, α preserves midpoints. $\square$

8.1.8 PROPOSITION. *If α is an affine transformation, the $n+1$ points $P_0, P_1, P_2, \dots, P_n$ divide the segment $\overline{P_0P_n}$ into n congruent segments $\overline{P_{i-1}P_i}$, and $P'_i = \alpha(P_i)$, then the $n+1$ points $P'_0, P'_1, P'_2, \dots, P'_n$ divide the segment $\overline{P'_0P'_n}$ into n congruent segments $\overline{P'_{i-1}P'_i}$.*

PROOF : Suppose α is an affine transformation and the $n+1$ points $P_0, P_1, P_2, \dots, P_n$ divide the segment $\overline{P_0P_n}$ into n congruent segments $\overline{P_{i-1}P_i}$. Let $P'_i = \alpha(P_i)$. Since $P_0P_1 = P_1P_2, P_1P_2 = P_2P_3, \dots$, then P_1 is the midpoint of P_0 and P_2 , point P_2 is the midpoint of P_1 and P_3 , etc. Hence, P'_1 is the midpoint of P'_0 and P'_2 , point P'_2 is the midpoint of P'_1 and P'_3 , etc. So the images $P'_0, P'_1, P'_2, \dots, P'_n$ divide the segment $\overline{P'_0P'_n}$ into n congruent segments $\overline{P'_{i-1}P'_i}$. $\square$

It follows from this last result that P between A and B implies $\alpha(P)$ between $\alpha(A)$ and $\alpha(B)$ provided that $\frac{AP}{PB}$ is rational.

NOTE : It would have to be a very strange collineation that allowed betweenness not to be preserved in general although preserving midpoints. Early geometers avoided such a “monster transformation” simply by incorporating the preservation of betweenness within the definition of an affine transformation. In 1880 GASTON DARBOUX (1842-1917) showed that the “monster transformation” does not exist. Thus the following result holds (but the proof will be omitted).

8.1.9 THEOREM. *If α is an affine transformation and point P is between points A and B , then point $\alpha(P)$ is between $\alpha(A)$ and $\alpha(B)$.*

As an immediate consequence of THEOREM 8.1.9, we know that *an affine transformation preserves all those geometric entities whose definition goes back only to the definition of betweenness*. Thus, an affine transformation preserves segments, rays, triangles, quadrilaterals, halfplanes, interiors of triangles, etc. In particular, the following result holds :

8.1.10 PROPOSITION. *If A', B', C' are the respective images of three non-collinear points A, B, C under affine transformation α , then*

$$\alpha(\overline{AB}) = \overline{A'B'} \quad \text{and} \quad \alpha(\triangle ABC) = \triangle A'B'C'.$$

8.1.11 PROPOSITION. *An affine transformation fixing two points on a line fixes that line pointwise.*

PROOF : Suppose affine transformation α fixes two points A and B . Assume there is a point C on $\overleftrightarrow{AB}$ such that $C' \neq C$ with $C' = \alpha(C)$. Without loss of generality, we may assume C is on $AB^\rightarrow$. As an intermediate step, we shall show C is between two fixed points A and D . Let $B_0 = B$ and define B_{i+1} so that B_i is the midpoint of A and B_{i+1} for $i = 0, 1, 2, \dots$. Since A and B_0 are given as fixed by α , then each of $B_1, B_2, B_3, \dots$ in turn must be fixed by α since α preserves midpoints. Let $D = B_k$ where k is an integer such that

$$AB_k = 2^k AB > AC.$$

Then C lies between fixed points A and D . So $\overline{AD}$ is then fixed and both C and C' lie in $\overline{AD}$. Now, let n be an integer large enough so that $nCC' > AD$. Let $P_0 = A$, $P_n = D$, and the $n+1$ points $P_0, P_1, \dots, P_n$ divide the segment $\overline{AD}$ into n congruent segments $\overline{P_{i-1}P_i}$. Each of the points P_i is fixed by α by PROPOSITION 8.1.7. So each $\overline{AP_i}$ and $\overline{P_iD}$ is fixed by α . However, integer n was chosen large enough so that for some integer j point P_j is between C and C' . So C and C' are in different fixed segments $\overline{AP_j}$ and $\overline{P_jD}$, contradiction. Therefore, $\alpha(C) = C$ for all points on $\overleftrightarrow{AB}$, as desired.

□

8.1.12 COROLLARY. *An affine transformation fixing three noncollinear points must be the identity. Given $\triangle ABC$ and $\triangle DEF$, there is at most one affine transformation α such that $\alpha(A) = D$, $\alpha(B) = E$, and $\alpha(C) = F$.*

NOTE : In the next section we shall see that there is also at least one affine transformation α as described in the corollary above. Thus *an affine transformation is completely determined once the images of any three noncollinear points are known.*

8.2 Affine Linear Transformations

We start by making an “ad hoc” definition.

8.2.1 DEFINITION. An **affine linear transformation** is any mapping

$$\alpha : \mathbb{E}^2 \rightarrow \mathbb{E}^2, \quad (x, y) \mapsto (ax + by + h, cx + dy + k) \quad \text{where } ad - bc \neq 0.$$

The number $ad - bc$ is called the **determinant** of α .

An affine linear transformation is actually a transformation since a given (x, y) obviously determines a unique (x', y') and, conversely, a given (x', y') determines a unique (x, y) precisely because the determinant is nonzero. As we might expect, affine linear transformations are related to affine transformations.

Exercise 126 If $P = (p_1, p_2)$, $Q = (q_1, q_2)$, and $R = (r_1, r_2)$ are vertices of a triangle, show that the area of $\triangle PQR$ is

$$\frac{1}{2} |(q_1 - p_1)(r_2 - p_2) - (q_2 - p_2)(r_1 - p_1)|.$$

(Hence the area of a triangle with vertices $(0, 0)$, (a, b) , (c, d) is half the absolute value of $ad - bc$.)

8.2.2 PROPOSITION. *An affine linear transformation is an affine transformation and, conversely, an affine transformation is an affine linear transformation.*

PROOF : Let α be an affine linear transformation and suppose line $\mathcal{L}$ has equation $px + qy + r = 0$. Since p and q are not both zero, then $ap + cq$ and $bp + dq$ are not both zero. So there is a line $\mathcal{M}$ with equation

$$(ap + cq)x + (bp + dq)y + r + hp + kq = 0.$$

Line $\mathcal{M}$ is introduced because each of the following implies the next, where $\alpha((x, y)) = (x', y')$:

- (1) (x', y') is on line $\mathcal{L}$.
- (2) $px' + qy' + r = 0$.
- (3) $p(ax + by + h) + q(cx + dy + k) + r = 0$.
- (4) $(ap + cq)x + (bp + dq)y + r + hp + kq = 0$.
- (5) (x, y) is on line $\mathcal{M}$.

We have shown that α^{-1} is a transformation that takes any line $\mathcal{L}$ to some line $\mathcal{M}$. So α^{-1} is a collineation. Hence, α is itself a collineation.

Conversely, suppose α is an affine transformation. Let

$$\alpha((0, 0)) = (p_1, p_2) = P, \quad \alpha((1, 0)) = (q_1, q_2) = Q, \quad \text{and} \quad \alpha((0, 1)) = (r_1, r_2) = R.$$

Since $(0, 0), (1, 0), (0, 1)$ are noncollinear, then P, Q, R are noncollinear. Hence the mapping β with equations

$$\begin{cases} x' = (q_1 - p_1)x + (r_1 - p_1)y + p_1 \\ y' = (q_2 - p_2)x + (r_2 - p_2)y + p_2 \end{cases}$$

is an affine linear transformation, since the absolute value of its determinant is twice the area of $\triangle PQR$ and therefore nonzero (see **Exercise 126**). Further,

$$\beta((0, 0)) = \alpha((0, 0)), \quad \beta((1, 0)) = \alpha((1, 0)), \quad \text{and} \quad \beta((0, 1)) = \alpha((0, 1)).$$

Therefore (COROLLARY 8.1.11), we have $\alpha = \beta$. So α is an affine linear transformation. $\square$

NOTE : Choosing the term *affine linear transformation* over its equivalents *collineation* and *affine transformation* can emphasize a coordinate viewpoint.

Given $\triangle ABC$ and $\triangle DEF$, we know that there is at most one affine transformation α such that $\alpha(A) = D$, $\alpha(B) = E$, and $\alpha(C) = F$. We can now show that there is at least one such transformation α .

8.2.3 PROPOSITION. *Given $\triangle ABC$ and $\triangle DEF$, there is a unique affine transformation α such that*

$$\alpha(A) = D, \quad \alpha(B) = E, \quad \text{and} \quad \alpha(C) = F.$$

PROOF : Given $\triangle ABC$ and $\triangle DEF$, we know (COROLLARY 8.1.12) there is at most one affine transformation α such that $\alpha(A) = D$, $\alpha(B) = E$ and $\alpha(C) = F$. We now show there is at least one such affine transformation α . From the preceding paragraph, we see how to find the equations for an affine linear transformation β_1 such that

$$\beta_1((0, 0)) = A, \quad \beta_1((1, 0)) = B, \quad \text{and} \quad \beta_1((0, 1)) = C.$$

Repeating the process, we see there is an affine linear transformation β_2 such that

$$\beta_2((0, 0)) = D, \quad \beta_2((1, 0)) = E, \quad \text{and} \quad \beta_2((0, 1)) = F.$$

The transformation $\beta_2\beta_1^{-1}$ is the desired affine transformation α that takes points A, B, C to points D, E, F , respectively. $\square$

Matrix representation

Let $\alpha : \mathbb{E}^2 \rightarrow \mathbb{E}^2$ be a transformation given by

$$(x, y) \mapsto (ax + by + h, cx + dy + k).$$

(α is an affine linear transformation.)

NOTE : Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

Hence the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ defines a mapping $(x, y) \mapsto (ax + by, cx + dy)$. Indeed, we write the pair (x, y) as a column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ (in fact, we identify points with geometric vectors) and so we get

$$(x, y) = \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = (ax + by, cx + dy).$$

This mapping is *linear* (i.e. preserves the vector structure of $\mathbb{E}^2$) and is invertible if (and only if) the matrix is invertible.

When the coefficients h and k vanish, α is linear and hence admits a matrix representation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

We say that the (invertible) matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *represents* the (linear) transformation α . In order to extend this representation to the general case, of affine linear transformations, we need to accommodate translations.

Exercise 127

(a) Verify that

$$\begin{bmatrix} 1 & 0 & 0 \\ h & a & b \\ k & c & d \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ ax + by + h \\ cx + dy + k \end{bmatrix}.$$

(b) Show that the matrix $\begin{bmatrix} 1 & 0 & 0 \\ h & a & b \\ k & c & d \end{bmatrix}$ is *invertible* if and only if $ad - bc \neq 0$, and then find its inverse.

If we “redefine” the concept of point – and write the pair (x, y) as a column matrix $\begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$ (this identification is more than just a “clever” notation) – then

we have

$$(x, y) = \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ h & a & b \\ k & c & d \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ ax + by + h \\ cx + dy + k \end{bmatrix} = (ax + by + h, cx + dy + k).$$

We see that the 3×3 matrix

$$[\alpha] = \begin{bmatrix} 1 & 0 & 0 \\ h & a & b \\ k & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathbf{v} & A \end{bmatrix}$$

(where $\mathbf{v} = \begin{bmatrix} h \\ k \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$) represents the transformation

$$\alpha : \mathbb{E}^2 \rightarrow \mathbb{E}^2, \quad (x, y) \mapsto (ax + by + h, cx + dy + k).$$

Exercise 128 Use matrix representation to show that the set of all linear affine transformations forms a group. (This group consists of all collineations, and is usually denoted by $\mathfrak{Aff}$.)

8.2.4 EXAMPLE. The *identity transformation* ι is represented by the ma-

trix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Thus

$$[\iota] = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}.$$

8.2.5 EXAMPLE. Consider the point $P = (h, k)$ and let $\mathbf{v} = \begin{bmatrix} h \\ k \end{bmatrix}$. The

translation $\tau = \tau_{O,P}$ is represented by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$. Thus

$$[\tau] = \begin{bmatrix} 1 & 0 \\ \mathbf{v} & I \end{bmatrix}.$$

8.2.6 EXAMPLE. Again, consider the point $P = (h, k)$. The *halfturn* $\sigma =$

σ_P is represented by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 2h & -1 & 0 \\ 2k & 0 & -1 \end{bmatrix}$. Thus

$$[\sigma] = \begin{bmatrix} 1 & 0 \\ 2\mathbf{v} & -I \end{bmatrix}.$$

Exercise 129 Let $P = (h, k)$ be a point. Determine the matrix which represents the dilation $\delta_{P,r}$ (of ratio $r \neq 0$) and hence verify the relations :

- (a) $\delta_{P,-r} = \sigma_P \delta_{P,r}$.
- (b) $\delta_{P,1} = \iota$.
- (c) $\delta_{P,-1} = \sigma_P$.
- (d) $\delta_{P,s} \delta_{P,r} = \delta_{P,rs}$ ($r, s \neq 0$).

Strains and shears

Some specific, basic affine transformations are introduced next.

8.2.7 DEFINITION. For number $k \neq 0$, the affine transformation

$$\varepsilon_{\mathcal{X},k} : (x, y) \mapsto (x, ky)$$

is called a **strain of ratio k about the x -axis**.

8.2.8 DEFINITION. For number $k \neq 0$, the affine transformation

$$\varepsilon_{\mathcal{Y},k} : (x, y) \mapsto (kx, y)$$

is called a **strain of ratio k about the y -axis**.

For fixed k , the product of the two affine transformations above is the familiar dilation about the origin $(x, y) \mapsto (kx, ky)$. Thus

$$\varepsilon_{\mathcal{X},k} \varepsilon_{\mathcal{Y},k} = \delta_{O,k}.$$

NOTE : The concept of a strain of ratio k about a given line $\mathcal{L}$ can be defined analogously. However, one can prove that *any dilation is the product of two strains about perpendicular lines*.

8.2.9 EXAMPLE. The strain with equations

$$\begin{cases} x' = 2x \\ y' = y \end{cases}$$

fixes the y -axis pointwise and stretches out the plane away from and perpendicular to the y -axis.

NOTE : As with similarity theory, the terminology here is *not* standardized. Each of the following words has been used for a strain or for a strain with positive ratio : enlargement, expansion, lengthening, stretch, compression.

8.2.10 DEFINITION. For number $k \neq 0$, the affine transformation

$$\zeta_{\mathcal{X},k} : (x, y) \mapsto (x + ky, y)$$

is called a **shear along the x -axis**.

Here the x -axis is fixed pointwise and every point is moved “horizontally” a directed distance proportional to its directed distance from the x -axis. We shall see below that *a shear has the property of preserving area*.

8.2.11 DEFINITION. An affine transformation that preserves area is said to be **equiaffine**.

8.2.12 PROPOSITION. *An affine transformation is the product of a shear, a strain, and a similarity.*

PROOF : We can see that the general affine linear transformation with equations

$$\begin{cases} x' = ax + by + h \\ y' = cx + dy + k \end{cases} \quad \text{with } ad - bc \neq 0$$

can be factored into the *similarity* with equations

$$\begin{cases} x' = ax - cy + h \\ y' = cx + ay + k \end{cases}$$

following the *strain* with equations

$$\begin{cases} x' = x \\ y' = \frac{ad - bc}{a^2 + c^2}y \end{cases}$$

following the *shear* with equations

$$\begin{cases} x' = x + \frac{ab + cd}{a^2 + c^2}y \\ y' = y. \end{cases}$$

Indeed, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ h & a & b \\ k & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ h & a & -c \\ k & c & a \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{ad-bc}{a^2+c^2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{ab+cd}{a^2+c^2} \\ 0 & 0 & 1 \end{bmatrix}.$$

□

8.2.13 PROPOSITION. *An affine transformation is a product of strains.*

PROOF : First, we see that the shear $(\zeta_{\mathcal{X},1})$ with equations

$$\begin{cases} x' = x + y \\ y' = y \end{cases}$$

can be factored into the *similarity* with equations

$$\begin{cases} x' = \frac{5-\sqrt{5}}{20}x + \frac{5-3\sqrt{5}}{20}y \\ y' = \frac{-5+3\sqrt{5}}{20}x + \frac{5-\sqrt{5}}{20}y \end{cases}$$

following the *strain* with equations

$$\begin{cases} x' = \frac{3+\sqrt{5}}{2}x \\ y' = y \end{cases}$$

following the *similarity* with equations

$$\begin{cases} x' = 2x + (1 + \sqrt{5})y \\ y' = -(1 + \sqrt{5})x + 2y. \end{cases}$$

Indeed, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{5-\sqrt{5}}{20} & \frac{5-3\sqrt{5}}{20} \\ 0 & \frac{-5+3\sqrt{5}}{20} & \frac{5-\sqrt{5}}{20} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3+\sqrt{5}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 + \sqrt{5} \\ 0 & -(1 + \sqrt{5}) & 2 \end{bmatrix}.$$

Secondly, we see that the nonidentity shear ($\zeta_{\mathcal{X},k}$, $k \neq 0$) with equations

$$\begin{cases} x' = x + ky \\ y' = y \end{cases}$$

can be factored into the *strain* of ratio k about the y -axis ($\varepsilon_{\mathcal{Y},k} : (x, y) \mapsto (kx, y)$) following the *shear* that just factored above following the *strain* of ratio $\frac{1}{k}$ about the y -axis ($\varepsilon_{\mathcal{Y},\frac{1}{k}} : (x, y) \mapsto (\frac{1}{k}x, y)$). The relation

$$\zeta_{\mathcal{X},k} = \varepsilon_{\mathcal{Y},k} \zeta_{\mathcal{X},1} \varepsilon_{\mathcal{Y},\frac{1}{k}}$$

holds since

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{k} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Putting these results together with PROPOSITION 8.2.13, we see that *an affine transformation is a product of strains and similarities*. Since a similarity

is an isometry following a dilation about the origin (PROPOSITION 7.1.12), and since a dilation about the origin is a product of two strains, then an affine transformation is a product of strains and isometries. However, isometries are products of reflections, which are special cases of strains. Thus affine transformations are products of strains.

□

NOTE : One can also prove that *an affine transformation is the product of a strain and a similarity.*

8.2.14 THEOREM. Suppose affine transformation α has equations

$$\begin{cases} x' = ax + by + h \\ y' = cx + dy + k \end{cases} \quad \text{with } ad - bc \neq 0.$$

Transformation α is equiaffine if and only if

$$|ad - bc| = 1.$$

Transformation α is a similarity (of ratio r) if and only if

$$a^2 + c^2 = b^2 + d^2 = r^2 \quad \text{and} \quad ab + cd = 0.$$

Transformation α is an isometry if and only if

$$a^2 + c^2 = b^2 + d^2 = 1 \quad \text{and} \quad ab + cd = 0.$$

PROOF : Suppose

$$\begin{cases} x' = ax + by + h \\ y' = cx + dy + k \end{cases}$$

are equations for affine transformation α . So the determinant $ad - bc$ of α is nonzero. What are the necessary and sufficient conditions for α to be equiaffine? In other words, when is area preserved by α ? Suppose P, Q, R are noncollinear points with

$$P = (p_1, p_2), \quad Q = (q_1, q_2), \quad R = (r_1, r_2),$$

$$P' = \alpha(P) = (p'_1, p'_2), \quad Q' = \alpha(Q) = (q'_1, q'_2), \quad R' = \alpha(R) = (r'_1, r'_2).$$

Recall that the area PQR of $\triangle PQR$ is given by

$$PQR = \pm \frac{1}{2} |(q_1 - p_1)(r_2 - p_2) - (q_2 - p_2)(r_1 - p_1)|$$

and similarly the area $P'Q'R'$ of $\triangle P'Q'R'$ is given by

$$P'Q'R' = \pm \frac{1}{2} |(q'_1 - p'_1)(r'_2 - p'_2) - (q'_2 - p'_2)(r'_1 - p'_1)|.$$

Substitution shows that

$$P'Q'R' = \pm(ad - bc)PQR.$$

Thus, *under an affine transformation with determinant t , area is multiplied by $\pm t$* . This result answers our question about preserving area : area is preserved by α when the determinant of α is ± 1 .

Continuing with the same notation for affine transformation α , we recall that α is a similarity if and only if there is a positive number r such that

$$P'Q' = rPQ \quad \text{for all points } P \text{ and } Q.$$

With substitution, this equation becomes

$$\begin{aligned} & \sqrt{(a^2 + c^2)(q_1 - p_1)^2 + (b^2 + d^2)(q_2 - p_2)^2 + 2(ab + cd)(q_1 - p_1)(q_2 - p_2)} = \\ & r \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}. \end{aligned}$$

This equation can hold for all p_1, p_2, q_1, q_2 if and only if

$$a^2 + c^2 = b^2 + d^2 = r^2 \quad \text{and} \quad ab + cd = 0.$$

Since a similarity of ratio r is an isometry if and only if $r = 1$, we obtain the last result. $\square$

NOTE : The matrix representing the given affine transformation α is

$$[\alpha] = \begin{bmatrix} 1 & 0 \\ \mathbf{v} & A \end{bmatrix}$$

where $\mathbf{v}$ is arbitrary and A is invertible (i.e. $ad - bc \neq 0$).

Transformation α is *equiaffine* if and only if $\det A = \pm 1$.

Transformation α is a *similarity* (of ratio r) if and only if $AA^T = r^2 I$.

Transformation α is an *isometry* if and only if $AA^T = I$ (such a matrix is called *orthogonal*).

8.3 Exercises

Exercise 130

- (a) For a given nonzero number k , find all fixed points and fixed lines for the affine transformations α_k and β_k with respective equations

$$\left\{ \begin{array}{l} x' = kx \\ y' = y \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} x' = x + ky \\ y' = y. \end{array} \right.$$

- (b) If $P = (-2, -1)$, $Q = (1, 2)$, and $R = (3, -6)$, what is the area of $\triangle PQR$?
- (c) What are the areas of the images of $\triangle PQR$ under the collineations α_k and β_k , respectively ?

Exercise 131 TRUE or FALSE ?

- (a) An affine transformation is a collineation; a collineation is an affine linear transformation; and an affine linear transformation is an affine transformation.
- (b) An affine transformation is determined once the images of three given points are known.
- (c) Strains and shears are equiaffine.
- (d) A shear is a product of strains and similarities.
- (e) A collineation is a product of strains and similarities.
- (f) A collineation is a product of strains and isometries.
- (g) A dilatation is a product of strains; a strain is a product of dilatations.

Exercise 132 Given nonzero number k and line $\mathcal{L}$, give a definition for the *strain* of ratio k about line $\mathcal{L}$. Using your definition, show that a dilation is a product of two strains.

Exercise 133 If $x' = ax + by + h$ and $y' = cx + dy + k$ are the equations of (affine linear) transformation α , find the equations of its inverse α^{-1} . Hence determine the matrices $[\alpha]$ and $[\alpha^{-1}]$ representing α and α^{-1} , respectively.

Exercise 134 PROVE or DISPROVE : If affine linear transformation α has determinant t , then α^{-1} has determinant t^{-1} .

Exercise 135 Suppose any affine transformation is the product of a strain and a similarity. Then show that an affine transformation is a product of two strains about perpendicular lines and an isometry. (To see that the perpendicular lines cannot be chosen arbitrarily, see the next exercise.)

Exercise 136 Show the shear with equations $x' = x + y$ and $y' = y$ is not the product of strains about the coordinate axes followed by an isometry.

Exercise 137 Show that the shears do not form a group.

Exercise 138 PROVE or DISPROVE : An equiaffine similarity is an isometry.

Exercise 139 PROVE or DISPROVE : An involutory affine transformation is a reflection or a halfturn.

Exercise 140 Give an example of an equiaffine transformation that is neither an isometry nor a shear.

DISCUSSION :

Perhaps the most fundamental concept of the earlier books of EUCLID's *Elements* is that of *congruence*. Intuitively, two plane geometrical figures (i.e. arbitrary subsets of the plane) are congruent if they differ only in the position they occupy in the plane; that is, if they can be made to coincide by the application of some "rigid motion" in the plane. Somewhat more precisely, two figures $\mathbf{F}_1$ and $\mathbf{F}_2$ are said to be congruent if there is a mapping α of the plane onto itself that leaves invariant the distance between each pair of points (i.e. $\alpha(\mathbf{F}_1) = \mathbf{F}_2$ and

$\alpha(P)\alpha(Q) = PQ$ for all P and Q). A mapping that preserves the distance between any pair of points is called an *isometry* and is the mathematical analog of a rigid motion; the study of congruent figures in the plane is, for this reason, often referred to as *plane Euclidean metric geometry*. If we construct an orthogonal Cartesian coordinate system in the plane, we can show that the isometries of the plane are precisely the mappings $(x, y) \rightarrow (x', y')$, where

$$\begin{cases} x' = ax - by + h \\ y' = \pm (bx + ay) + k \end{cases} \quad \text{with } a^2 + b^2 = 1.$$

Observe that the product (composition) of any two isometries is again an isometry and that each isometry has an inverse that is again an isometry. Now, any collection of invertible mappings of a set $\mathbf{S}$ onto itself that is closed under the formation of compositions and inverses is called a *group of transformations* on $\mathbf{S}$; the collection of all isometries is therefore referred to as the *group of plane isometries*. From the point of view of plane Euclidean metric geometry the only properties of a geometric figure $\mathbf{F}$ that are of interest are those that are possessed by all figures congruent to $\mathbf{F}$; that is, those properties that are invariant under the group of plane isometries. Since any isometry carries lines onto lines, the property of being a line is one such property. Similarly, the property of being a square or, more generally, a polygon of a particular type is invariant under the group of plane isometries, as is the property of being a conic of a particular type. The length of a line segment, the area of a polygon, and the eccentricity of a conic are likewise all invariants and thus legitimate objects of study in plane Euclidean geometry.

Of course, the point of view of plane Euclidean metric geometry is not the only point of view. Indeed, in Book VI of the *Elements* itself, emphasis shifts from congruent to similar figures. Roughly speaking, two geometric figures are *similar* if they have the same shape, but not necessarily the same size. In order to formulate a more precise definition, let us refer to a map α of the plane onto itself under which each distance is multiplied by the same positive constant r (i.e. $\alpha(P)\alpha(Q) = rPQ$ for all P and Q) as a *similarity transformation* with similarity ratio r . It can be shown that, relative to an orthogonal Cartesian coordinate system, each such map has the

form

$$\begin{cases} x' = ax - by + h \\ y' = \pm (bx + ay) + k \end{cases} \quad \text{where } a^2 + b^2 = r^2.$$

Two plane geometric figures $\mathbf{F}_1$ and $\mathbf{F}_2$ are then said to be similar if there exists a similarity transformation that carries $\mathbf{F}_1$ onto $\mathbf{F}_2$. Again, the set of all similarity transformations is easily seen to be a transformation group, and we might reasonably define *plane Euclidean similarity geometry* as the study of those properties of geometric figures that are invariant under this group; that is, those properties that, if possessed by some figure, are necessarily possessed by all similar figures. Since any isometry is also a similarity transformation (with $r = 1$), any such property is necessarily an invariant of the group of plane isometries; but the converse is false since, for example, the length of a line segment and area of a polygon are not preserved by all similarity transformations.

At this point it is important to observe that in each of the two geometries discussed thus far, certain properties of geometric figures were of interest while others were not. In plane Euclidean metric geometry we were interested in the shape and size of a given figure, but not in its position or orientation in the plane, while similarity geometry concerns itself with only the shape of the figure. Those properties that we deem important depend entirely on the particular sort of investigation we choose to carry out. Similarity transformations are, of course, capable of “distorting” geometric figures more than isometries, but this additional distortion causes no concern as long as we are interested only in properties that are not effected by such distortions. In other sorts of studies the permissible degree of distortion may be even greater. For example, in the mathematical analysis of perspective it was found that the “interesting” properties of a geometric figure are those that are invariant under a class of maps called *plane projective transformations*, each of which can be represented, relative to an orthogonal Cartesian coordinate system, in the following form

$$\begin{cases} x' = \frac{a_1x + a_2y + a_3}{c_1x + c_2y + c_3} \\ y' = \frac{b_1x + b_2y + b_3}{c_1x + c_2y + c_3} \end{cases} \quad \text{where} \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0.$$

The collection of all such maps can be shown to form a transformation group, and we define *plane projective geometry* as the study of those properties of geometric

figures that are invariant under this group. Two figures are said to be “projectively equivalent” if there is a projective transformation that carries one onto the other. Since any similarity transformation is also a projective transformation, any invariant of the projective group is also an invariant of the similarity group. The converse, however, is false since projective transformations are capable of greater distortions of geometric figures than are similarities. For example, two conics are always projectively equivalent, but they are similar only if they have the same eccentricity.

Needless to say, the approach we have taken here to these various geometrical studies is of a relatively recent vintage. Indeed, it was FELIX KLEIN (1849-1925) – in his famous *Erlanger Programm* of 1872 – who first proposed that a “geometry” be defined quite generally as the *study of those properties of a set $\mathbf{S}$ that are invariant under some specified group of transformations of $\mathbf{S}$* . Plane Euclidean metric, similarity, and projective geometries and their obvious generalizations to three and higher dimensional spaces all fit quite nicely into Klein’s scheme, as did various other offshoots of classical Euclidean geometry known at the time. Despite the fact that, during this century, our conception of geometry has expanded still further and now includes studies that cannot properly be considered “geometries” in the Kleinian sense, the influence of the ideas expounded in the *Erlangen Program* has been great indeed. Even in theoretical physics KLEIN’s emphasis on the study of invariants of transformation groups has had a profound impact. The *special theory of relativity*, for example, is perhaps best regarded as the invariant theory of the so-called *Lorentz group of transformations* on Minkowski space.

Based on his appreciation of the importance of BERNHARD RIEMANN’s work in *complex function theory*, KLEIN was also able to anticipate the rise of a new branch of geometry that would concern itself with those properties of a geometric figure that remain invariant when the figure is bent, stretched, shrunk or deformed in any way that does not create new points or fuse existing points. Such a deformation is accomplished by any bijective mapping that, roughly speaking, “sends nearby points to nearby points”; that is, a *continuous* one. In dimension two, then, the relevant group of transformations is the collection of all one-to-one mappings of the plane onto itself that are continuous and have continuous inverse; such transformations are called *homeomorphisms* (or *topological transformations*) of the plane. What sort of properties of a plane geometric figure are preserved by homeomorphisms? Certainly, the property of being a line is not. Topological transformations are clearly capa-

ble of a very great deal of distortion. Indeed, virtually all of the properties you are accustomed to associating with plane geometric figures are destroyed by such transformations. Nevertheless, homeomorphisms do preserve a number of very important, albeit less obvious properties. For example, although a line need not be mapped onto another line, its image must be “one-dimensional” and consist of one “connected” piece. Properties of plane geometric figures such as these that are invariant under the group of topological transformations of the plane are called *topological properties*.

During the past one hundred years topology has outgrown its geometrical origins and today stands alongside analysis and algebra as one of the most fundamental branches of mathematics.

Geometry is not true, it is advantageous.

HENRI POINCARÉ

Appendix A

Answers and Hints to Selected Exercises

Geometric transformations

1. (M_1) and (M_2) are immediate, but (M_3) requires some work. (For a “clever” solution, you may think of the *dot product* of two points (vectors) in $\mathbb{R}^2$).
2. TRUE. Find the equation of the line.
3. The lines are *parallel* if and only if their direction vectors are collinear, and are *perpendicular* if and only if their direction vectors are orthogonal. Thus

$$\mathcal{L} \parallel \mathcal{M} \iff \begin{bmatrix} -b \\ a \end{bmatrix} = r \begin{bmatrix} -e \\ d \end{bmatrix} \quad \text{for some } r \in \mathbb{R} \setminus \{0\} \iff ae - bd = 0$$

and

$$\mathcal{L} \perp \mathcal{M} \iff \begin{bmatrix} -b \\ a \end{bmatrix} \bullet \begin{bmatrix} -e \\ d \end{bmatrix} = 0 \iff ad + be = 0.$$

4. TRUE. Find the equation of the line.
5. The line passing through P_2 and P_3 has equation

$$\begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix} = 0.$$

6. The set *cannot* be finite.
7. The mapping is invertible. (One can solve uniquely for x and y in terms of x' and y').
8. TRUE.

9. Recall (and use) the fact that a line is determined by two points.
 10. Yes. (The mapping is invertible and coincides with its inverse).
 11. Relation $PQ + QR = PR$ (equality in the triangle inequality) implies

$$Q - P = s(R - Q) \quad \text{for some } s > 0.$$

Conversely, we have $PQ = tPR$, $QR = (1 - t)PR$, etc.

12. (a), (d), (f), (h), (i).
 13. (d) $3ax + 2by + 6c = 0$; (f) $bx + 3ay + 3(c - 2a) = 0$; (h) $ax + by - c = 0$; (i) $ax + by + (c - 2a + 3b) = 0$.
 14. (a) $y = -5x + 7$; (b) $y = -5x - 7$; (c) $y = 5x - 7$; (d) $x - 9y - 32 = 0$.
 15. $TTTT \quad TTFT \quad TF$.
 16. For instance, examples (8), (9), and (10) from **1.2.2**. (Find other examples.)
 17. $x - 10y - 2 = 0$.
 18. The necessary and sufficient condition for α to be a transformation is $ad - bc \neq 0$. Such a transformation is *always* a collineation.
 19. Straightforward verification.
 20. (a) $(x, y) \mapsto (x, y) + x(0, 1)$ is a shear (about the y -axis); the image of the unit square is a parallelogram. (b) $(x, y) \mapsto (y, x)$ is a reflection (in the angle bisector of the first quadrant); the image of the unit square is also the unit square. (c) $(x, y) \mapsto (x, y) + x^2(0, 1)$ is a generalized shear; the image of the unit square is a curvilinear quadrilateral (with two sides line segments). (d) $(x, y) \mapsto (x, \frac{y}{2}) \mapsto (x, -x + \frac{y}{2}) \mapsto (-x + \frac{y}{2}, x) \mapsto (-x + \frac{y}{2}, x + 2)$ is a product of transformations (strain + shear + reflection + translation); the image of the unit square is a parallelogram. (The decomposition is not unique. Find other decompositions, for instance : strain + shear + rotation + reflection.)
 21. (a) $\beta\alpha = \gamma\alpha \Rightarrow \beta\alpha(\alpha^{-1}) = \gamma\alpha(\alpha^{-1}) \Rightarrow \beta(\alpha\alpha^{-1}) = \gamma(\alpha\alpha^{-1}) \Rightarrow \beta\iota = \gamma\iota \Rightarrow \beta = \gamma$. In particular, for $\gamma = \iota$, one has (c) $\beta\alpha = \alpha \Rightarrow \beta = \iota$. The parts (b), (d), and (e) can be proved analogously.
 22. $TTTT \quad FF$.
 23. True. (The group generated by the rotation of 1 rad is an infinite cyclic group.)
 24. $TTFF$.
 25. $a = b \in \mathbb{R} \setminus \{0\}$.

Translations and halfturns

26. Express condition (4) in coordinates : $(x_B - x_A)^2 + (y_B - y_A)^2 = (x_D - x_C)^2 + (y_D - y_C)^2$ and $\frac{y_B - y_A}{y_D - y_C} = \frac{x_B - x_A}{x_D - x_C} = t > 0$, etc.
27. If the points A, B , and C are collinear, then the parallelogram $\square CABD$ becomes a “degenerate” one. (What is a *degenerate* parallelogram ?)
28. A translation : τ^{-1} .
29. The LHS is a product of five halfturns that fixes Q .
30. $FTTT \quad TTTF \quad TT$.
31.
 - The halfturn σ_A , where $A = (\frac{3}{2}, -4)$.
 - $x' = x + a - g$ and $y' = y + c - h$.
32. TRUE.
33. $x' = x + a_5$ and $y' = y + b_5$.
34. $5x - y + 27 = 0$.
35.
 - $\sigma_M \alpha \sigma_P(P) = P \Rightarrow \alpha \sigma_P = \sigma_M$.
 - $\sigma_P \alpha = \sigma_N$ (P is the midpoint of M and N).
36. $\sigma_P(\mathcal{L})$ is the line with equation $y = 5x - 21$.
37. For $n \in \mathbb{Z} \setminus \{0\}$, $\tau_{P,Q}^n \neq \iota$.
38. $\tau_{P,Q} \in \langle \tau_{R,S} \rangle \Rightarrow \exists m \in \mathbb{Z} : \tau_{P,Q} = \tau_{R,S}^m = \tau_{S,R}^{-m}$.
39. TRUE.
40. (a) $X = (0, -1)$; (b) $Y = (0, \frac{1}{2})$; (c) $Z = (0, -2)$.

Reflections and rotations

41. If a product $\alpha_2 \alpha_1$ is invertible, then α_1 is one-to-one and α_2 is onto.
42. Show that certain angles are supplementary.
43. Yes.
45. A line is uniquely determined by two (distinct) points.
46. $TFFF \quad TF$.
47. The given reflection has the equations $x' = \frac{1}{5}(-3x + 4y) + 4$, $y' = \frac{1}{5}(4x + 3y) - 2$.
 $(0, 0) \mapsto (4, -2)$, $(1, -3) \mapsto (1, -3)$, $(-2, 1) \mapsto (6, -3)$, $(2, 4) \mapsto (6, 2)$.
48. Reflection in the line through O and orthogonal to $\overleftrightarrow{OO'}$.
49. (a) FALSE. (Notice that the statement “ $\sigma_{\mathcal{L}} \sigma_{\mathcal{M}} = \sigma_{\mathcal{M}} \sigma_{\mathcal{L}} \iff \mathcal{L} = \mathcal{M}$ or $\mathcal{L} \perp \mathcal{M}$ ” is TRUE.) (b) TRUE.

50. FALSE. (Find a counterexample.)

Isometries I

52. No. (Find a simple counterexample.)

53. $2x + y = 5$ and $4x - 3y = 10$.

54. $x' = -x + 4$, $y' = -y + 6$ (halfturn) and $x' = x$, $y' = y + 4$ (translation).

55. TRUE. $\sigma_{\mathcal{L}}\sigma_{\mathcal{L}} = \iota$

56. FALSE. (Consider an equilateral triangle.)

57. $TTTF \quad TFFF \quad F$.

58. If $Q = \sigma_{\mathcal{M}}(P) = \sigma_{\mathcal{N}}(P) \neq P$ then $\mathcal{M}$ and $\mathcal{N}$ are both perpendicular bisectors of $\overline{PQ}$, contradiction. Hence, $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}(P) = P \Rightarrow P \in \mathcal{M} \cap \mathcal{N}$.

59. TRUE.

61. $x' = x + 4$, $y' = y + 2$.

62. $\sigma_{\mathcal{L}}\rho_{C,r}\sigma_{\mathcal{L}} = \sigma_{\mathcal{L}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{L}}\sigma_{\mathcal{M}} = \rho_{C,-r}$.

63. $TFTT \quad FTFT \quad TT$.

64. Consider the perpendicular bisector of $\overline{AC}$. There are two cases; in each case construct the desired rotation.

65. TRUE. $\tau = \sigma_{\mathcal{N}}\sigma_{\mathcal{M}} = (\sigma_{\mathcal{N}}\sigma_{\mathcal{L}})(\sigma_{\mathcal{L}}\sigma_{\mathcal{M}})$.

66. TRUE.

67. Let $C \in \mathcal{L}$. If $\rho_{C,r}(\mathcal{L}) = \mathcal{L}$, then let $\mathcal{M} \perp \mathcal{L}$ and $\rho_{C,r} = \sigma_{\mathcal{N}}\sigma_{\mathcal{M}}$. We have

$$\mathcal{L} = \rho_{C,r}(\mathcal{L}) = \sigma_{\mathcal{N}}(\mathcal{L}) \Rightarrow (1) \mathcal{L} = \mathcal{N} \quad \text{or} \quad (2) \mathcal{L} \perp \mathcal{N}.$$

In the first case, $\rho_{C,r}$ is a halfturn (and so any line through C is fixed), whereas in the second case, $\rho_{C,r}$ is the identity transformation (and so any line is fixed).

68. $(\mathcal{M}) \quad 2x - y + c = 0$ and $(\mathcal{N}) \quad 4x - 2y + 2c - 15 = 0$ (parallel lines).

69. If the lines are neither concurrent nor parallel, then $\sigma_{\mathcal{C}}\sigma_{\mathcal{B}}\sigma_{\mathcal{A}}$ is the product of a (nonidentity) rotation and a reflection (in whatever order). Such a product *cannot* be a reflection : assume the contrary and derive a contradiction.

70. $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = (\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}})^{-1} = \sigma_{\mathcal{L}}\sigma_{\mathcal{M}}\sigma_{\mathcal{N}}$.

Isometries II

71. There are several cases.

72. $(\alpha\beta\alpha^{-1})^2 = \iota \iff \alpha\beta^2\alpha^{-1} = \iota \iff \beta^2 = \iota$.

73. Consider the identity $\alpha\rho_{C,r}\alpha^{-1} = \rho_{\alpha(C),\pm r}$ (where α is *any* isometry).
74. Any translation is a product of two (special) rotations.
75. A translation fixing line $\mathcal{C}$ commutes with (the reflection) $\sigma_{\mathcal{C}}$.
76. $TTTT \quad FT$.
77. TRUE.
78. Let $\rho_1 = \rho_{C_1,r} = \sigma_{\mathcal{C}}\sigma_{\mathcal{A}}$ with $\{C_1\} = \mathcal{A} \cap \mathcal{C}$, and $\rho_2 = \rho_{C_2,s} = \sigma_{\mathcal{B}}\sigma_{\mathcal{C}}$ with $\{C_2\} = \mathcal{B} \cap \mathcal{C}$. Then

- $\rho_2\rho_1 = \sigma_{\mathcal{B}}\sigma_{\mathcal{A}} = \rho_{C,r+s}$ and
- $\rho_2^{-1}\rho_1 = \sigma_{\mathcal{B}'}\sigma_{\mathcal{A}} = \rho_{C',r-s} \quad (r \neq s)$.

The points C_1 , C , and C' are collinear : they all lie on the line $\mathcal{A}$.

79. Let $\tau = \tau_{A,B}$ (that is, $\tau(A) = B$). Then

$$\tau\sigma_{\mathcal{C}} = \sigma_{\mathcal{C}}\tau \iff \sigma_{\mathcal{C}}\tau\sigma_{\mathcal{C}}^{-1} = \tau \iff \tau_{\sigma_{\mathcal{C}}(A),\sigma_{\mathcal{C}}(B)} = \tau_{A,B} \iff \overleftrightarrow{AB} \parallel \mathcal{C} \iff \tau(\mathcal{C}) = \mathcal{C}.$$

80. $FFFT \quad FF$.

81. TRUE. Let $\gamma = \sigma_{\mathcal{C}}\tau_{A,B}$. We can choose $A, B \in \mathcal{C}$ such that the given point M is the midpoint of the segment $\overline{AB}$.

82. TRUE. $\gamma = \sigma_P\sigma_{\mathcal{L}} = \sigma_N\sigma_M\sigma_{\mathcal{L}}$.

83. Translations and halfturns are dilatations. Reflections are not. Neither are glide reflections : if $\sigma_{\mathcal{L}}\sigma_P$ where a dilatation δ , then $\sigma_{\mathcal{L}}$ would be the dilatation $\delta\sigma_P$.

84. Let $\tau = \tau_{A,B}$ (that is, $\tau(A) = B$). Then, for the glide reflection $\gamma = \sigma_{\mathcal{L}}\sigma_A$,

$$\gamma^2 = \sigma_{\mathcal{L}}\sigma_A\sigma_{\mathcal{L}}\sigma_A = (\sigma_{\mathcal{L}}\sigma_A\sigma_{\mathcal{L}}^{-1})\sigma_A = \sigma_{\sigma_{\mathcal{L}}(A)}\sigma_A = \sigma_M\sigma_A = \tau_{A,B} = \tau.$$

85. • $\rho_{O,90} : \quad x' = -y \text{ and } y' = x.$
 • $\rho_{O,180} : \quad x' = -x \text{ and } y' = -y.$
 • $\rho_{O,270} : \quad x' = y \text{ and } y' = -x.$

86. Since α is an odd isometry, we have that α is a reflection $\iff \alpha = \alpha^{-1}$. The equations for α^{-1} are $x' = ax + by - (ah + bk)$ and $y' = bx - ay + ak - bh$.

87. $TFTT \quad TTT$.

88. $r = 150$. Let $P = (u, v)$. Then $1 = (1 - \cos r)u + (\sin r)v$ and $-\frac{1}{2} = (1 - \cos r)v - (\sin r)u$ imply $u = \frac{4-\sqrt{3}}{4}$ and $v = \frac{3-2\sqrt{3}}{4}$.

89. C is the *only* point fixed by $\rho_{C,r}$ with $r \neq 0$. The coordinates of C are

$$x_C = \frac{(1 - \cos r)h - (\sin r)k}{2(1 - \cos r)} \quad \text{and} \quad y_C = \frac{(1 - \cos r)k + (\sin r)h}{2(1 - \cos r)}.$$

90. $\mathcal{L}$ is the *only* line fixed by $\sigma_{\mathcal{L}}$. An equation for $\mathcal{L}$ is $(a-1)x + by + h = 0$ or, equivalently, $bx - (a+1)y + k = 0$.

Symmetry

92. (a) Yes. (b) No.
93. A *bounded* figure cannot have two points of symmetry, because if it had, it would be invariant under a nonidentity translation.
94. (a) $\mathfrak{C}_1$. (b) $\mathfrak{D}_1$.
95. (a) $\mathfrak{C}_2$ or $\mathfrak{D}_2$.
97. $\mathfrak{D}_2$.
98. $\mathfrak{D}_2$.
99. $FTTF$.
100. $\mathfrak{D}_3$.
103. There are *ten* equivalence classes :
- **A, M, T, U, V, W.**
 - **B, C, D, E, K.**
 - **F, G, J, P, R.**
 - **H, I.**
 - **L.**
 - **N, S, Z.**
 - **O.**
 - **Q.**
 - **X.**
 - **Y.**

104. (c) Use the relation $\sigma\rho^k = \rho^{-k}\sigma$ to show that $\sigma\rho^{n-1} \neq \rho^{n-1}\sigma$.
105. The polygons cannot be regular. (Find more than one example in each case.)

Similarities

107. The inverse of a similarity is also a similarity.
109. The function is invertible.
112. In each case we have

$$\bullet \quad C = rA + (1-r)B = B + r(A-B) \in \overleftrightarrow{AB}, \quad C \neq B.$$

- $C = \frac{s(1-r)}{1-rs}A + \frac{1-s}{1-rs}B \in \overleftrightarrow{AB}$.
 - $C = \frac{1}{1-r}B + \frac{r}{r-1}A \in \overleftrightarrow{AB}$.
113. Let $P = (h, k)$. Then $x' = -2(x - h) + h$, $y' = -2(y - k) + k \Rightarrow h = 1$ and $k = -\frac{4}{3}$.
114. A stretch reflection fixes *exactly one* point and *exactly two* lines. A stretch rotation fixes *no* line.
115. *TTTF FTT*.
116. FALSE.
117. TRUE. α is 1-1 and onto ($\alpha\beta$ is onto $\Rightarrow \alpha$ is onto). See NOTE 2 : Distance-preserving mappings.
118. (a) $P = (-\frac{7}{2}, \frac{5}{2})$; (b) $t = 5$; (c) $x = -15$; (d) $P + \delta_{C,r}(B) = Q + \delta_{C,r}(A)$; in particular, $P = \delta_{C,r}(A)$ and $Q = \delta_{C,r}(B)$; (e) $\delta_{B,s}(A)$; (f) $x = r$; (g) $C = \tau_{B,A}^2(B)$.
119. FALSE.
120. $r = \frac{5\sqrt{2}}{4}$.
121. α is a direct similarity with equations $x' = x - 2y + 1$, $y' = 2x + y$. $\alpha((-1, 6)) = (-12, 4)$. (α is a stretch rotation; find its center and the angle of rotation.)
122. An involutory similarity is an isometry ($\alpha = \beta\delta \Rightarrow \delta = \iota$).
123. TRUE. (Consider first the case when the circles are equal (i.e. $A = C$ and $AB = CD$)).
124. $\sigma_{\mathcal{L}} = \delta_{P,r}\sigma_{\mathcal{L}}\delta_{P,r}^{-1} = \sigma_{\delta_{P,r}(\mathcal{L})} \iff \mathcal{L} = \delta_{P,r}(\mathcal{L}) \iff P \in \mathcal{L}$.

Affine transformations

126. You may use (in a clever way) the *cross product* of two vectors in $\mathbb{R}^3$.
127. (b) $\begin{vmatrix} 1 & 0 & 0 \\ h & a & b \\ k & c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
128. $[\alpha\beta] = [\alpha][\beta]$ (closure property); $[\alpha^{-1}] = [\alpha]^{-1}$ (inverse property).
- 129.
- $$[\delta_{P,r}] = \begin{bmatrix} 1 & 0 \\ (1-r)\mathbf{v} & rI \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} h \\ k \end{bmatrix}.$$
130. (a) For α_k : if $k = 1$ ($\alpha_k = \iota$), then every point is fixed; if $k \neq 1$, then only the points on the y -axis are fixed. (b) $PQR = 15$. (c) $\alpha_k(P)\alpha_k(Q)\alpha_k(R) = 15 \cdot |k|$; $\beta_k(P)\beta_k(Q)\beta_k(R) = 15$.

131. $TFFT \rightarrow TTF$.

132. $\underline{P_0 P'} = k \cdot \underline{P_0 P}$, $k \neq 0$.

133. $x = \frac{1}{\Delta}(dx' - by')$ and $y = \frac{1}{\Delta}(-cx' + ay')$ ($\Delta = ad - bc \neq 0$). We have

$$[\alpha] = \begin{bmatrix} 1 & 0 \\ \mathbf{v} & A \end{bmatrix} \quad \text{and} \quad [\alpha^{-1}] = [\alpha]^{-1} = \begin{bmatrix} 1 & 0 \\ -A^{-1}\mathbf{v} & A^{-1} \end{bmatrix}.$$

134. TRUE.

135. A similarity is a product of a stretch and an isometry, whereas a stretch is a product of two strains (about perpendicular lines).

136. The equations of the given transformation (shear) are *not* of the form

$$x' = (ar)x - (bs)y + h \quad \text{and} \quad y' = \pm((br)x + (as)y) + k.$$

137. $x' = x - y$, $y' = y$ and $x' = x$, $y' = x + y$. Only one point is fixed by the product of these two transformations (shears).

138. TRUE. (A similarity that preserves area must be an isometry.)

139. TRUE. (An involutory affine transformation is an isometry.)

140. $x' = 2x$ and $y' = \frac{1}{2}y$. (Find other examples.)

Appendix B

Revision Problems

1. Find all triangles such that three given noncollinear points are the midpoints of the sides of the triangle.
2. Prove that :

i. $\sigma_A\sigma_B = \sigma_B\sigma_C \iff B$ is the midpoint of $\overline{AC}$.

ii. $\sigma_A\sigma_{\mathcal{L}} = \sigma_{\mathcal{L}}\sigma_B \iff \mathcal{L}$ is the perpendicular bisector of $\overline{AB}$.

3. Prove that any distance-preserving *mapping* $\alpha : \mathbb{E}^2 \rightarrow \mathbb{E}^2$ is a bijection (hence an isometry).
4. If $\square ABCD$ and $\square EFGH$ are congruent rectangles and $AB \neq BC$, then how many isometries are there that take one rectangle to the other ?
5. Show that the product of the reflections in the three angle bisectors of a triangle is a reflection in a line perpendicular to a side of the triangle.
6. If $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are the perpendicular bisector of sides $\overline{AB}, \overline{BC}, \overline{CA}$, respectively, of $\triangle ABC$, then $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$ is a reflection in which line ?
7. If $\mathcal{L}$ and $\mathcal{M}$ are distinct intersecting lines, find the locus of all points P such that $\rho_{P,r}(\mathcal{L}) = \mathcal{M}$ for some r .
8. Let A, B, C be three noncollinear points such that $m(\angle ABC) = 45$, $m(\angle BCA) = 105$ and $m(\angle CAB) = 30$. Find the fixed point of

$$\rho_{B,90}\rho_{A,60}.$$

9. Given a figure consisting of two points P and Q , sketch a construction of the fixed point of $\tau_{P,Q}\rho_{P,45}$.
10. Given a figure consisting of three points A, B, C , sketch a construction of the fixed point of $\tau_{B,C}\rho_{A,120}$.
11. Indicate all pairs of *commuting* elements in the dihedral groups $\mathfrak{D}_3$ and $\mathfrak{D}_4$.
12. Show that any two parabolas are similar.
13. Find all stretch reflections taking point A to point B . Also, find all stretch rotations taking point A to point B .
14. Show that any given ellipse is the image of the unit circle under some affine transformation.
15. If $x' = ax + by + h$ and $y' = cx + dy + k$ are the equations of mapping α and $ad - bc = 0$, then show that α is not a collineation since all images are collinear.
16. Find equations for the strain of ratio r about the line with equation $y = mx$.
17. Find
 - (a) the equations for $\tau_{P,Q}^{-1}$ if $P = (a, b)$ and $Q = (h, k)$.
 - (b) the equations for $\sigma_{\mathcal{L}}$ if $\mathcal{L}$ has equation $y = -x + 1$.
 - (c) the image of the line with equation $x + y = 3$ under the transformation $\sigma_{\mathcal{L}}$, where $\mathcal{L}$ has equation $y = x - 1$.
 - (d) the image of the line with equation $x = 2$ under $\sigma_A\sigma_B$, where $A = (2, 1)$ and $B = (1, 2)$.

Class test, March 1999

18. Find
 - (a) the equations for the rotation $\rho_{C,30}$ where $C = (2, -1)$.
 - (b) $\mathcal{L}$, if $x' = \frac{3}{5}x + \frac{4}{5}y$ and $y' = \frac{4}{5}x - \frac{3}{5}y$ are equations for the isometry $\sigma_{\mathcal{L}}$.
 - (c) the image of the point $P = (a, -b)$ and of the line $\mathcal{L}$ with equation $bx + ay = 0$ under the isometry

$$x' = ax - by - 1, \quad y' = bx + ay \quad (\text{with } a^2 + b^2 = 1).$$

Class test, May 1999

19. (a) Find equations of the dilation $\delta_{P,-3}$ about the point $P = (1, -1)$.
 (b) If $\sigma_P((x, y)) = (-2x + 3, -2y - 4)$, find P .
 (c) Find equations of the strain of ratio k about the line with equation $y = 2x$.

Exam, June 1999

20. (a) If $\rho_{C,r}((x, y)) = ((\cos r)x - (\sin r)y + m, (\sin r)x + (\cos r)y + n)$, find C .
 (b) If $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}((x, y)) = (x + 6, y - 3)$, find equations for lines $\mathcal{L}$ and $\mathcal{M}$.
 (c) Find all direct similarities α such that

$$\alpha((1, 0)) = (0, 1) \quad \text{and} \quad \alpha((0, 1)) = (1, 0).$$

Exam, June 1999

21. (a) Determine the preimage of the point $(-1, 3)$ under the mapping $(x, y) \mapsto (-x + \frac{y}{3}, 3x - y)$.
 (b) Write the equations for $\tau_{P,Q}^{-1}\sigma_P$ if $P = (1, 1)$ and $Q = (3, 3)$.
 (c) Find the image of the line with equation $x = 2$ under $\sigma_A\sigma_B$, when $A = (2, 1)$ and $B = (1, 2)$.

Class test, March 2000

22. (a) What is the symmetry group of the capital letter **H** (written in most symmetric form) ? Describe this group (e.g. order of the group, generators, etc.).
 (b) Given the lines $(\mathcal{A}) x + y = 0$ and $(\mathcal{B}) x + y = 1$, find points A and B such that $\sigma_{\mathcal{B}}\sigma_{\mathcal{A}} = \sigma_B\sigma_A$.
 (c) Find equations for the rotation $\rho_{C,30}$ where $C = (2, -1)$.

Class test, May 2000

23. (a) What is the image of the line with equation $x - y + 1 = 0$ under the reflection in the line with equation $2x + y = 0$?
 (b) If $\rho_{C,r}((x, y)) = (-y + h, x + k)$, find C and r .
 (c) Find equations of the dilation $\delta_{P,-3}$ about the point $P = (1, -1)$.

- (d) Find all stretch reflections taking point $A = (1, 1)$ to point $B = (2, 0)$.
- (e) If $x' = (1 - ab)x - b^2y$, $y' = a^2x + (1 + ab)y$ are equations for a shear about the line $\mathcal{L}$, find $\mathcal{L}$. What is the ratio of this shear ?

Exam, June 2000

24. Consider the transformation

$$\alpha : \mathbb{E}^2 \rightarrow \mathbb{E}^2, \quad (x, y) \mapsto (x + y, y).$$

Show that α is a collineation. Is α an isometry ? Motivate your answer.

Class test, March 2001

25. (a) Write the equations for $\sigma_P \tau_{P,Q}^{-1}$ if $P = (a, b)$ and $Q = (c, d)$. What is this transformation ?
- (b) Find the image of the line with equation $x = 2$ under $\sigma_A \sigma_B$, when $A = (2, 1)$ and $B = (1, 2)$.

Class test, March 2001

26. (a) What is the symmetry group of the capital letter **H** (written in most symmetric form) ? Describe this group (e.g. order of the group, generators, etc.).
- (b) If $\sigma_A \sigma_B((x, y)) = (x - 4, y + 2)$, find equations for lines $\mathcal{A}$ and $\mathcal{B}$.
- (c) Find equations for
- the reflection $\sigma_{\mathcal{L}}$
 - the rotation $\rho_{C,r}$

that map the x -axis onto the line $\mathcal{M}$ with equation $x - y + 2 = 0$.

Class test, May 2001

27. (a) What is the image of the line with equation $x - y + 1 = 0$ under the reflection in the line with equation $2x + y = 0$?
- (b) If $\rho_{C,r}((x, y)) = (-y + h, x + k)$, find C and r .
- (c) Find all dilatations taking the circle with equation $x^2 + y^2 = 1$ to the circle with equation $(x - 1)^2 + (y - 2)^2 = 5$.
- (d) Find equations for the strain of ratio k about the line with equation $2x - y = 0$.

- (e) If $x' = (1 - ab)x - b^2y$, $y' = a^2x + (1 + ab)y$ are equations for a shear about the line $\mathcal{L}$, find $\mathcal{L}$. What is the ratio of this shear ?

Exam, June 2001

28. Consider the mapping

$$\alpha : \mathbb{E}^2 \rightarrow \mathbb{E}^2, \quad (x, y) \mapsto (x, x^2 + y).$$

- (a) Show that α is a transformation.
- (b) Is α a collineation ? Motivate your answer.
- (c) What is the *preimage* of the parabola with equation $y - x^2 = 0$ under the transformation α ?
- (d) Find the points and lines *fixed* by the transformation α .

Class test, August 2002

29. (a) Write the equations for the reflection σ in the line with equation

$$(\sin r)x - (\cos r)y = 0.$$

Use these equations to verify that σ is a transformation.

- (b) Compute the (Cayley table for) the symmetry group of an equilateral triangle.

Class test, August 2002

30. (a) If $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}((x, y)) = (x + 6, y - 3)$, find equations for lines $\mathcal{L}$ and $\mathcal{M}$.
 (b) If

$$x' = -\frac{\sqrt{3}}{2}x - \frac{1}{2}y + 1, \quad y' = \frac{1}{2}x - \frac{\sqrt{3}}{2}y - \frac{1}{2}$$

are equations for the rotation $\rho_{C,r}$, then find the centre C and the angle r .

Class test, October 2002

31. (a) Compute (the Cayley table for) the symmetry group of an equilateral triangle.
 (b) Show that if

$$x' = (\cos r)x - (\sin r)y + h, \quad y' = (\sin r)x + (\cos r)y$$

are equations for nonidentity rotation $\rho_{C,r}$, then

$$x_C = \frac{h}{2} \quad \text{and} \quad y_C = \frac{h}{2} f\left(\frac{r}{2}\right)$$

where f is a function to be determined.

- (c) Find all dilatations taking the circle with equation $x^2 + y^2 = 1$ to the circle with equation $(x - 1)^2 + (y - 2)^2 = 5$.
- (d) Find equations for the strain of ratio k about the line with equation $3x - y = 0$.

Exam, November 2002

32. Consider the mapping $\alpha : \mathbb{E}^2 \rightarrow \mathbb{E}^2$ given by

$$(x, y) \mapsto \left(\frac{1}{5}(-3x + 4y) + 4, \frac{1}{5}(4x + 3y) - 2 \right).$$

- (a) Verify that α is a transformation (on $\mathbb{E}^2$).
- (b) Investigate whether α is a collineation. If so, find the image of the line $\mathcal{L}$ with equation $2x - y = 5$ under α .
- (c) Determine all points P that are fixed by α (i.e. $\alpha(P) = P$).
- (d) Find the image and the preimage of (the origin) $O = (0, 0)$ under α .
- (e) Write down the equations of the translation $\tau_{O, \alpha(O)}$ and the halfturn $\sigma_{\alpha(O)}$.

Class test, August 2003

33. Let $c, d \in \mathbb{R}$ and consider the transformation $a : \mathbb{E}^2 \rightarrow \mathbb{E}^2$ be given by the equations

$$x' = x + c \quad \text{and} \quad y' = -y + d.$$

- (a) Look at the form of these equations and conclude that they represent an *odd isometry*. Explain.
- (b) For what values of the parameters c and d is the given isometry α
 - a glide reflection ?
 - a reflection ?

In each case, find the (equations of the) line fixed by α (i.e. the *axis* of the glide reflection and the *mirror* of the reflection, respectively).

Class test, October 2003

34.

- (a) Given the point $C = (3, -2)$ and the line $\mathcal{L}$ with equation $x + y - 1 = 0$, find equations for $\delta_{C,2}$ (stretch), $\alpha_{\mathcal{L}}$ (reflection), and $\rho_{C,45}$ (rotation). Compute $\delta_{C,2}(\mathcal{L})$ and $\rho_{C,45}(\mathcal{L})$.

- (b) Determine the collineation α such that

$$\alpha((0, 0)) = (-1, 4), \quad \alpha((-1, 4)) = (-9, 6), \quad \text{and} \quad \alpha((-9, 16)) = (-13, 22).$$

Is this collineation a similarity? If yes, find its ratio. Identify α .

- (c) Find equations for α^2 and then identify this transformation.

Exam, November 2003

35. Consider the mapping $\alpha : \mathbb{E}^2 \rightarrow \mathbb{E}^2$ given by

$$(x, y) \mapsto \left(\frac{1}{2}(x + \sqrt{3}y), \frac{1}{2}(\sqrt{3}x - y) \right).$$

- (a) Verify that α is a transformation.
- (b) Determine whether α is a collineation. If so, find the image of the line $\mathcal{L}$ with equation $\sqrt{3}x + y + 1 = 0$ under α .
- (c) Find *all* points P that are fixed by α (i.e. $\alpha(P) = P$).
- (d) Given the points $O = (0, 0)$, $A = (1, 0)$, $B = (1, \frac{1}{\sqrt{3}})$, and $C = (0, \frac{1}{\sqrt{3}})$
- verify that the quadrilateral $\square OABC$ is a rectangle.
 - determine *with justification* the image of $\square OABC$ under α .

Class test, September 2004

36. Consider the points

$$O = (0, 0), \quad A = (1, 1) \quad \text{and} \quad B = (-1, 1).$$

- (a) Find the equations of all *four* isometries sending the segment $\overline{OA}$ onto the segment $\overline{OB}$.
- (b) Identify these transformations. (Specify clearly, for a rotation : the centre and the directed angle, and for a reflection or glide reflection : the axis.)

(HINT : Use the general equations of an isometry.)

Class test, October 2004

37. Consider the points

$$A = (2, 1), \quad B = (2, -2), \quad C = (-2, 3), \quad D = (4, 3), \quad P = \left(\frac{4}{5}, \frac{3}{5}\right), \quad Q = (0, -1)$$

and the line $\mathcal{L}$ with equation

$$x - y - 1 = 0.$$

(a) Find the equations of the following transformations :

- i. the stretch $\delta_{P,2}$;
- ii. the rotation $\rho_{P,90}$;
- iii. The stretch rotation $\alpha_1 = \rho_{P,90}\delta_{P,2}$;
- iv. the stretch $\delta_{Q,2}$;
- v. the reflection $\sigma_{\mathcal{L}}$;
- vi. the stretch reflection $\alpha_2 = \sigma_{\mathcal{L}}\delta_{Q,2}$.

(b) How many similarities are there sending the segment $\overline{AB}$ onto the segment $\overline{CD}$? Justify your claim.

(c) Find the equations of the unique *direct similarity* such that $A \mapsto C$ and $B \mapsto D$.

(d) Find the equations of the unique *opposite similarity* such that $A \mapsto D$ and $B \mapsto C$.

Exam, November 2004

38. Consider the points $O = (0, 0)$ and $P = (0, 2)$, the line $\mathcal{L}$ with equation $y - 1 = 0$, and the mappings

$$\begin{aligned} \alpha_1 : (x, y) &\mapsto (x, x^2 + y) \\ \alpha_2 : (x, y) &\mapsto (x, -y + 2) \\ \alpha_3 : (x, y) &\mapsto \left(x - \frac{1}{2}y, -2x + y\right). \end{aligned}$$

(a) Find the point $\alpha_2(P)$ and the line $\alpha_2^{-1}(\mathcal{L})$.

(b) Find $\alpha_1(\mathcal{L})$ and $\alpha_3^{-1}(O)$.

- (c) Which of the given mappings are *transformations* ? Which, if any, of these transformations is a *collineation* ? Justify your answers.
- (d) Find the equations for the following (transformations) :
- i. $\tau_{O,P}$.
 - ii. $\sigma_{\mathcal{L}}$.
 - iii. σ_P .
 - iv. $\tau_{O,P} \sigma_{\mathcal{L}}$.

Class test, September 2005

39. (a) Let $a, b, h, k \in \mathbb{R}$ such that $a^2 + b^2 = 1$. If

$$\begin{aligned} x' &= ax + by + h \\ y' &= bx - ay + k \end{aligned}$$

are the equations for isometry α , show that α is a *reflection* if and only if

$$ah + bk + h = 0 \quad \text{and} \quad ak - bh = k.$$

(HINT : Find the equations for α^{-1} by using *Cramer's rule*.)

- (b) Determine the fixed points (if any) of the following isometry

$$(x, y) \mapsto \left(\frac{1}{5}(-3x + 4y) + 4, \frac{1}{5}(-4x - 3y) + 2 \right).$$

Hence identify this isometry.

- (c) Find the *glide reflection* γ such that

$$\gamma^2 = \tau_{A,B}, \quad \text{where } A = (1, 1), B = (2, 2).$$

Class test, October 2005

40. Consider the points

$$A = (4, 2), \quad P = (h, k)$$

and the lines

$$\mathcal{L} : y = 0, \quad \mathcal{M} : 2x + y = 3, \quad \mathcal{N} : 2x + y = 8.$$

- (a) Write the equations for the following transformations :

-
- i. the halfturn σ_P ;
 - ii. the reflection $\sigma_{\mathcal{L}}$;
 - iii. the translation $\tau_{O,A}$;
 - iv. the rotation $\rho_{P,30}$;
 - v. the glide reflection $\sigma_{\mathcal{M}}\sigma_A$;
 - vi. the translation $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}$.
- (b) Under what conditions the transformations σ_P and $\sigma_{\mathcal{L}}$ do commute ?
- (c) Find the point C such that

$$\delta_{P,3}\delta_{A,\frac{1}{3}} = \tau_{A,C}.$$

- (d) Let $B = \tau_{O,A}((0, -4))$. (O is the origin.) Find the equations of all four isometries sending segment $\overline{OA}$ onto the segment $\overline{OB}$.
(HINT : Use the general equations for an isometry.)

Exam, November 2005

Appendix C

Miscellany

The Greek Alphabet

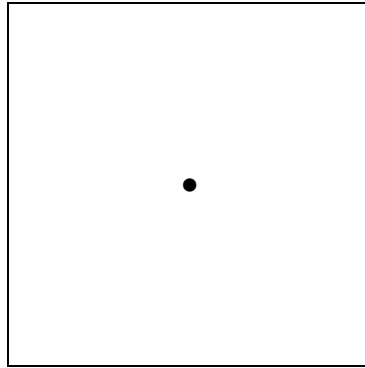
Letters	Names	Letters	Names	Letters	Names
A, α	alpha	I, ι	iota	$P, \rho(\varrho)$	rho
B, β	beta	K, κ	kappa	$\Sigma, \sigma(\varsigma)$	sigma
Γ, γ	gamma	Λ, λ	lambda	T, τ	tau
Δ, δ	delta	M, μ	mu	Υ, υ	upsilon
$E, \epsilon(\varepsilon)$	epsilon	N, ν	nu	$\Phi, \phi(\varphi)$	phi
Z, ζ	zeta	Ξ, ξ	xi	X, χ	chi
H, η	eta	O, o	omicron	Ψ, ψ	psi
$\Theta, \theta(\vartheta)$	theta	$\Pi, \pi(\varpi)$	pi	Ω, ω	omega

The Gothic (Fraktur) Alphabet

ⱱ, a	Ɱ, h	Ɀ, o	ⱼ, u
Ⱳ, b	Ȿ, i	Ɀ, p	ⱻ, v
ⱼ, c	Ȿ, j	Ɀ, q	ⱼ, w
Ɀ, d	Ɱ, k	Ɱ, r	ⱼ, x
ⱼ, e	ⱼ, l	ⱼ, s	ⱼ, y
ⱼ, f	Ɱ, m	ⱼ, t	ⱼ, z
ⱼ, g	Ɱ, n		

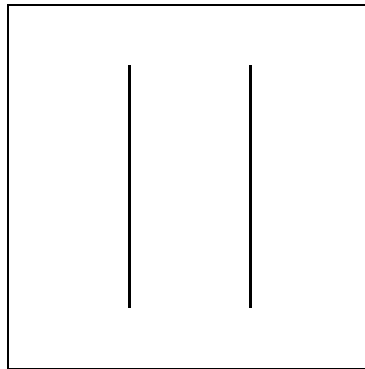
Affine Transformations on the Euclidean Plane

1. The identity transformation



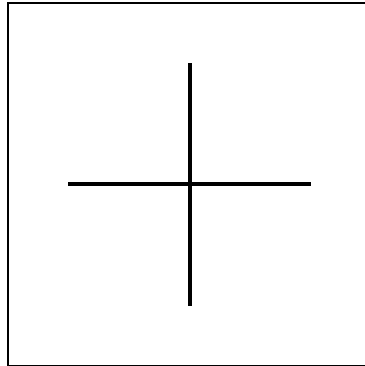
The identity transformation ι

2. Translation



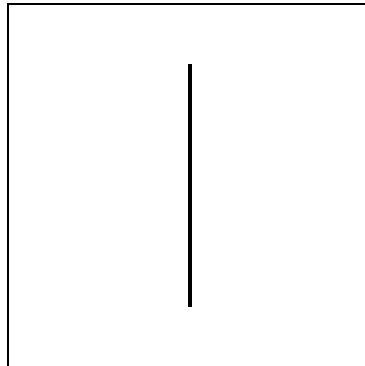
The translation $\tau_{P,Q}$

3. Halfturn (about a point)

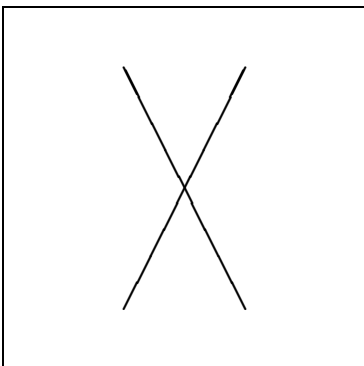


The halfturn σ_P

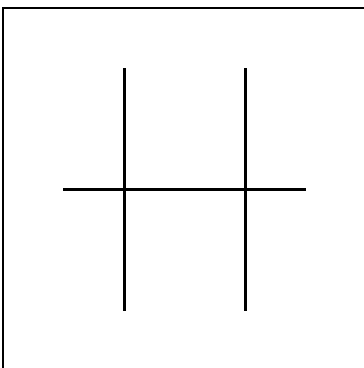
4. Reflection (in a line)



The reflection $\sigma_{\mathcal{L}}$

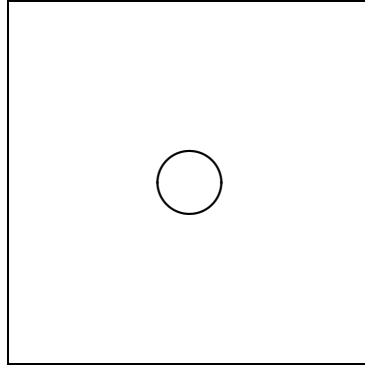
5. Rotation (about a point)

The rotation $\rho_{P,r}$

6. Glide reflection (along an axis)

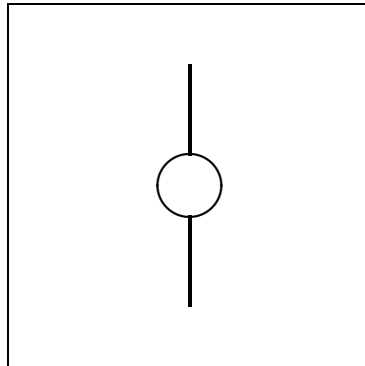
The glide reflection $\gamma = \sigma_B \sigma_A = \sigma_B \sigma_{\mathcal{A}}$

7. Stretch (about a point)

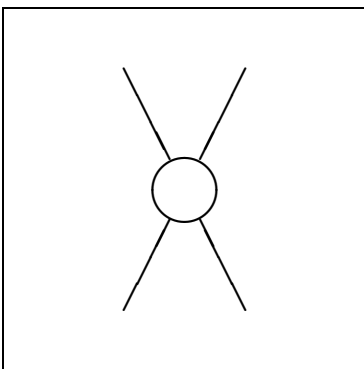


The stretch $\delta_{P,r}$

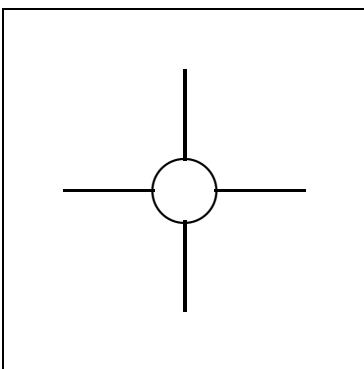
8. Stretch reflection



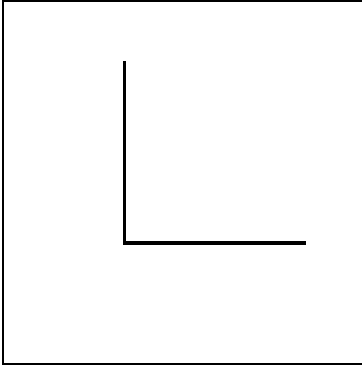
The stretch reflection $\sigma_{\mathcal{L}}\delta_{P,r}$

9. Stretch rotation

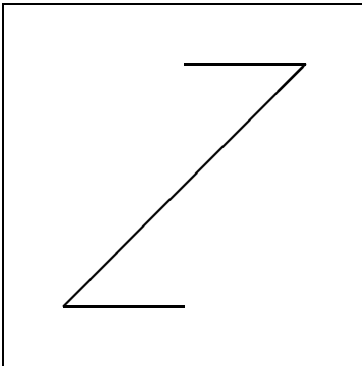
The stretch rotation $\rho_{P,s}\delta_{P,r}$

10. Dilation

The (nonisometric) dilation $\sigma_P\delta_{P,r}$ ($r \neq 1$)

11. Strain (about a line)

The strain $\varepsilon_{\mathcal{L},r}$

12. Shear (along a line)

The shear $\zeta_{\mathcal{L},r}$

Similarities on the Euclidean Plane

