

A line is a straight, continuous arrangement of infinitely many points. It is infinitely long and extends in two directions but has no width or thickness. It is represented and named by any two distinct points that lie on it.

A plane has length and width but no thickness. It is a flat surface that extends infinitely along its length and width. It can be named with a script capital letter, such as $\mathcal{P}$.

Space has length, width and thickness, extending infinitely along all three dimensions.

Define and/or draw representations of the following terms using conventional symbols. Possible definitions are provided below. Participants should include diagrams showing conventional symbols, such as matching tick marks for congruent figures, matching arrows for parallel lines and right angle and perpendicular signs.

I.

Collinear points lie on the same line.

Coplanar points or lines lie on the same plane.

A line segment is that part of a line that consists of two points, called endpoints, and all the points between them. It is designated $\overline{AB}$.

An endpoint is a point at the end of a segment or ray.

The measure of a line segment is designated AB .

An angle is formed by two rays that share a common endpoint, designated $\angle ABC$ or $\angle D$.

A vertex is the common endpoint of the rays forming the angle.

The measure of an angle is designated $m\angle ABC$ or $m\angle D$.

Congruent (angles, segments, polygons, circles, solids) are identical in size and shape. e.g., $\angle ABC \cong \angle ATC$.

Equal measures of segments or angles are designated $AB = CD$ or $m\angle ABC = m\angle D$.

A midpoint of a segment is a point that divides a segment into two congruent segments.

II.

A bisector is a line, segment, or ray that divides a figure into two congruent figures.

A right angle is an angle that measures 90° .

An acute angle is an angle whose measure is between 0° and 90° .

An obtuse angle is an angle whose measure is between 90° and 180° .

A straight angle is an angle whose measure is 180° ; the rays forming the angle are collinear.

A pair of vertical angles is a pair of non-adjacent angles formed by two intersecting lines.

A linear pair of angles consists of two adjacent angles whose sum is 180° .

A pair of complementary angles is a pair of angles whose sum is 90° .

A pair of supplementary angles is a pair of angles whose sum is 180° .

III.

A polygon is a closed figure in a plane formed by connecting line segments endpoint to endpoint.

Consecutive angles in a polygon share one side of the polygon.

Consecutive sides in a polygon share one vertex of the polygon.

A convex polygon has all of its diagonals within the polygon.

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A convex polygon has all of its diagonals within the polygon.

A concave polygon has at least one diagonal lying outside the polygon.

A diagonal of a polygon is a segment that connects two non-consecutive vertices.

A polygon in which all sides are congruent is an equilateral polygon.

A polygon in which all angles are congruent is an equiangular polygon.

A regular polygon is equilateral and equiangular.

Common polygon names: triangle, quadrilateral, pentagon, hexagon, heptagon, octagon, nine-gon, decagon, dodecagon.

IV.

Perpendicular lines, segments, rays or planes intersect at right angles to each other.

Parallel lines, in the same plane, are equidistant from each other. Parallel planes never intersect.

A right triangle is a triangle that has one right angle.

An acute triangle is a triangle that has three acute angles.

An obtuse triangle is a triangle that has one obtuse angle.

A scalene triangle is a triangle with no congruent sides.

An equilateral triangle is a triangle that has three congruent sides.

An isosceles triangle is a triangle that has at least two congruent sides.

V.

A trapezoid is a quadrilateral with exactly one pair of parallel sides.

A kite is a quadrilateral with two distinct pairs of consecutive congruent sides.

A parallelogram is a quadrilateral with two pairs of parallel sides.

A rhombus is a quadrilateral with four congruent sides.

A rectangle is a quadrilateral with four right angles.

A square is a regular quadrilateral; it has four congruent sides and four right angles.

VI.

A circle is a set of points a given distance (radius) from a given point (center) in the plane.

A diameter is a segment with endpoints on the circle that contains the center of the circle.

An arc of a circle is that part of the circle that consists of two points on the circle and all the points between them. The two points are called endpoints.

A semicircle is an arc of a circle whose endpoints are the endpoints of a diameter.

A chord is a segment whose endpoints lie on the circle.

A tangent is a line that intersects a circle at only one point.

A secant is a line that intersects a circle at two points.

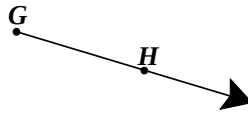
Terms and Definitions

The following four terms are undefined in the Euclidean axiomatic system. Describe each term in your own words.

point
line
plane
space

Define and/or draw representations of the following terms using conventional symbols.

Example: ray $\overrightarrow{GH}$



A ray is the part of a line that contains point G and all of the points on the same side of point G as point H .

I. collinear coplanar line segment endpoint measure of a line segment angle vertex measure of an angle congruent equal measure midpoint	II. bisector right angle acute angle obtuse angle straight angle pair of vertical angles linear pair of angles pair of complementary angles pair of supplementary angles
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III. polygon consecutive angles consecutive sides convex polygon concave polygon diagonal equilateral polygon equiangular polygon regular polygon common polygon names	IV. perpendicular parallel right triangle acute triangle obtuse triangle scalene triangle equilateral triangle isosceles triangle
V. trapezoid kite parallelogram rhombus rectangle square	VI. circle radius center diameter arc of a circle semicircle chord tangent secant

What is a Translation?

Overview: The properties of translation vectors are determined for 2-dimensional figures in the plane and on the coordinate plane.

Objective: **TE_xES Mathematics Competencies**
II.006.A. The beginning teacher understands the concept of slope as a rate of change and interprets the meaning of slope and intercept in a variety of situations.
III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.014.A. The beginning teacher identifies transformations (i.e., reflections, translations, glide-reflections, rotations, dilations) and explores their properties.
III.014.D. The beginning teacher applies transformations in the coordinate plane.
III.014.H. The beginning teacher explores the relationship between geometric and algebraic representations of vectors and uses this relationship to solve problems.
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

Geometry TEKS

d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
d.2.B. The student uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.

Background: Identify and graph coordinates and slopes in coordinate figures.
Recognize parallel lines.

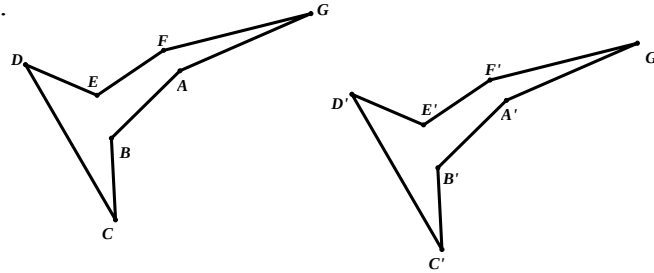
Materials: easel paper, centimeter ruler, colored markers

New Terms: image, isometry, pre-image, translation vector

Procedures:

The activity has been divided into parts, represented by the different phases of learning in the van Hiele model, which will be explained to participants after they have experienced the activities on translations and reflections. The activity begins with participants sharing prior knowledge about translations, using 1, in their groups. Post a large sheet of easel paper with the heading “Translations.”

- The polygon $ABCDEFGG$, transformed into polygon $A'B'C'D'E'F'G'$ below, represents a translation. The original figure, $ABCDEFGG$, is called the *pre-image*. The resultant figure, $A'B'C'D'E'F'G'$, is called the *image*. Each point on the image is labeled with the same letter as its corresponding pre-image point, with the addition of a prime mark.



Ask participants to add the new terms pre-image and image to their glossaries.

In your group, discuss the figure and make a list of properties for a translation.

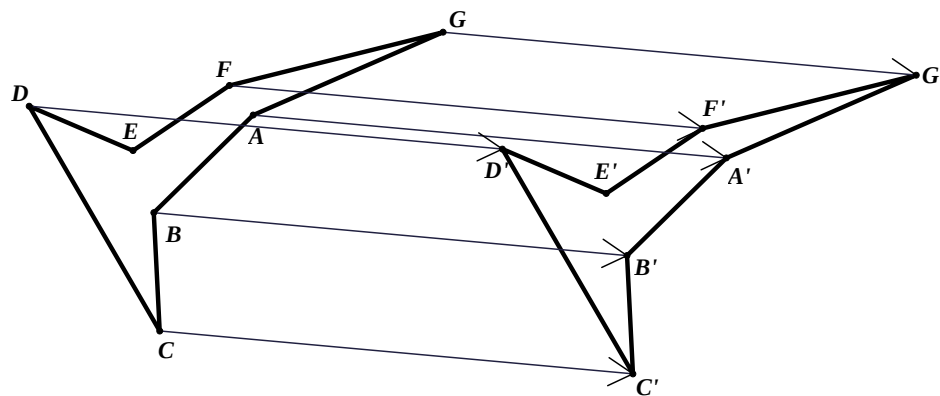
After a few minutes, during whole group discussion, ask a participant to record a list of translation properties supplied by the class. Do not discuss or critique the statements. Explain that, after completing 2 – 5, these statements will be critiqued for validity.

Where are translations seen in the real world?

Some examples of translations can be seen in decorative friezes, rows of apartment units, or hotel rooms facing the same direction, and some tessellations.

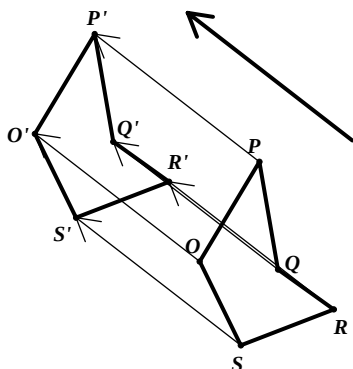
Participants work on 2 – 5. Move around the room listening and facilitating while participants work.

- Use a ruler and colored pencil or marker to connect each pre-image point to its corresponding image point. In your group, discuss and record your findings on paper.



The connecting segments are all parallel and have the same magnitude.

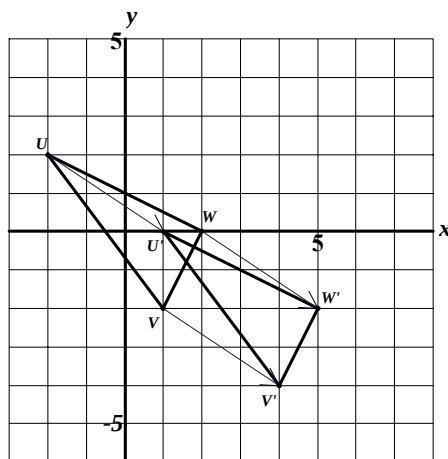
3. A *translation vector* resembles a ray and is used to define the magnitude (by the length of the vector) and the direction (by the direction of the arrow) of the translation. Transform pentagon $OPQRS$ using the translation vector given. Show the segments connecting pre-image to corresponding image points. Label the image points appropriately.



Remind participants to add the new term translation vector to their glossaries.

4. Draw and number the x - and y - axes on the grid below.
Write the coordinates of the vertices of $\triangle UVW$.
Apply the rule $(x, y) \rightarrow (x + 3, y - 2)$ to the coordinates for U , V , and W .
Draw the image that results from this transformation.

An example:



	Pre-image (x, y)	Image ($x + 3, y - 2$)
U	$(-2, 2)$	$(1, 0)$
V	$(1, -2)$	$(4, -4)$
W	$(2, 0)$	$(5, -2)$

Using a colored pencil or marker, connect corresponding pre-image and image points.
Describe what happened in terms of the transformation rule $(x, y) \rightarrow (x + 3, y - 2)$.
The figure moved 3 units to the right and 2 units down.

The slope of the translation vector is $-\frac{2}{3}$.

Generalize: Describe the translation for the rule $(x, y) \rightarrow (x + a, y + b)$.
If a and b are positive, each point moves a units to the right and b units up. If a and b are negative, then the point moves in the opposite direction.

The slope of the translation vector is $\frac{b}{a}$.

5. In your group, discuss and record on paper the properties of translations. Be prepared to share during whole class discussion.

After most participants have completed 2 – 5, facilitate a whole class discussion on the properties of translations.

Ask a volunteer to record the class-proposed properties of translations on the easel paper. Discuss each item for validity from the list produced at the beginning of the activity. Cross out invalid properties.

The following represent possible responses.

- *The pre-image and image are congruent.*
- *The pre-image and image have exactly the same orientation.*
- *The pre-image points all move in exactly the same direction and distance.*
- *Parallel congruent segments connect the pre-image points to the corresponding image points.*
- *The connecting segments with directional arrows are called translation vectors.*
- *The translation vector determines the magnitude and direction of movement for each point in the figure.*
- *If a and b are both positive, for the general coordinate rule $(x, y) \rightarrow (x + a, y + b)$, each point moves a units to the right and b units up. If a and b are negative, then points move a units to the left and b units down.*

The slope of the translation vector is $\frac{b}{a}$.

Participants should discuss any additional properties and knowledge emerging from the activity.

Look at 3. It shows a figure with a translation vector.

Does a translation vector have to be attached to the figure?

No, it can lie anywhere. However, if the connecting segments are drawn, they should all be parallel to, and of the same magnitude as, the given translation vector.

What is an isometry?

An *isometry* is a transformation that preserves congruence.

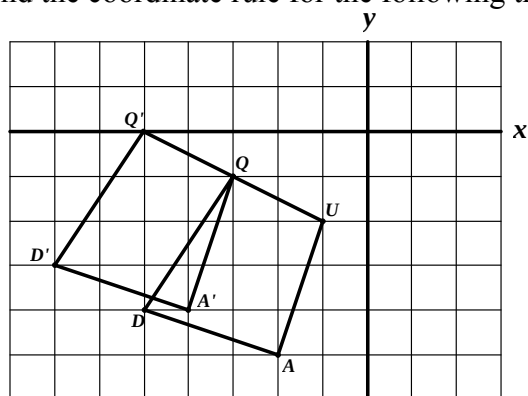
Is translation a type of isometry?

Yes, the pre-image and image are congruent.

Point out that students in secondary school may prefer to use the term slide instead of the term translation. Slide is acceptable at the informal Visual Level. However, following any discussion, when formal terms and symbols have been introduced, only formal terms should be used.

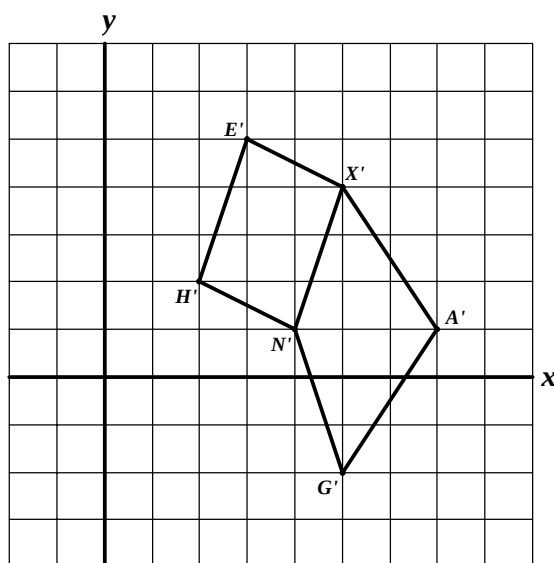
Participants work independently on 6 and 7, applying the properties of translations.

6. Find the coordinate rule for the following translation:



$$(x, y) \rightarrow (x - 2, y + 1)$$

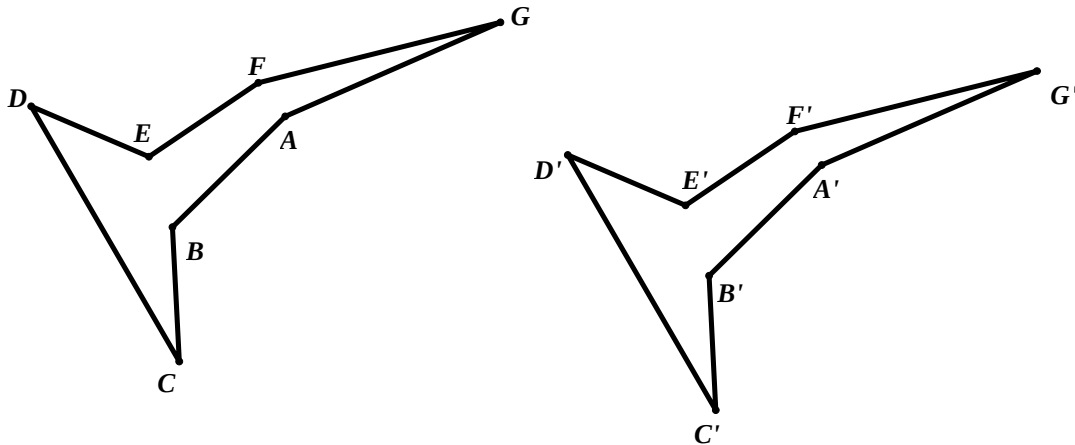
7. Polygon $H'E'X'A'G'N'$ is the image resulting from the translation rule $(x, y) \rightarrow (x + 7, y - 4)$. Find the coordinates of the pre-image.



$$\begin{aligned} H &(-5, 6) \\ E &(-4, 9) \\ X &(-2, 8) \\ A &(0, 5) \\ G &(-2, 2) \\ N &(-3, 5) \end{aligned}$$

What is a Translation?

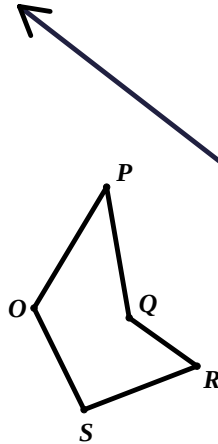
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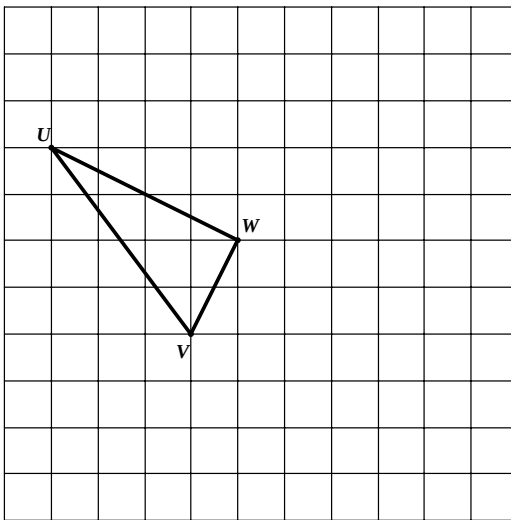
In your group, discuss the figure and make a list of properties for a translation.

2. Use a ruler and colored pencil or marker to connect each pre-image point to its corresponding image point. In your group, discuss and record your findings on paper.

3. A translation vector resembles a ray, and is used to define the magnitude (by the length of the vector) and the direction (by the direction of the arrow) of the translation. Transform pentagon $OPQRS$ using the translation vector given. Show the segments connecting pre-image to corresponding image points. Label the image points appropriately.



4. Draw and number the x - and y - axes on the grid below. Write the coordinates of the vertices of $\triangle UVW$. Apply the rule $(x, y) \rightarrow (x + 3, y - 2)$ to the coordinates for U , V , and W . Draw the image that results from this transformation.

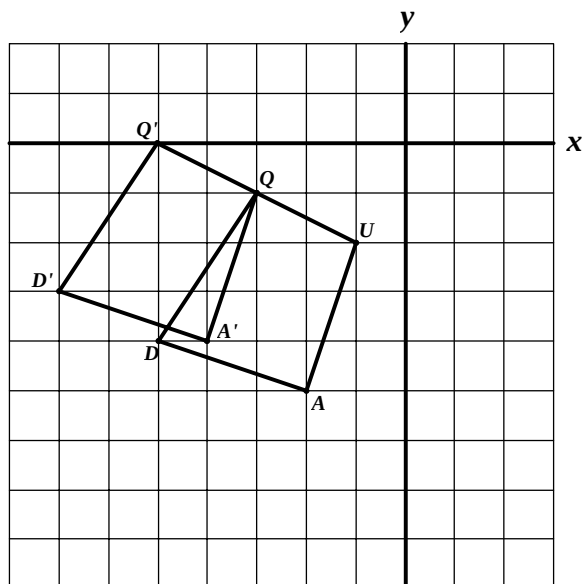


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W		

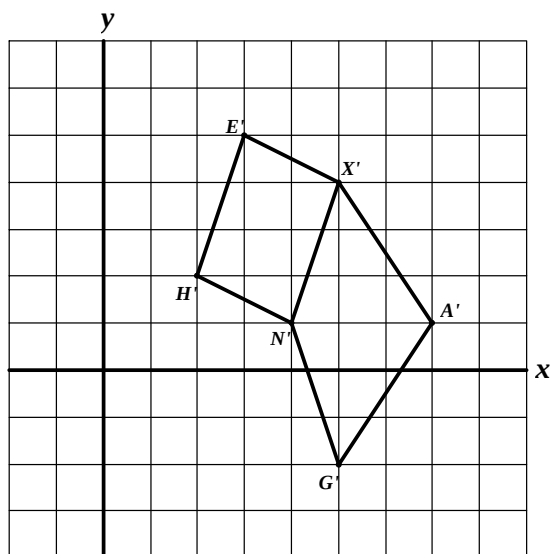
Using a colored pencil or marker, connect corresponding pre-image and image points. Describe what happened in terms of the transformation rule $(x, y) \rightarrow (x + 3, y - 2)$.

Generalize: Describe the translation for the rule $(x, y) \rightarrow (x + a, y + b)$.

5. In your group, discuss and record on paper the properties of translations.
Be prepared to share during whole class discussion.
6. Find the coordinate rule for the following translation:



7. Polygon $H'E'X'A'G'N'$ is the image resulting from the translation rule $(x, y) \rightarrow (x + 7, y - 4)$. Find the coordinates of the pre-image.



Reflections

Overview: The properties of reflections are determined for 2-dimensional figures in the plane and on the coordinate plane.

Objective: **TEXES Mathematics Competencies**

II.006.A. The beginning teacher understands the concept of slope as a rate of change and interprets the meaning of slope and intercept in a variety of situations.

III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.

III.014.A. The beginning teacher identifies transformations (i.e., reflections, translations, glide-reflections, rotations, dilations) and explores their properties.

III.014.C. The beginning teacher uses transformations to explore and describe reflectional, rotational, and translational symmetry.

III.014.D. The beginning teacher applies transformations in the coordinate plane.

III.014.H. The beginning teacher explores the relationship between geometric and algebraic representations of vectors and uses this relationship to solve problems.

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Geometry TEKS

d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.

d.2.B. The student uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons.

e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.

Background: Participants should be able to identify slopes and graph coordinates in the coordinate plane and recognize parallel and perpendicular lines.

Materials: easel paper, centimeter ruler, colored markers, patty paper, graph paper

New Terms: line of reflection or line of symmetry

Procedures:

Post a large sheet of easel paper with the heading “Reflections.” Distribute the activity sheet. Ask participants to work only on 1 for about 5 minutes.

1. The figures below represent reflections across the dotted lines. In your group, discuss and record the properties of reflections. Be prepared to share all of your observations at the end of this activity during whole class discussion.

Ask a participant to record reflection properties as provided by the class on the easel paper. Do not critique the statements.

Where are reflections seen in the real world?

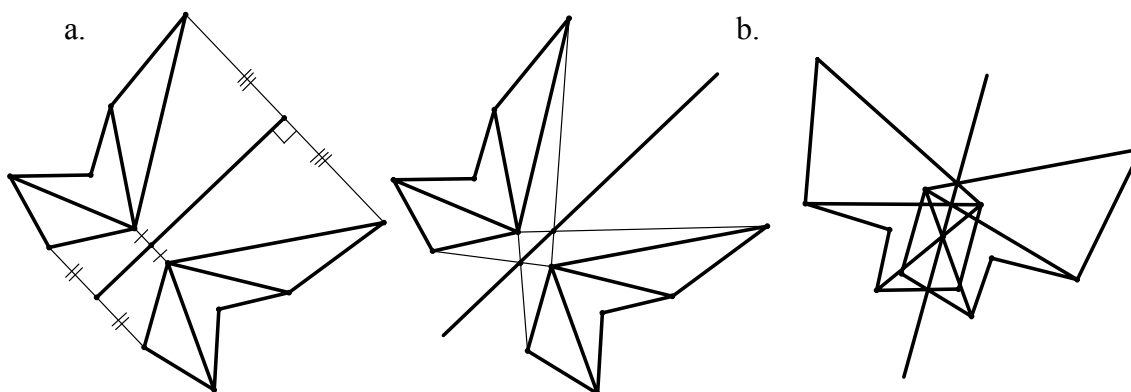
Some examples are: mirrors, adjacent hotel rooms, the word “AMBULANCE” written on the front of an ambulance as seen in a rear-view mirror, works of art (e.g., M.C. Escher).

The mathematical definition of a *line of reflection* or a *line of symmetry* is the line over which a figure is reflected resulting in a figure that coincides exactly with the original figure.

Ask participants to add the terms line of reflection and line of symmetry to their glossaries.

Participants work on 2 – 5. Walk around the room listening and facilitating while participants work. When most participants have completed these items, conduct a group discussion to bring about consensus.

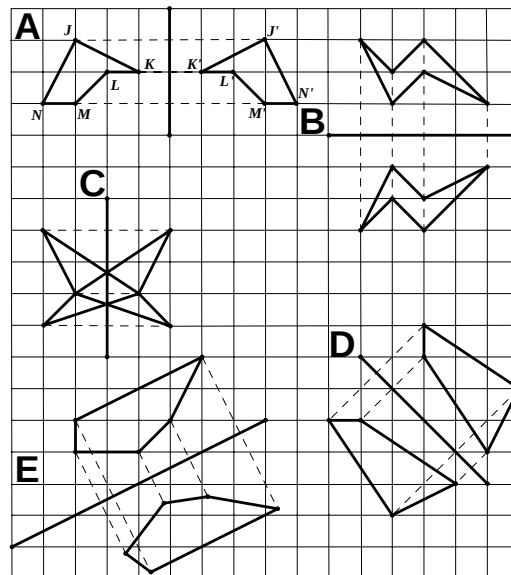
2. Find the lines of reflection or lines of symmetry for the following figures.
The line of reflection can be obtained by connecting the midpoints of segments drawn between corresponding pre-image and image points. Alternately, connect a pre-image point to a non-corresponding image point and vice versa. The intersection point is on the line of reflection. Repeat for a different pair of points. Draw a line, the line of reflection, through the intersection points.



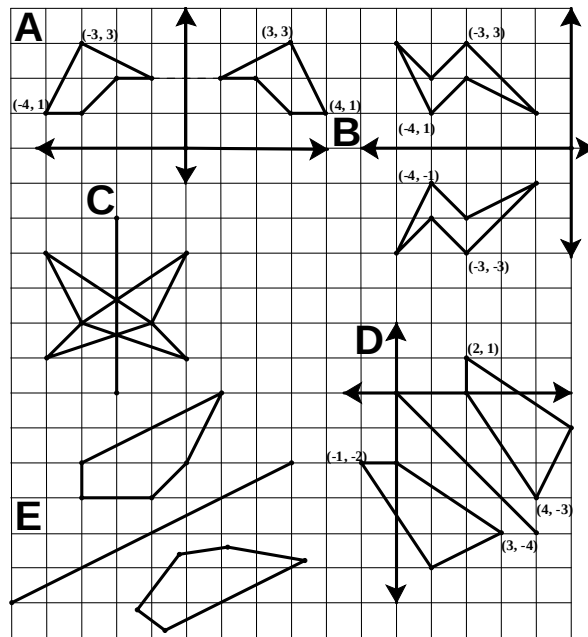
3. Reflect each polygon below across its line of reflection. Label at least one polygon and its image, using prime notation, e.g., A transforms to A' .

In your group, share and write down observations. Be prepared to share all of your observations at the end of this activity during whole class discussion.

If anyone needs further assistance, remind them to use the property discovered in 2, or provide patty paper. Have participants trace the figure and the reflection line, fold across the reflection line, and trace the folded figure. They should then copy the result onto the activity sheet. Figure E is difficult. The grid is a hindrance. Give participants adequate time to work on E. Discourage use of patty paper for this item.



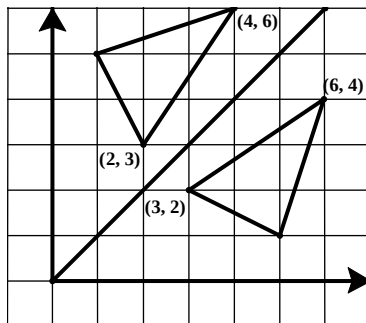
4. For each of the figures A – D from 2, let one end of the reflection segment represent the origin. Examples are illustrated below. Work with a partner to find the coordinates of each pre-image point and its corresponding image point. Then complete the following:
- When a figure is reflected across the y -axis, $(x, y) \rightarrow (-x, y)$.
 - When a figure is reflected across the x -axis, $(x, y) \rightarrow (x, -y)$.
 - When a figure is reflected across the line $y = -x$, $(x, y) \rightarrow (-y, -x)$.



- d. Predict: When a figure is reflected across the line $y = x$, $(x, y) \rightarrow (y, x)$.
On a coordinate grid, draw a shape, reflect it across the line $y = x$, and verify your prediction.

With your group, share and record on paper your observations.

An example is shown.

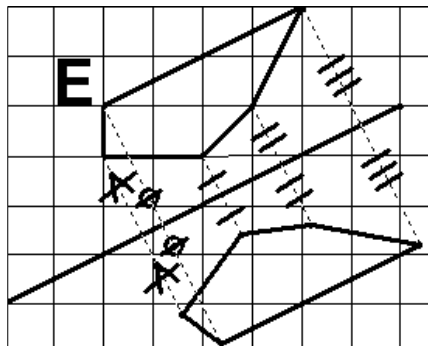


5. Describe at least three ways in which the position of the line of reflection, or line of symmetry, can be determined.
- Trace the pre-image and the image. Fold so that the figures coincide. The fold line is the line of reflection.
 - Draw a segment from a pre-image point to its corresponding image point. Construct the perpendicular bisector, which is the line of symmetry.
 - Draw at least two line segments connecting points to the corresponding image points. Find the midpoints. Draw a line through the midpoints, forming the line of reflection.
 - Connect a pre-image point to a non-corresponding image point and vice versa. The intersection point is on the line of reflection. Repeat for a different pair of points. Draw a line, the line of reflection, through the intersection points.

Lead a whole class discussion during which participants share their observations. Discuss the validity of each of the statements provided at the start of the lesson. Ask a participant to record new properties on the poster, and be sure to include all of the following points in the discussion.

- *Reflections are congruence transformations.*
- *Reflections are a type of isometry, because the pre-image and its image are congruent.*
- *The image and its pre-image are the same distance from the line of reflection but on opposite sides.*
- *The lines connecting corresponding pre-image to image points are parallel to each other and the line of reflection is their perpendicular bisector.*

When these properties are provided, participants should label congruence and perpendicular symbols on their papers. For example:

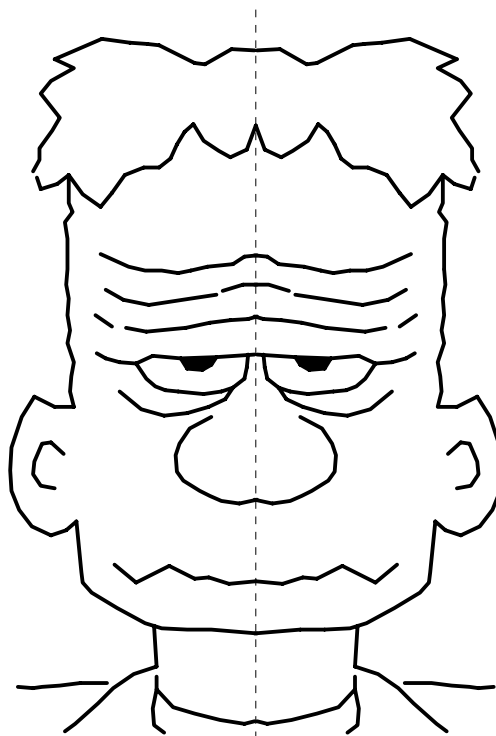
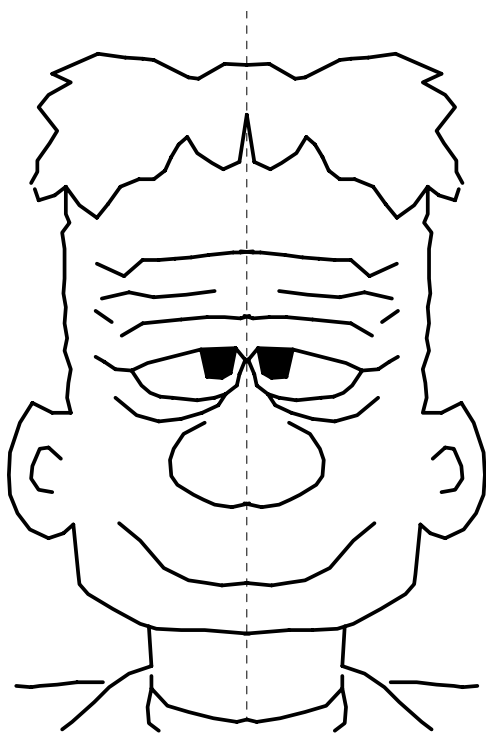


- *If a line is perpendicular to two distinct lines in the plane, then the two lines are parallel to each other.*
- *If the slope of the line of reflection is m , then the slopes of the connecting parallel lines are all $-\frac{1}{m}$.*
- *Coordinate rules:*
When a figure is reflected across the y-axis, $(x, y) \rightarrow (-x, y)$.
When a figure is reflected across the x-axis, $(x, y) \rightarrow (x, -y)$.
When a figure is reflected across the line $y = -x$, $(x, y) \rightarrow (-y, -x)$.
When a figure is reflected across the line $y = x$, $(x, y) \rightarrow (y, x)$.

Point out that the term flip is commonly used in secondary classrooms, but this informal term should be replaced with the formal term reflection.

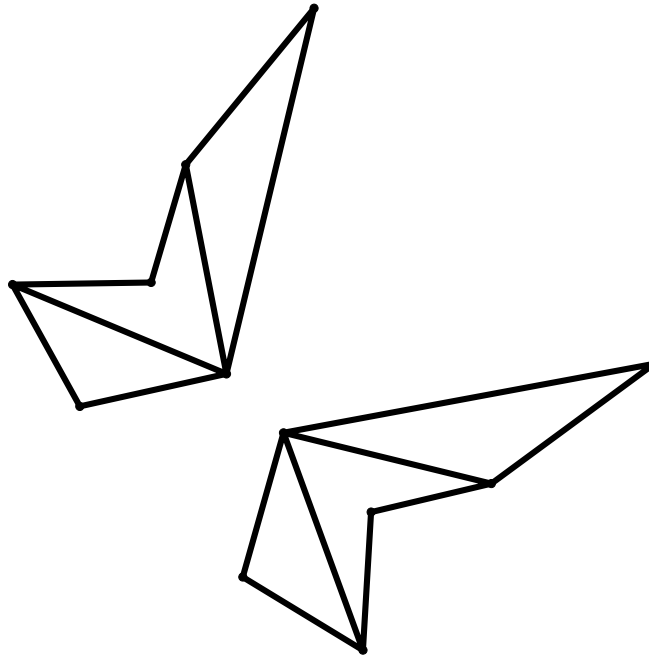
Reflections

1. The figures below represent reflections across the dotted lines. In your group, discuss and record the properties of reflections. Be prepared to share all of your observations at the end of this activity during whole class discussion.

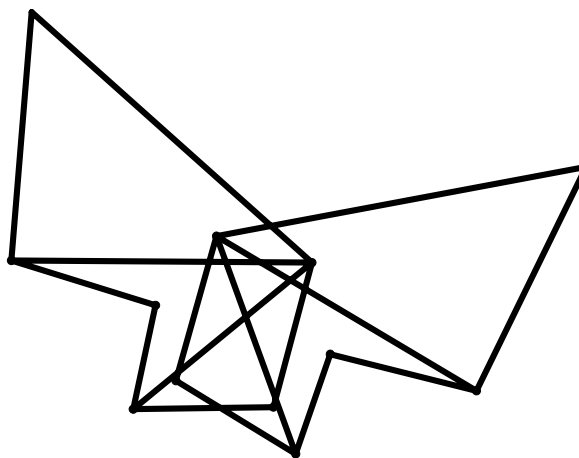


2. Find the lines of reflection, or lines of symmetry, for the following figures.

a.

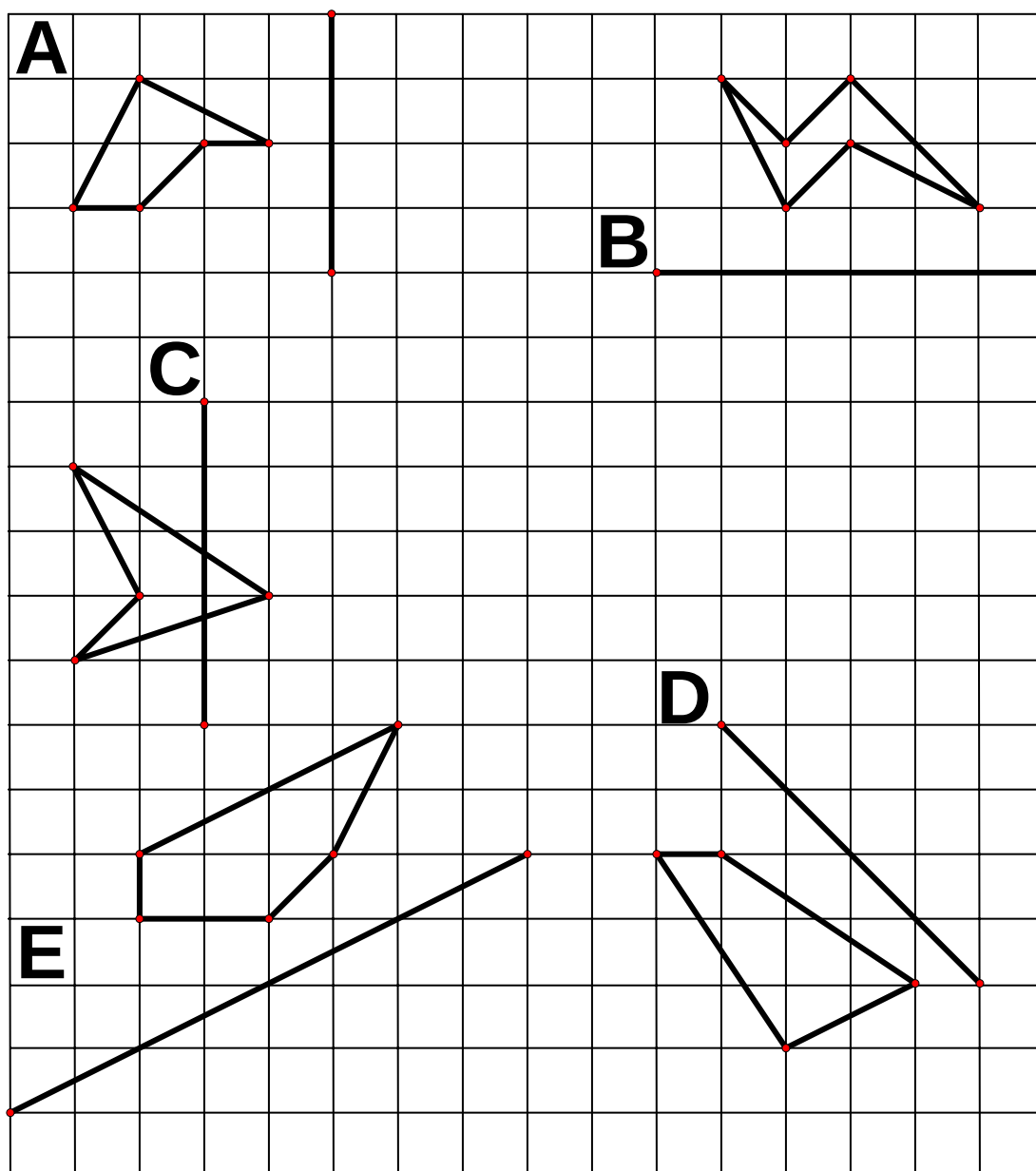


b.



3. Reflect each polygon below across its line of reflection. Label at least one polygon and its image using prime notation, e.g., A transforms to A' .

In your group, share and write down observations. Be prepared to share all of your observations at the end of this activity during whole class discussion.



4. For each of the figures A – D from 2, let one end of the reflection segment represent the origin. Work with a partner to find the coordinates of each pre-image point and its corresponding image point. Then complete the following:

a. When a figure is reflected across the y -axis, $(x, y) \rightarrow$ _____

b. When a figure is reflected across the x -axis, $(x, y) \rightarrow$ _____

c. When a figure is reflected across the line $y = -x$, $(x, y) \rightarrow$ _____

d. Predict:

When a figure is reflected across the line $y = x$, $(x, y) \rightarrow$ _____

On a coordinate grid, draw a shape, reflect it across the line $y = x$, and verify your prediction.

With your group, share and record on paper your observations.

5. Describe at least three ways in which the position of the line of reflection, or line of symmetry, can be determined.

Theoretical Framework: The van Hiele Model of Geometric Thought

Overview: Participants view and take notes on the PowerPoint presentation about the van Hiele Model of Geometric Thought, the theoretical framework on which this geometry module is based.

Objective: **TEXES Mathematics Competencies**
VI.020.A. The beginning teacher applies research-based theories of learning mathematics to plan appropriate instructional activities for all students.
VI.020.B. The beginning teacher understands how students differ in their approaches to learning mathematics.
VI.020.C. The beginning teacher uses students' prior mathematical knowledge to build conceptual links to new knowledge and plans instruction that builds on students' strengths and addresses students' needs.
VI.020.E. The beginning teacher understands how to provide instruction along a continuum from concrete to abstract.
VI.020.F. The beginning teacher understands a variety of instructional strategies and tasks that promote students' abilities to do the mathematics described in the TEKS.
VI.020.G. The beginning teacher understands how to create a learning environment that provides all students, including English Language Learners, with opportunities to develop and improve mathematical skills and procedures.
VI.020.H. The beginning teacher understands a variety of questioning strategies to encourage mathematical discourse and to help students analyze and evaluate their mathematical thinking.

Background:

Materials: PowerPoint presentation: van Hiele Model of Geometric Thought or transparencies of PowerPoint slides

New Terms:

Procedures:

This activity describes the theoretical framework known as the van Hiele model for geometric thought as it relates to the *Geometry Module*. Awareness of the model helps teachers understand how geometry should be taught and identify reasons for possible students' lack of success in learning high school geometry.

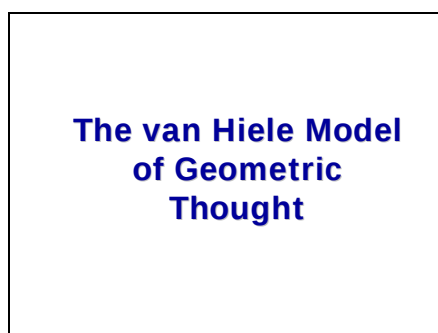
Students first learn to recognize whole shapes and then to analyze the relevant properties of the shapes. Later they see relationships between shapes and make simple deductions. Curriculum development and instruction must consider this hierarchy because although

learning can occur at several levels simultaneously, the learning of more complex concepts and strategies requires a firm foundation of prior skills.

The van Hiele model underscores the importance of the Learning Principle in NCTM's *Principles and Standards for School Mathematics* (2000), p. 11, which states that "students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge." Students learn more mathematics and learn it better when they can take control of their learning by defining their goals and monitoring their progress. Effective learners recognize the importance of reflecting on their thinking and learning from their mistakes.

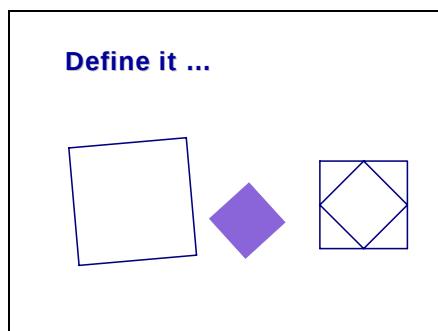
Distribute the activity sheet to participants to help them focus on the important ideas from the PowerPoint presentation "The van Hiele Model of Geometric Thought" as they take notes. They will need additional paper for note-taking. Show the PowerPoint presentation. Use the following notes pages to elaborate on the content of each slide.

Slide
1



The van Hiele model of geometric thought outlines the hierarchy of levels through which students progress as they develop geometric ideas. The model clarifies many of the shortcomings in traditional instruction and offers ways to improve it. Pierre van Hiele and his wife, Dina van Hiele-Geldof, focused on getting students to the appropriate level to be successful in high school geometry.

Slide
2



Language is the basis for understanding and communicating. Before we can find out what a student knows we must establish a common language and vocabulary.

Slide
3

**When is it appropriate
to ask for a definition?**

A definition of a concept is only possible if one knows, to some extent, the thing that is to be defined.

Pierre van Hiele

Only through the use of a common language can we develop the network of structures and relations which help produce good definitions.

Slide
4

Definition?

How can you define a thing before you know what you have to define? Most definitions are not preconceived but the finished touch of the organizing activity. The child should not be deprived of this privilege...

Hans Freudenthal

Slide
5

**Levels of Thinking in
Geometry**

- Visual Level
- Descriptive Level
- Relational Level
- Deductive Level
- Rigor

The development of geometric ideas progresses through a hierarchy of levels. The research of Pierre van Hiele and his wife, Dina van Hiele-Geldof, clearly shows that students first learn to recognize whole shapes, then to analyze the properties of a shape. Later they see relationships between the shapes and make simple deductions. Only after these levels have been attained can they create deductive proofs.

Slide
6

Levels of Thinking in Geometry

- Each level has its own network of relations.
- Each level has its own language.
- The levels are sequential and hierarchical. The progress from one level to the next is more dependent upon instruction than on age or maturity.

The hierarchy for learning geometry described by the van Hieles parallels Piaget's stages of cognitive development. One should note that the van Hiele model is based on instruction, whereas Piaget's model is not.

The van Hiele model supports Vygotsky's notion of the "zone of proximal development" which is the "distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1978, p. 85-86).

Slide
7

Visual Level Characteristics

- The student
- identifies, compares and sorts shapes on the basis of their appearance as a whole.
 - solves problems using general properties and techniques (e.g., overlaying, measuring).
 - uses informal language.
 - does NOT analyze in terms of components.

Language at the Visual Level serves to make possible communication for the whole group about the structures that students observe. Formal or conventional vocabulary is used to describe the figures. Any misconceptions identified may be clarified by the use of appropriate language. The language of the next level (e.g., congruence) will not be understood by students who are at the Visual Level.

Slide
8

Visual Level Example

It turns!



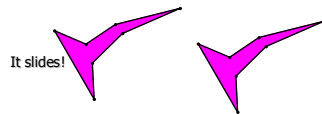
In this example, the term rotation is introduced.

Slide
9

Where and how is the Visual Level represented in the translation and reflection activities?

Slide
10

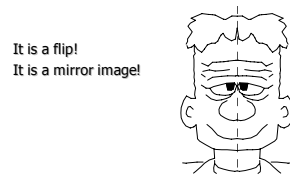
Where and how is the Visual Level represented in this translation activity?



The term translation is introduced. The observation that the two figures look the same and keep the same orientation may emerge.

Slide
11

Where and how is the Visual Level represented in this reflection activity?



The term reflection is introduced. Students may note that a mirror might produce a reflection.

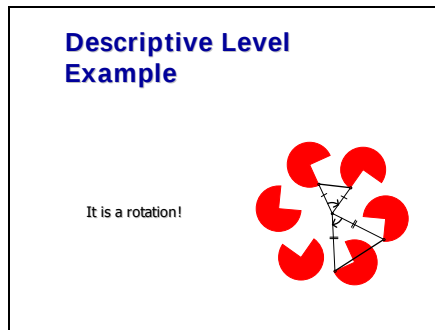
Slide
12

Descriptive Level Characteristics

- The student
- recognizes and describes a shape (e.g., parallelogram) in terms of its properties.
 - discovers properties experimentally by observing, measuring, drawing and modeling.
 - uses formal language and symbols.
 - does NOT use sufficient definitions. Lists many properties.
 - does NOT see a need for proof of generalizations discovered empirically (inductively).

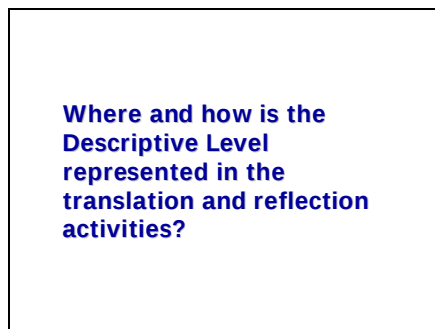
The language of the Descriptive Level includes words relating to properties within a given figure (e.g., parallel, perpendicular, and congruent). Various properties can be used to describe or define a figure, but this is usually a very uneconomical list of properties. A concise definition, using a sufficient number of properties rather than an exhaustive list, is not possible at this level. Students functioning at the Visual Level are not able to understand the properties of the Descriptive Level even with the help of pictures.

Slide
13

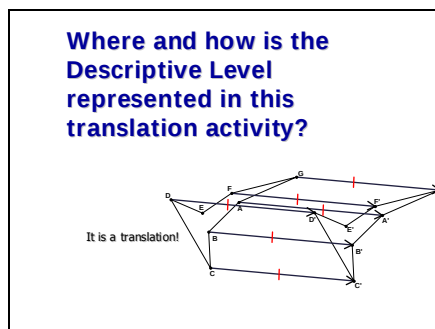


The isosceles triangles, with congruent vertices at the point or center of rotation, become an accepted property of rotations. Students at the Descriptive Level recognize a rotation by these properties.

Slide
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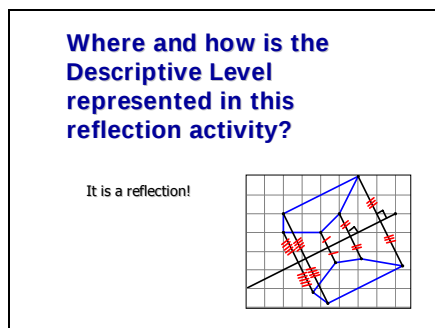


Slide
15



The congruent, parallel translation vectors become the defining property behind a translation. The symbols used to describe the vectors (e.g., parallel and congruent), both on the figure and in written geometric language, become a part of the formal language of the Descriptive Level.

Slide
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The line of reflection bisects the parallel segments that connect corresponding points on the pre-image and image. This property describes and defines a reflection. The congruency symbols, the perpendicular symbol, and the corresponding symbols used in written descriptions become part of the language of the Descriptive Level.

Slide
17

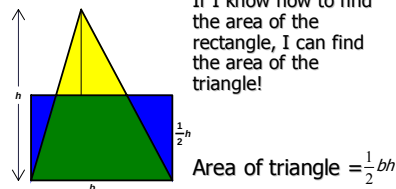
Relational Level Characteristics

- The student
- can define a figure using minimum (sufficient) sets of properties.
 - gives informal arguments, and discovers new properties by deduction.
 - follows and can supply parts of a deductive argument.
 - does NOT grasp the meaning of an axiomatic system, or see the interrelationships between networks of theorems.

The language of the Relational Level is based on ordering arguments which may have their origins at the Descriptive Level. For example, a figure may be described by an exhaustive list of properties at the Descriptive Level. At the Relational Level, it is possible to select one or two properties of the figure to determine whether these are sufficient to define the figure. The language is more abstract including causal, logical and other relations of the structure. A student at the Relational Level is able to determine relationships among figures, and to arrange arguments in an order in which each statement except the first one is the outcome of previous statements.

Slide
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Relational Level Example



In this example, the triangle is divided into parts which transform to form the rectangle. Properties of rotation, congruence, and conservation of area are utilized.

Slide
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Deductive Level

My experience as a teacher of geometry convinces me that all too often, students have not yet achieved this level of informal deduction. Consequently, they are not successful in their study of the kind of geometry that Euclid created, which involves formal deduction.

Pierre van Hiele

If the structures of the Visual, Descriptive, and Relational Levels are not sufficiently understood, then the student is not able to learn successfully from a traditional deductive approach as expected in most high school textbooks.

Slide
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Deductive Level Characteristics

- The student
- recognizes and flexibly uses the components of an axiomatic system (undefined terms, definitions, postulates, theorems).
 - creates, compares, contrasts different proofs.
 - does NOT compare axiomatic systems.

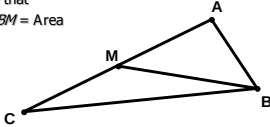
The Deductive Level is sometimes called the Axiomatic Level. The language of this level uses the symbols and sequence of formal logic.

Slide
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Deductive Level Example

In $\triangle ABC$, $\overline{BM}$ is a median.

I can prove that
Area of $\triangle ABM$ = Area
of $\triangle MBC$.



At this level, students are able to construct the steps of a proof using appropriate symbolic language.

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Rigor

The student

- compares axiomatic systems (e.g., Euclidean and non-Euclidean geometries).
- rigorously establishes theorems in different axiomatic systems in the absence of reference models.

The student is able to establish proofs and reach conclusions using the symbolic language of the system without the aid of visual cues.

Slide
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Phases of the Instructional Cycle

- Information
- Guided orientation
- Explicitation
- Free orientation
- Integration

The van Hiele model defines the different levels along with the accompanying language (symbols) for each level. The phases in the instructional cycle help students progress from lower to higher levels of thinking.

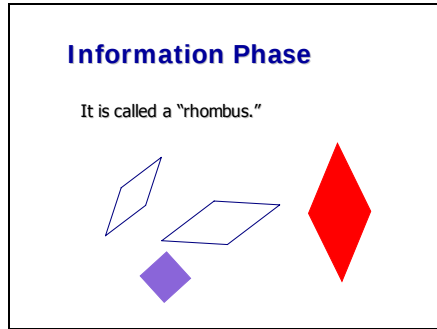
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Information Phase

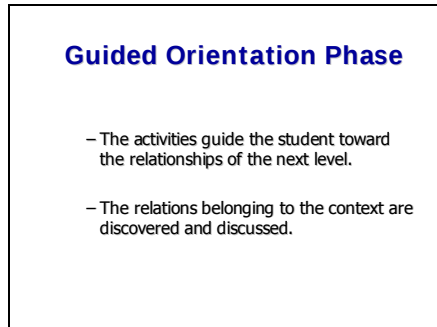
The teacher holds a conversation with the pupils, in well-known language symbols, in which the context he wants to use becomes clear.

In other models, this phase might be called knowledge, i.e., what the students already know about the context. Formal vocabulary associated with the concept is clarified.

Slide
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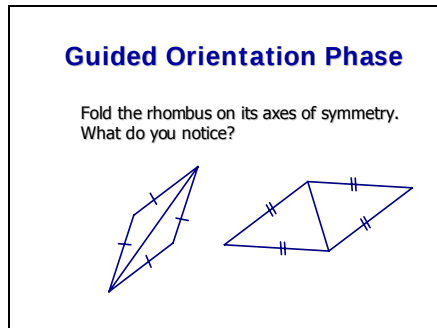


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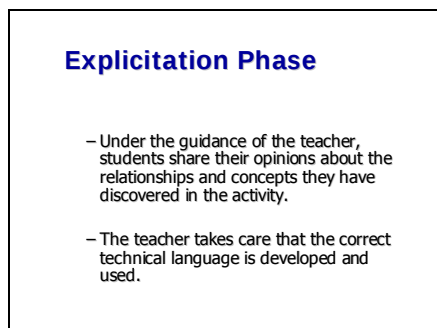


The teacher provides instructional activities in which students explore and discuss the concept, preferably in small groups, and come to a consensus about the concept within their groups. The teacher's role is to facilitate, provide hints, and ask scaffolding questions rather than to provide answers. Students should construct their own knowledge from their own thinking rather than rely on the teacher for direct information.

Slide
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Slide
28



After the class has completed the guided orientation activity, the teacher leads a whole class discussion. Each group shares its findings, and through discussion misconceptions are re-conceptualized. Even if groups share the same conclusions, much can be gained when students hear explanations in different words or from slightly different viewpoints. The van Hiele model emphasizes that this step is often short-changed in mathematics classrooms at the expense of student understanding and learning.

Slide
29**Explicitation Phase**

Discuss your ideas with your group, and then with the whole class.

- The diagonals lie on the lines of symmetry.
- There are two lines of symmetry.
- The opposite angles are congruent.
- The diagonals bisect the vertex angles.
- ...

Sometimes students arrive at the wrong conclusions. During the discussion, these misconceptions can be corrected, especially if other students, rather than the teacher, are able to explain to their peers.

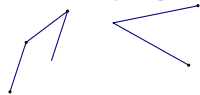
Slide
30**Free Orientation Phase**

- The relevant relationships are known.
- The moment has come for the students to work independently with the new concepts using a variety of applications.

After the whole class discussion, the teacher provides independent practice using the newly discovered relationships. This can be in the form of homework problems or extended investigations.

Slide
31**Free Orientation Phase**

The following rhombi are incomplete. Construct the complete figures.



Using the newly found properties of the rhombus, students should be able to work backwards to sketch these rhombi.

Slide
32**Integration Phase**

The symbols have lost their visual content and are now recognized by their properties.

Pierre van Hiele

The knowledge gained by completing the instructional cycle now forms the basis for the Information Phase of the next level of thinking.

Slide
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Integration Phase

Summarize and memorize the properties of a rhombus.

At the next level of thinking, students must know these properties for immediate recall.

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What we do and what we do not do...

- It is customary to illustrate newly introduced technical language with a few examples.
- If the examples are deficient, the technical language will be deficient.
- We often neglect the importance of the third stage, explicitation. Discussion helps clear out misconceptions and cements understanding.

The use of the appropriate language and its symbols for each level is critical if students are to progress to higher levels. If the language is mismatched for the level at which the student functions, then he/she will not be able to progress, even if the teacher attempts to use visual structures for explanation.

Slide
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What we do and what we do not do...

- Sometimes we attempt to inform by explanation, but this is useless. Students should learn by doing, not be informed by explanation.
- The teacher must give guidance to the process of learning, allowing students to discuss their orientations and by having them find their way in the field of thinking.

The traditional form of teaching, by modeling and explanation, is time efficient but not effective. Students learn best through personal exploration and active thinking. By answering students' questions, we rob students of the opportunity to develop good thinking habits. It is best to guide by asking probing or scaffolding questions, making suggestions, and waiting.

Slide
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Instructional Considerations

- Visual to Descriptive Level
 - Language is introduced to describe figures that are observed.
 - Gradually the language develops to form the background to the new structure.
 - Language is standardized to facilitate communication about observed properties.
 - It is possible to see congruent figures, but it is useless to ask why they are congruent.

Slide
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Instructional Considerations

- Descriptive to Relational Level
 - Causal, logical or other relations become part of the language.
 - Explanation rather than description is possible.
 - Able to construct a figure from its known properties but not able to give a proof.

Language is critical in moving students through the hierarchy.

Slide
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Instructional Considerations

- Relational to Deductive Level
 - Reasons about logical relations between theorems in geometry.
 - To describe the reasoning to someone who does not "speak" this language is futile.
 - At the Deductive Level it is possible to arrange arguments in order so that each statement, except the first one, is the outcome of the previous statements.

Slide
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Instructional Considerations

- Rigor
 - Compares axiomatic systems.
 - Explores the nature of logical laws.

"Logical Mathematical Thinking"

Slide
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Consequences

- Many textbooks are written with only the integration phase in place.
- The integration phase often coincides with the objective of the learning.
- Many teachers switch to, or even begin, their teaching with this phase, a.k.a. "direct teaching."
- Many teachers do not realize that their information cannot be understood by their pupils.

Slide
41

Children whose geometric thinking you
nurture carefully will be better able to
successfully study the kind of
mathematics that Euclid created.

Pierre van Hiele

The following table provides more detailed descriptors for what a student should be able to do at each van Hiele level. These are for your reference or for anyone requiring more information. The descriptors were adapted from Fuys, Geddes, & Tischler (1988).

Level	Descriptors
Visual Level	1. Identifies instances of a shape by its appearance as a whole in ▪ a simple drawing, diagram or set of cut-outs. ▪ different positions. ▪ a shape or other more complex configurations. 2. Constructs (using craft or commercial materials), draws, or copies a shape. 3. Names or labels shapes and other geometric configurations and uses standard and/or nonstandard names and labels appropriately. 4. Compares and sorts shapes on the basis of their appearance as a whole. 5. Verbally describes shapes by their appearance as a whole. 6. Solves routine problems by operating on shapes rather than by using properties which apply in general (e.g., by overlaying, measuring, counting). 7. Identifies parts of a figure but ▪ does NOT analyze a figure in terms of its components. ▪ does NOT think of properties as characterizing a class of figures. ▪ does NOT make generalizations about shapes or use related language.

Descriptive Level	1. Identifies and tests relationships among components of figures (e.g., congruence of opposite sides of a parallelogram; congruence of angles in a tiling pattern). 2. Recalls and uses appropriate vocabulary for components and relationships (e.g., corresponding angles are congruent, diagonals bisect each other). 3. Compares two shapes according to relationships among their components. Sorts shapes in different ways according to certain properties. This may include sorting figures by class or non-class. 4. Interprets and uses verbal description of a figure in terms of its properties and uses this description to draw/construct the figure. Interprets verbal or symbolic statements of rules and applies them. 5. Discovers properties of specific figures empirically and generalizes properties for that class of figures. 6. Describes a class of figures (e.g., parallelograms) in terms of its properties. Given certain properties, identifies a shape. 7. Identifies which properties used to characterize one class of figures also apply to another class of figures and compares classes of figures according to their properties. 8. Discovers properties of an unfamiliar class of figures. 9. Solves geometric problems by using known properties of figures or by insightful approaches. 10. Formulates and uses generalizations about properties of figures (guided by teacher/material or spontaneously on own) and uses related language (e.g., all, every, none) but ▪ does NOT explain how certain properties of a figure are interrelated. ▪ does NOT formulate and use formal definitions (e.g., defines a figure by listing many properties rather than identifying a set of necessary or sufficient properties). ▪ does NOT explain subclass relationships beyond checking specific instances against given list of properties (e.g., after listing properties of all quadrilaterals cannot explain why “all rectangles are not squares”). ▪ does NOT see a need for proof or logical explanations of generalizations discovered empirically and does NOT use related language (e.g., if-then, because) correctly.
---------------------------------	--

Relational Level	1. Identifies different sets of properties that characterize a class of figures and tests that these are sufficient. Identifies minimum sets of properties that can characterize a figure. Formulates and uses a definition for a class of figures. 2. Gives informal arguments (using diagrams, cutout shapes that are folded, or other materials). ▪ Having drawn a conclusion from given information, justifies the conclusion using logical relationships. ▪ Orders classes of shapes. ▪ Orders two properties. ▪ Discovers new properties by deduction. ▪ Interrelates several properties in a family tree. 3. Gives informal deductive arguments. ▪ Follows a deductive argument and can supply parts of an argument. ▪ Gives a summary or variation of a deductive argument. ▪ Gives deductive arguments on own. 4. Gives more than one explanation to prove something and justifies these explanations by using family trees. 5. Informally recognizes difference between a statement and its converse. 6. Identifies and uses strategies or insightful reasoning to solve problems. 7. Recognizes the role of deductive argument and approaches problems in a deductive manner but ▪ does NOT grasp the meaning of deduction in an axiomatic sense (e.g., does NOT see the need for definitions and basic assumptions). ▪ does NOT formally distinguish between a statement and its converse. ▪ does NOT yet establish interrelationships between networks of theorems.
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Deductive Level	1. Recognizes the need for undefined terms, definitions, and basic assumptions (postulates). 2. Recognizes characteristics of a formal definition (e.g., necessary and sufficient conditions) and equivalence of definitions. 3. Proves in axiomatic setting relationships that were explained informally in the relational level. 4. Proves relationships between a theorem and related statements (e.g., converse, inverse, contrapositive). 5. Establishes interrelationships among networks of theorems. 6. Compares and contrasts different proofs of theorems. 7. Examines effects of changing an initial definition or postulate in a logical sequence. 8. Establishes a general principle that unifies several different theorems. 9. Creates proofs from simple sets of axioms, frequently using a model to support arguments. 10. Gives formal deductive arguments but does NOT investigate the axiomatics themselves or compare axiomatic systems.
Rigor	1. Rigorously establishes theorems in different axiomatic systems. 2. Compares axiomatic systems (e.g., Euclidean and non-Euclidean geometries); spontaneously explores how changes in axioms affect the resulting geometry. 3. Establishes consistency of a set of axioms, independence of an axiom, and equivalency of different sets of axioms; creates an axiomatic system. 4. Invents generalized methods for solving classes of problems. 5. Searches for the broadest context in which a mathematical theorem/principle will apply. 6. Does in-depth study of the subject logic to develop new insights and approaches logical inference.

The van Hiele Model of Geometric Thought

Use this sheet and additional paper if needed to take notes from the van Hiele PowerPoint presentation.

Levels of Geometric Understanding

Student characteristics:

Visual Level

Descriptive Level

Relational Level

Deductive Level

Rigor

Phases in the Instructional Cycle:

Knowledge

Guided Orientation

Explicitation

Free Orientation

Integration

Important key instructional considerations:

Rotations

- Overview:** In this activity, participants explore the properties of rotations by rotating a variety of figures about different rotation points using different angles of rotation.
- Objective:** **TEExES Mathematics Competencies**
III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.014.A. The beginning teacher identifies transformations (i.e., reflections, translations, glide-reflections, rotations, dilations) and explores their properties.
III.014.C. The beginning teacher uses transformations to explore and describe reflectional, rotational, and translational symmetry.
III.014.D. The beginning teacher applies transformations in the coordinate plane.
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.
VI.020.A. The beginning teacher applies research-based theories of learning mathematics to plan appropriate instructional activities for all students.
- Geometry TEKS**
b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
d.2.A. The student uses one-and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.
- Background:** Participants must be able to recognize and apply different angle measurements and to know and apply clockwise and counter-clockwise movements. An understanding of translations and reflections is helpful.
- New Terms:** identity rotation, point of rotation
- Materials:** centimeter ruler, patty paper, protractor, colored pencils, centimeter grid paper, small colored dot, transparency sheets (1 per group of 4), two overhead pens of different colors (for each group)

Procedures:

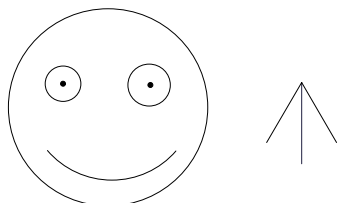
Begin by asking a participant to demonstrate a rotation about a given point, represented by the colored dot on the floor. There are two possible ways to rotate about this point. The participant may stand on the point and turn in a circular motion or he/she may stand off the point and walk around it in a circular path. Depending on the method chosen by the participant, ask for a second volunteer to demonstrate the other possible rotation.

What is a rotation?

For now, accept all definitions from participants. The purpose is just to get participants thinking about rotations. Possible responses may include a spin on a point or a turn about a point.

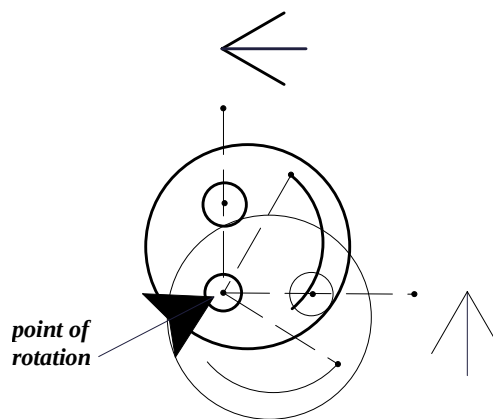
Where are rotations used in the real world?

Possible answers are: architecture, art, kaleidoscopes, tri-mirrors in dressing rooms at clothing stores, Ferris wheels, and an ice skater spinning about a point.

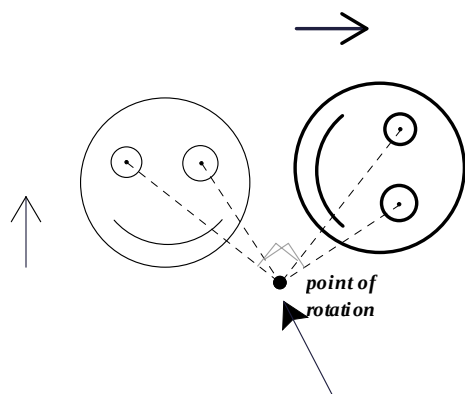


Model how to draw a rotation using patty paper. Ask participants to draw a smiling circular face on patty paper. This is the pre-image. A small arrow pointing upwards can be drawn on the paper to indicate the orientation of the pre-image.

Model or discuss how to rotate the figure 90° counterclockwise about one of the eyes. The *point of rotation* is the vertex of the angle of rotation; it is a fixed point. Place the pencil on one of the dots in the middle of an eye to indicate the point of rotation. The angle of rotation can be directed counter-clockwise (having a positive measure) or clockwise (having a negative measure).



Place a second piece of patty paper on top of the smiling face and trace the face. With the pencil on the point of rotation (the dot within one of the eyes), turn the top piece of patty paper 90° left (counter-clockwise) while keeping the bottom piece of patty paper still. The resulting figure is the image of a 90° counter-clockwise rotation. Trace the pre-image on the top sheet of patty paper, and draw the angle of rotation with the point of rotation as the vertex of the angle. Using a colored pencil, draw in other angles of rotation by connecting points on the pre-image to the point of rotation and then to the corresponding points on the image.



Next, model how to rotate the face from a point not on the figure. Place a point off of the smiling face and use it as the point of rotation. Ask participants to rotate the smiling face about a point not on the smiling face -90° , which is a 90° clockwise rotation. Use a colored pencil to draw the angles of rotation.

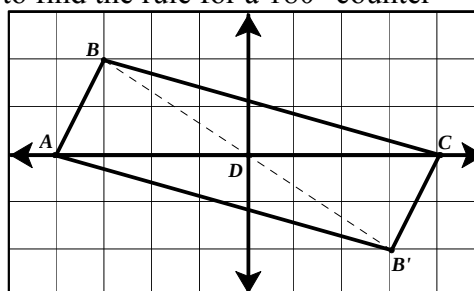
Participants work 1 – 4 to identify the properties of rotations. When participants have completed 1 – 4, conduct a whole group discussion on the properties of rotations.

Possible properties participants may identify are:

- Rotations preserve congruence of the two shapes.
- The distance (side length), angle measure, perpendicularity and parallelism are all preserved.
- Rotations are isometries because congruence of size and shape are preserved.
- 180° clockwise and 180° counter-clockwise rotations are identical to each other.

- Rotate $\triangle ABC$ 180° counter-clockwise about point D . Label the corresponding vertices. Write the coordinates in the table below in order to find the rule for a 180° counter-clockwise rotation.

Pre-image	Image
$A (-4, 0)$	$A' (4, 0)$
$B (-3, 2)$	$B' (3, -2)$
$C (4, 0)$	$C' (-4, 0)$
Rule: $(x, y) \rightarrow$	$(-x, -y)$

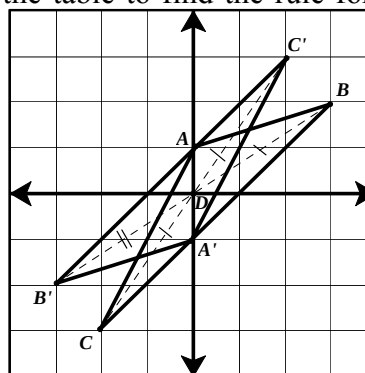


Angles of rotation: $\angle BDB'$, $\angle CDC'$ and $\angle ADA'$

- Rotate $\triangle ABC$ 180° clockwise about point D . Write the coordinates of the vertices of the pre-image and the image in the table below. Use the table to find the rule for an 180° clockwise rotation.

Pre-image	Image
$A (0, 1)$	$A' (0, -1)$
$B (3, 2)$	$B' (-3, -2)$
$C (-2, -3)$	$C' (2, 3)$
Rule: $(x, y) \rightarrow$	$(-x, -y)$

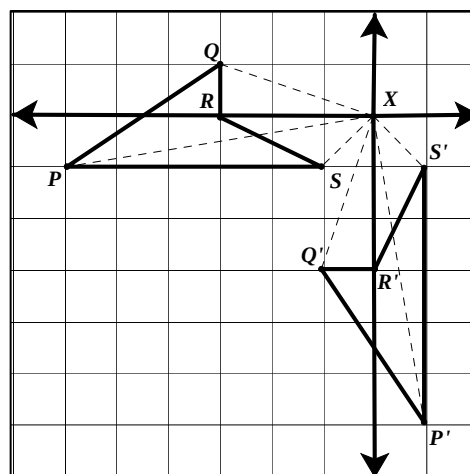
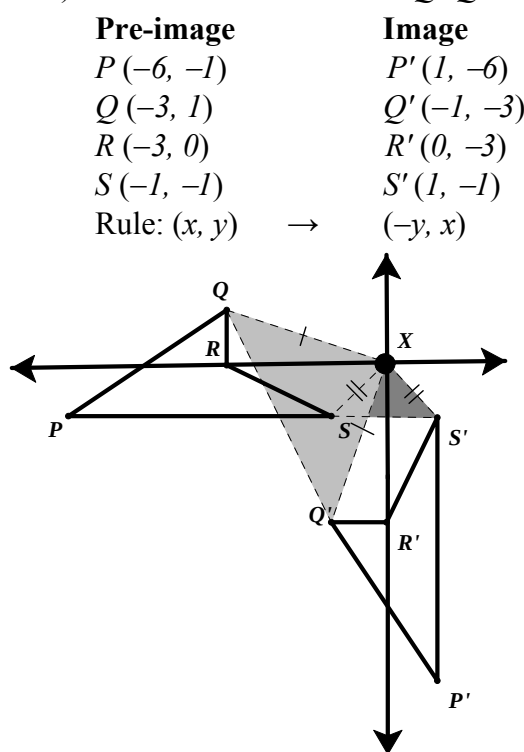
Angles of rotation: $\angle BDB'$, $\angle CDC'$ and $\angle ADA'$



In order to fully understand the geometry of rotations, carefully draw the isosceles triangles for at least one of the rotations in 2, 3 or 4. Connect a pre-image point to a corresponding image point, in addition to the associated angle of rotation. The vertex of this isosceles triangle is at the point of rotation. Repeat for a second pair of points on the same figure. See the answers below for a full development of this property of rotations, which is required for 6 and 7.

3. Rotate quadrilateral $PQRS$ 90° counter-clockwise about point X .

- Write the coordinates of the vertices of the pre-image and the image in the table below. Use the table to find the rule for a 90° counter-clockwise rotation.
- Draw isosceles triangles QXQ' and SXS' . Shade each triangle in a different color. Mark congruent segments.
- Write the measures of $\angle QXQ'$ and $\angle SXS'$.

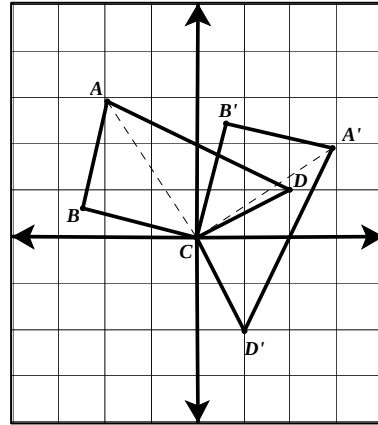
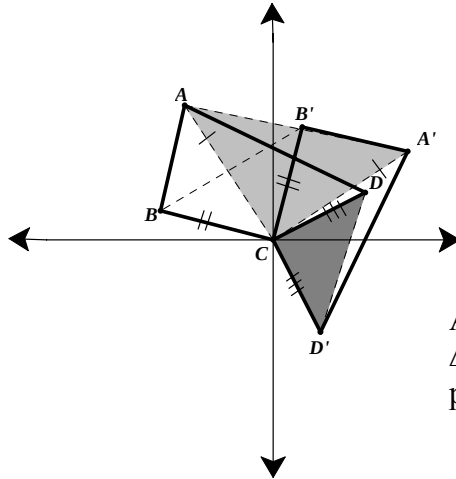


In this solution, two of the four possible isosceles triangles have been drawn. Triangle QXQ' and $\triangle SXS'$ are isosceles triangles. The vertices of both triangles meet at the point of rotation, X .

4. Rotate quadrilateral $ABCD$ 90° clockwise about point C .

- Write the coordinates of the vertices of the pre-image and the image in the table below. Use the table to find the rule for a 90° clockwise rotation.
- Draw isosceles triangles ACA' and DCD' . Shade each triangle in a different color. Mark congruent segments.
- Write the measures of $\angle ACA'$ and $\angle DCD'$.

Pre-image	Image
$A (-2, 3)$	$A' (3, 2)$
$B (-2.5, 0.5)$	$B' (0.5, 2.5)$
$C (0, 0)$	$C' (0, 0)$
$D (2, 1)$	$D' (1, -2)$
Rule: $(x, y) \rightarrow$	$(y, -x)$



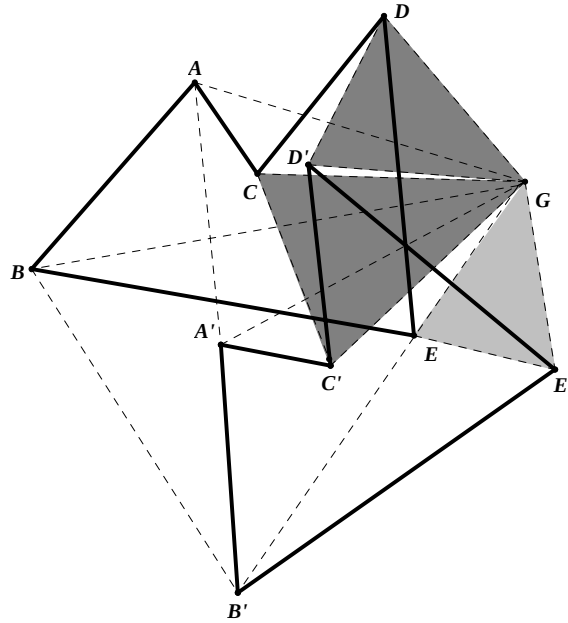
All three isosceles triangles, $\triangle ACA'$, $\triangle BCB'$, and $\triangle DCD'$, have a common vertex, C , which is the point of rotation.

Participants work 5 – 9. Construct the rotations using a ruler and protractor without the use of patty paper. In 8 and 9, participants determine the point of rotation and the angle of rotation.

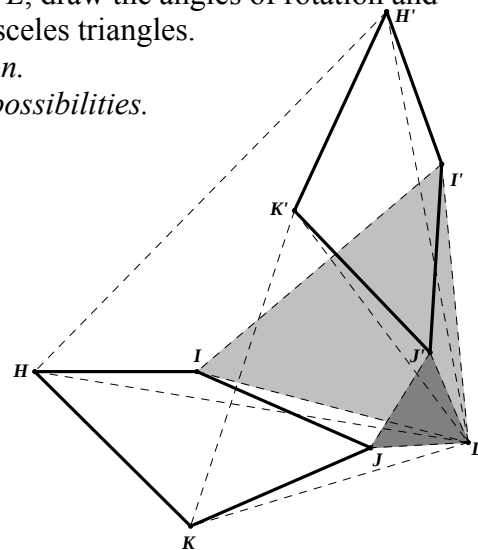
The role of the instructor is to walk around the room, listen and facilitate. Try *not* to directly answer participants' questions, but rather provide minimal prompts and/or ask questions to help participants clarify their thinking. When most of the participants have completed 5 – 9, give each group an overhead transparency sheet and two different colored overhead pens. Each group prepares an answer to one of 5 – 9 for presentation to the class.

- Rotate the pentagon $ABEDC$ 45° counter-clockwise about point G , draw the angles of rotation and connect the corresponding vertices to form isosceles triangles.

The figure below represents the correct rotation. The isosceles triangles shown are three of five possibilities.



6. Rotate the quadrilateral $HIJK$ -70° about point L , draw the angles of rotation and connect the corresponding vertices to form isosceles triangles.
The figure below represents the correct rotation.
The isosceles triangles shown are two of four possibilities.

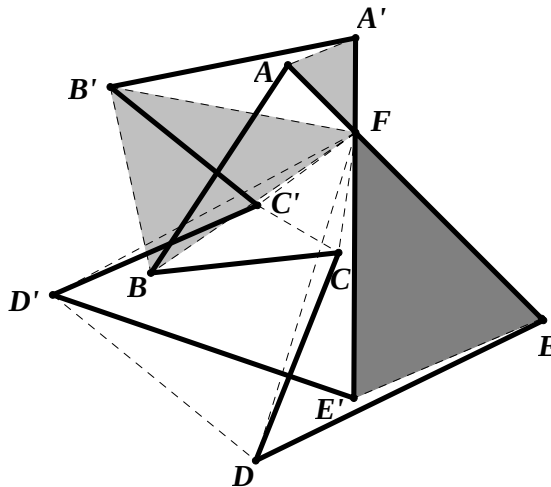


7. Describe the effect of rotations of magnitude 0° and 360° on a figure.
The image coincides with its pre-image. Each pre-image point is mapped on the corresponding image point. A rotation of magnitude of 0° or 360° is the identity rotation.

Find the point of rotation and the angle of rotation in 8 and 9.

Draw a segment connecting a pre-image point to its corresponding image point. Find the midpoint. Using the corner of a sheet of paper, draw a line through the midpoint, perpendicular to the segment. Repeat for another pair of points. The two perpendicular lines pass through the vertex of the isosceles triangles. The vertex is the point of rotation. Measure the vertex angles of the isosceles triangles to find the angle of rotation.

8. Point F is the point of rotation because all of the isosceles triangles share a vertex at F . The angle of rotation is 45° clockwise.



$$m\angle AFA' = 45^\circ$$

$$m\angle BFB' = 45^\circ$$

$$m\angle CFC' = 45^\circ$$

$$m\angle DFD' = 45^\circ$$

$$m\angle EFE' = 45^\circ$$

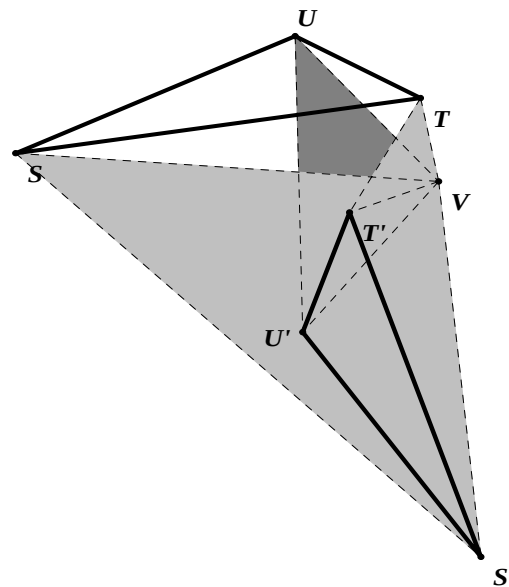
9. Point V is the point of rotation, the vertex of the isosceles triangles $\triangle UVU'$, $\triangle TVT'$, and $\triangle SVS'$.

The angle of rotation is 100° .

$$m\angle UVU' = 100^\circ$$

$$m\angle TVT' = 100^\circ$$

$$m\angle SVS' = 100^\circ$$



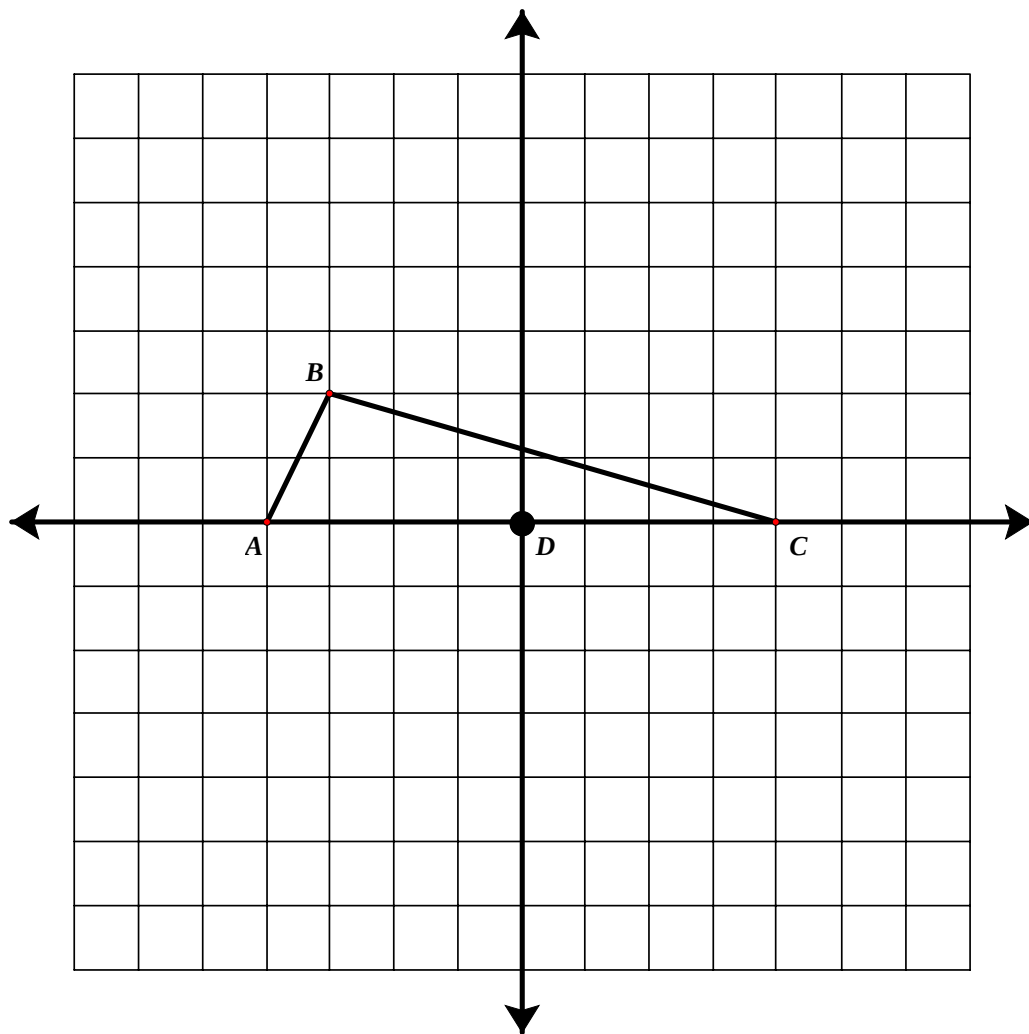
At the end of the activity, summarize by asking participants to discuss which van Hiele levels are represented.

The Visual Level is represented in the walking demonstration as participants are asked to experience rotation holistically. Success with the rest of the activity indicates participants are at the Descriptive Level as they are developing and applying properties of transformations.

Rotations

For each of the following figures construct the given rotations. Draw each angle of rotation in a different color.

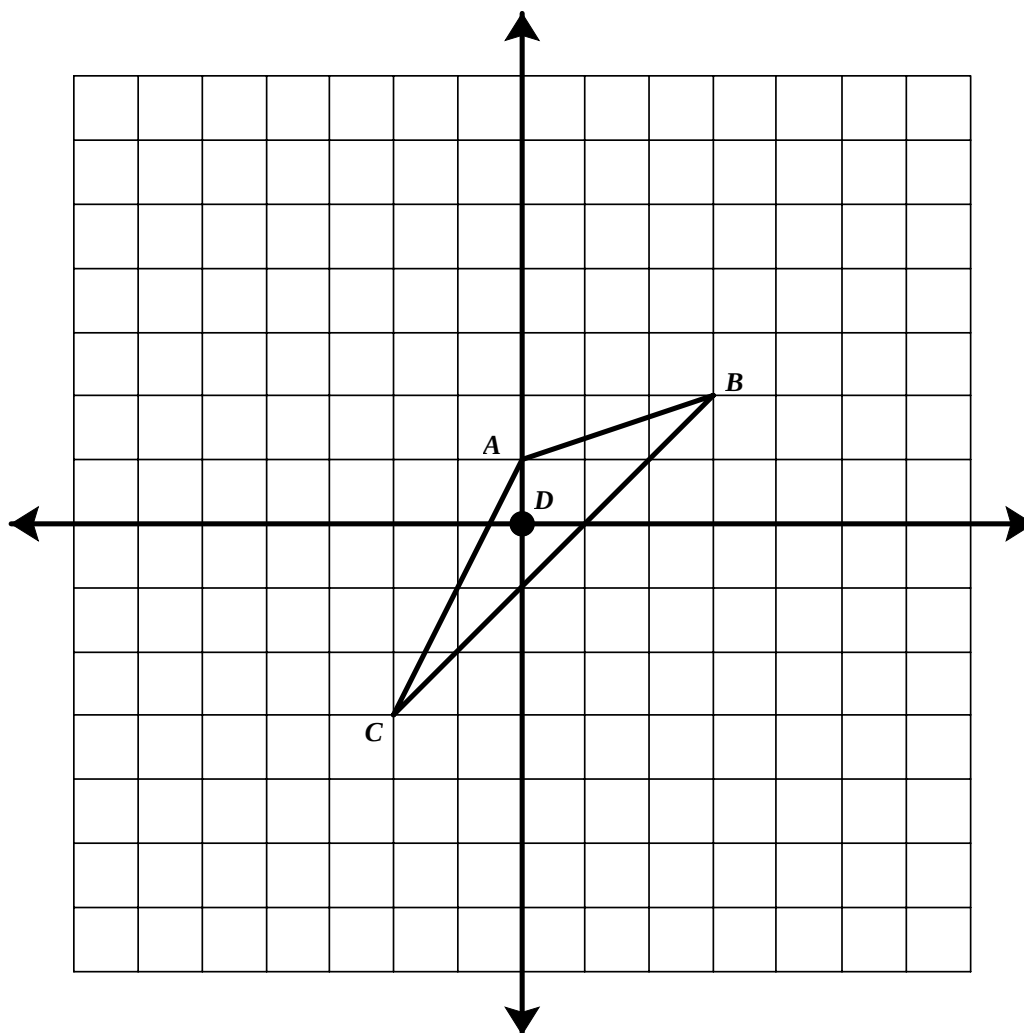
1. Rotate $\triangle ABC$ 180° counter-clockwise about point D . Label the corresponding vertices. Write the coordinates in the table below in order to find the rule for a 180° counter-clockwise rotation.



	Pre-image	Image
A		
B		
C		

Rule: $(x, y) \rightarrow$

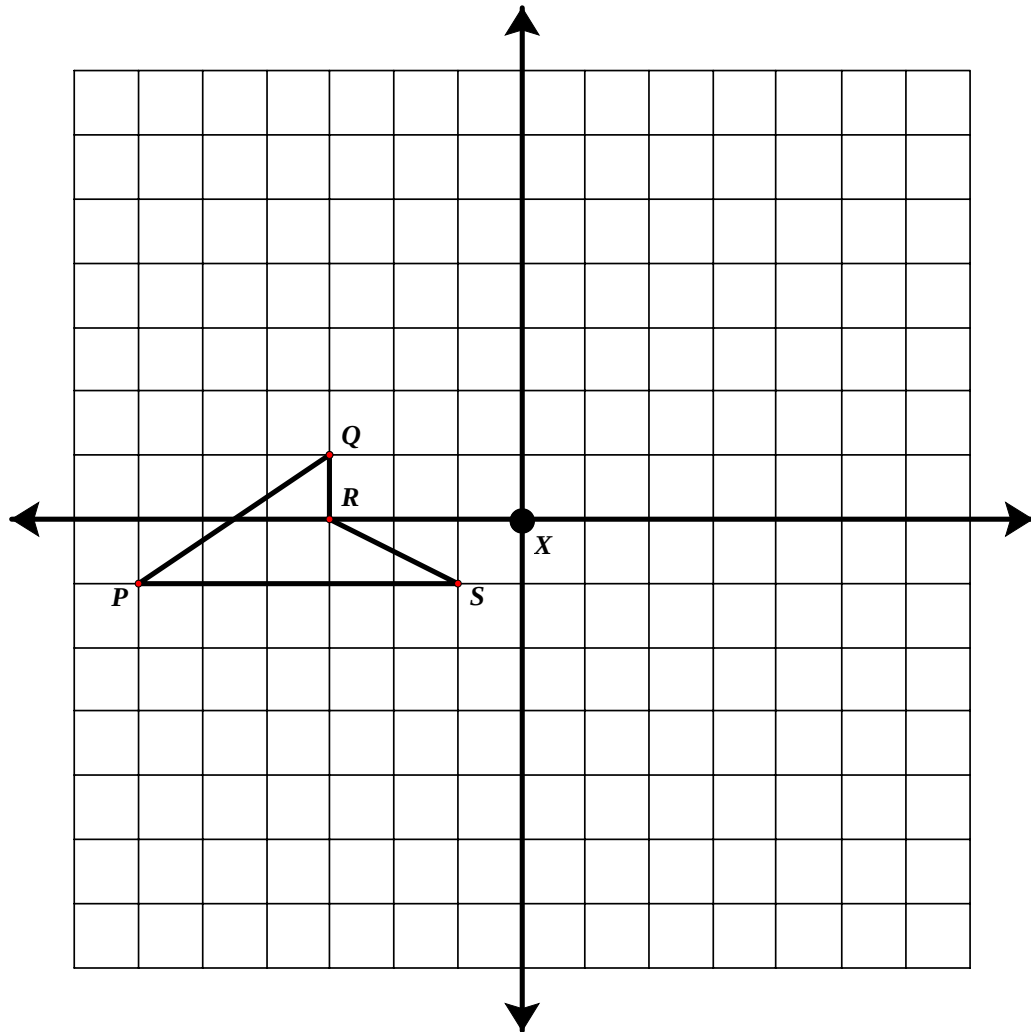
2. Rotate $\triangle ABC$ 180° clockwise about point D . Write the coordinates of the vertices of the pre-image and the image in the table below. Use the table to find the rule for an 180° clockwise rotation.



	Pre-image	Image
A		
B		
C		

Rule: $(x, y) \rightarrow$

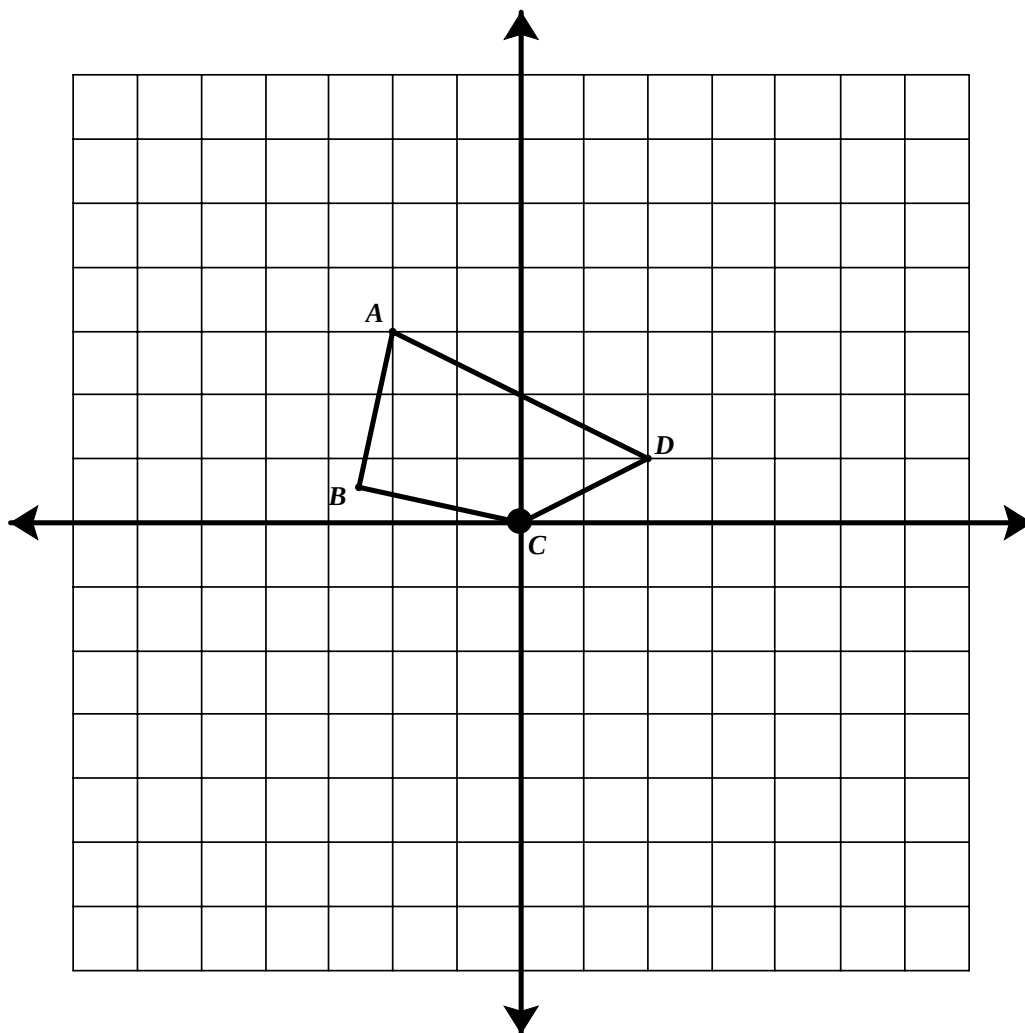
3. Rotate quadrilateral $PQRS$ 90° counter-clockwise about point X .
- Write the coordinates of the vertices of the pre-image and the image in the table below. Use the table to find the rule for a 90° counter-clockwise rotation.
 - Draw isosceles triangles QXQ' and SXS' . Shade each triangle in a different color. Mark congruent segments.
 - Write the measures of $\angle QXQ'$ and $\angle SXS'$.



	Pre-image	Image
P		
Q		
R		
S		

Rule: $(x, y) \rightarrow$

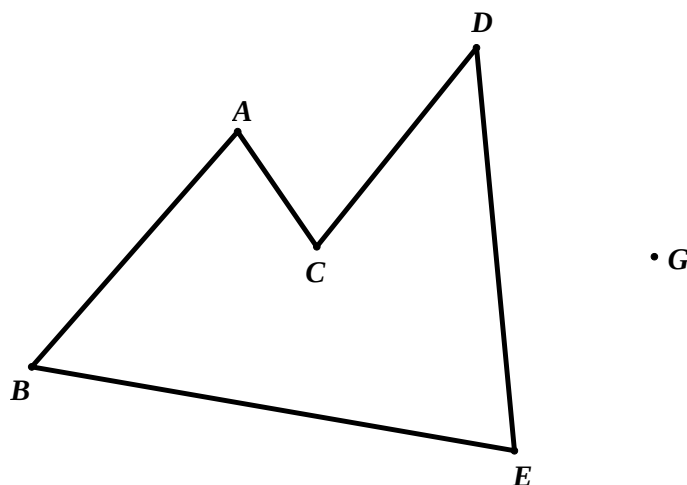
4. Rotate quadrilateral $ABCD$ 90° clockwise about point C .
- Write the coordinates of the vertices of the pre-image and the image in the table below. Use the table to find the rule for a 90° clockwise rotation.
 - Draw isosceles triangles ACA' and DCD' . Shade each triangle in a different color. Mark congruent segments.
 - Write the measures of $\angle ACA'$ and $\angle DCD'$.



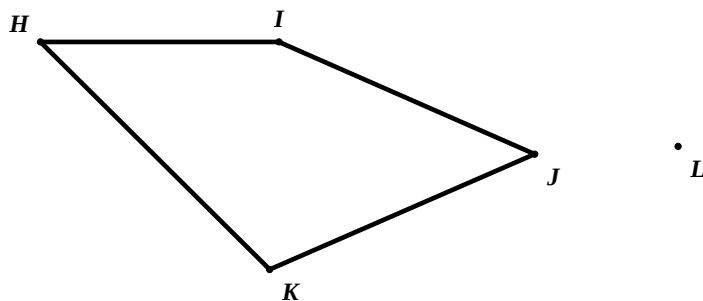
	Pre-image	Image
A		
B		
C		
D		

Rule: $(x, y) \rightarrow$

5. Rotate the pentagon $ABEDC$ 45° counter-clockwise about point G , draw the angles of rotation and connect the corresponding vertices to form isosceles triangles.



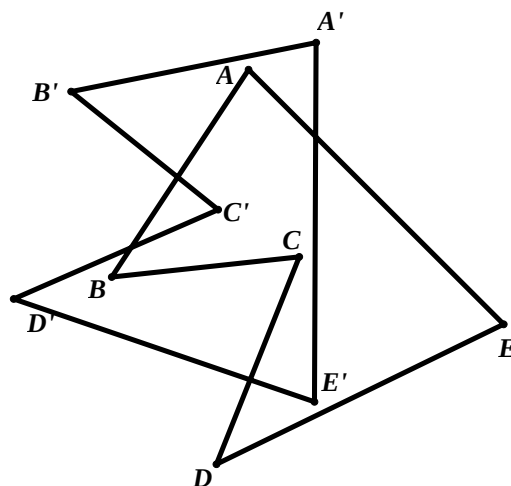
6. Rotate the quadrilateral $HIJK$ -70° about point L , draw the angles of rotation and connect the corresponding vertices to form isosceles triangles.



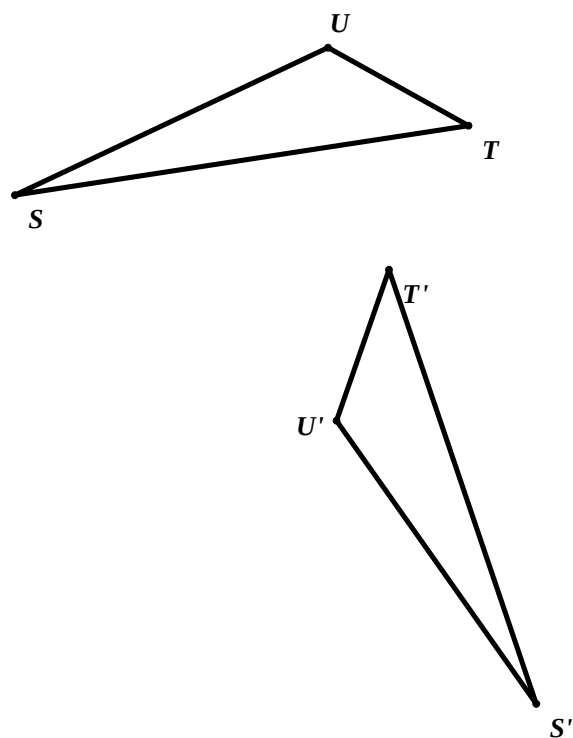
7. Describe the effect of rotations of magnitude 0° and 360° on a figure.

Find the points of rotation and the angles of rotation in 8 and 9.

8.



9.



Composite Transformations

Overview: Participants reflect figures over two lines of reflection to produce a composite transformation.

Objective: **TEExES Mathematics Competencies**
III.014.A. The beginning teacher identifies transformations (i.e., reflections, translations, glide-reflections, rotations, dilations) and explores their properties.
III.014.D. The beginning teacher applies transformations in the coordinate plane.
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

Geometry TEKS
c.2. The student uses properties of transformations and their compositions to make connections between mathematics and the real world in applications such as tessellations or fractals.
d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.

Background: Participants need to have a general knowledge of transformations.

Materials: protractor, centimeter ruler, transparency sheets (1 per group of 4), transparency pens in at least two different colors

New Terms: composite transformation or glide-reflection

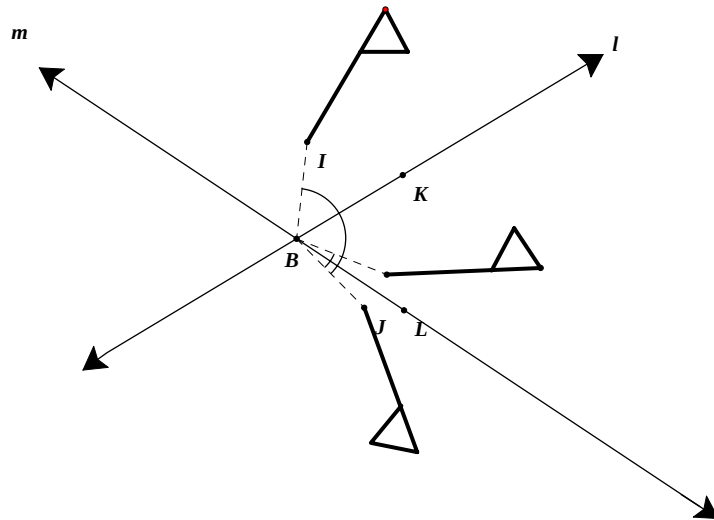
Procedures

This activity may be used as a homework assignment. If time permits, have participants complete the activity in class. Participants work individually, and then compare answers with their group members.

A *composite transformation* is created when one transformation follows another transformation. For example, a reflection is followed by a reflection, or a reflection is followed by a translation.

1. Create a composite reflection by reflecting the flag pre-image over line l and then reflecting the image over line m .
2. Measure the acute angle created by the intersection of lines l and m . For each of the two reflections, measure the angles created between a pre-image point, the point of intersection of l and m , and the corresponding image point. What conclusion can be drawn?

When a figure is reflected about two intersecting lines, the resulting image is the same as that from a rotation. The measure of the angle of rotation is twice the measure of the angle between the intersecting lines. The point of intersection between the two lines is the point of rotation.

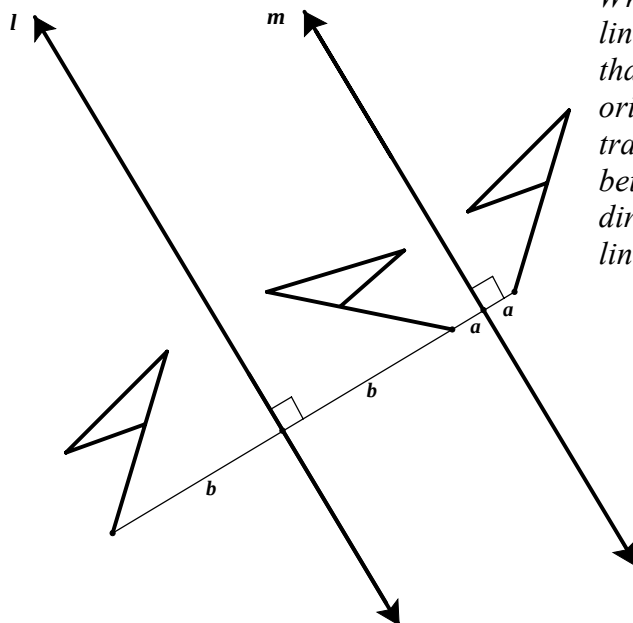


$$m\angle IBJ = 129^\circ$$

$$m\angle KBL = 64.5^\circ$$

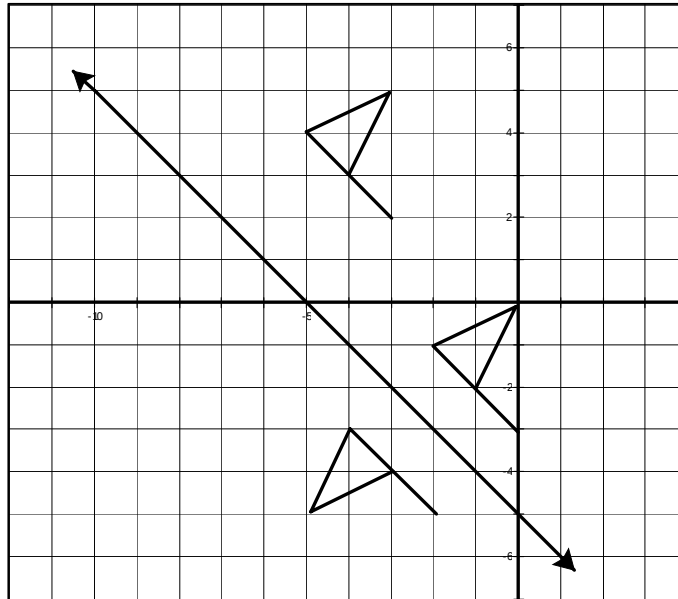
$$m\angle IBJ = 2m\angle KBL$$

3. Create a composite reflection over parallel lines. Reflect the flag figure about line m and then reflect the image over line l . Write a conclusion about a composite reflection over two parallel lines.



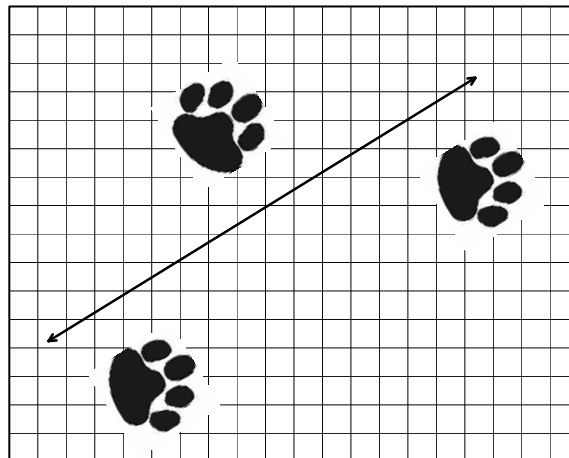
When a figure is reflected over parallel lines, the resulting image is the same as that created from translating the original figure. The magnitude of the translation vector is twice the distance between the parallel reflection lines. The direction of the translation is along the line perpendicular to the parallel lines.

4. Translate the flag figure with the rule $(x, y) \rightarrow (x + 3, y - 5)$. Then reflect the image across line m . The composite transformation is called a *glide-reflection*. Describe the properties of a glide-reflection.



A glide-reflection resembles footprints on either side of a reflection line. The translation and reflection can be performed in any order resulting in the same image.

5. Use the grid below to create “footprints in the sand”. Draw a foot at one end of the grid below and create a set of footprints using the properties of glide reflections. A possible answer is shown.



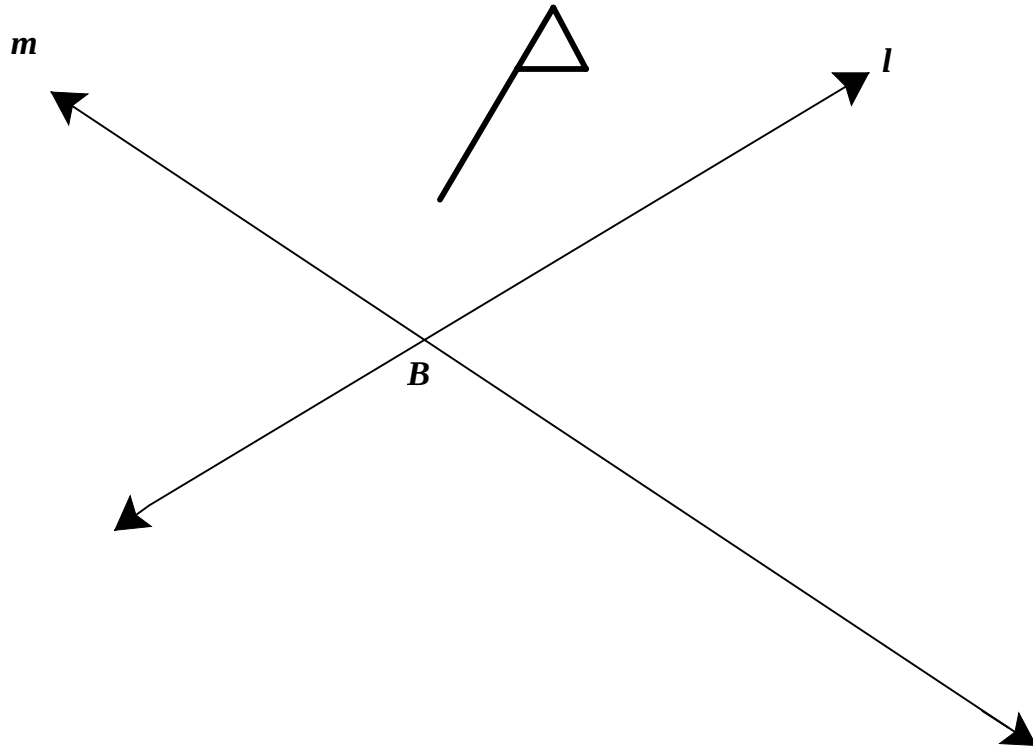
Ask participants to add the new terms composite transformation and glide-reflection to their glossaries.

Success with this activity indicates that participants are at the Descriptive Level with respect to congruence transformations as a whole, because they apply properties of each transformation in each of the steps and then identify a transformation based on its properties.

Composite Transformations

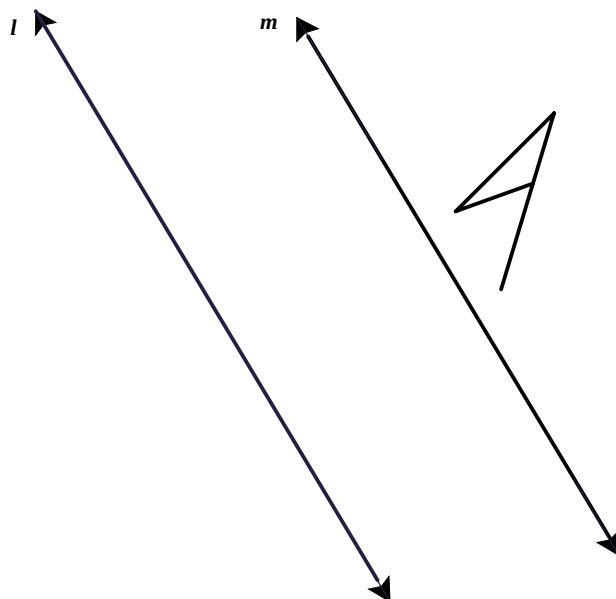
A composite transformation is created when one transformation follows another transformation. For example, a reflection is followed by a reflection, or a reflection is followed by a translation.

1. Create a composite reflection by reflecting the flag pre-image over line l and then reflecting the image over line m .

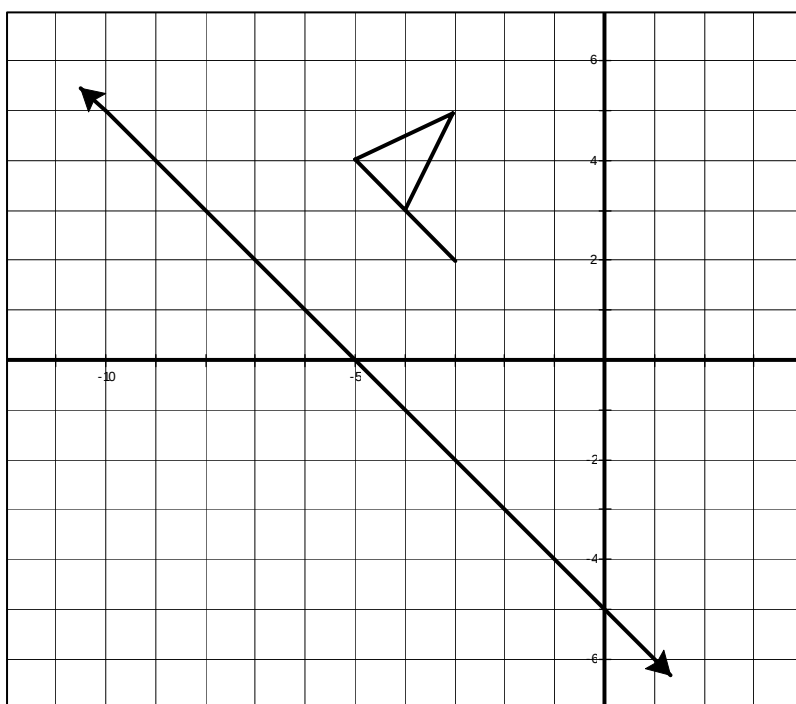


2. Measure the acute angle created by the intersection of lines l and m . For each of the two reflections, measure the angles created between a pre-image point, the point of intersection of l and m , and the corresponding image point. What conclusion can be drawn?

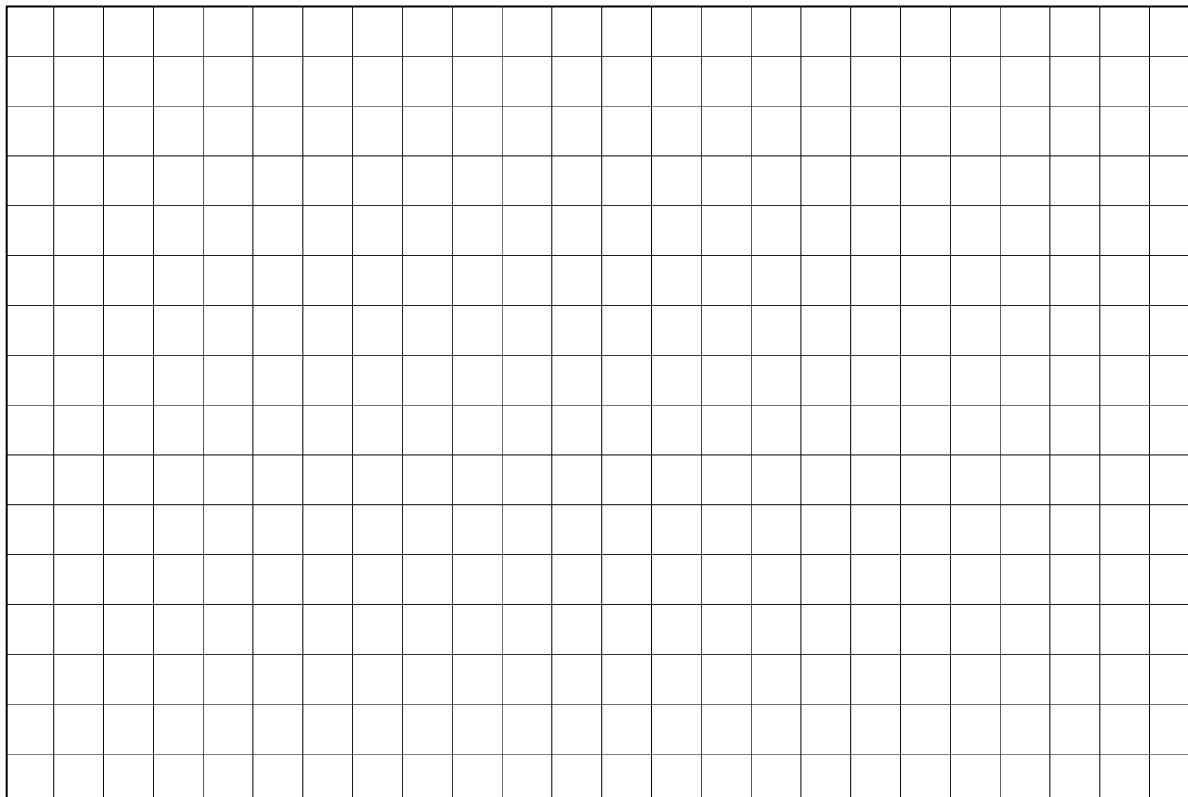
3. Create a composite reflection over parallel lines. Reflect the flag figure about line m and then reflect the image over line l . Write a conclusion about a composite reflection over two parallel lines.



4. Translate the flag figure with the rule $(x, y) \rightarrow (x + 3, y - 5)$. Then reflect the image across line m . The composite transformation is called a glide-reflection. Describe the properties of a glide-reflection.



5. Use the grid below to create “footprints in the sand”. Draw a foot at one end of the grid below and create a set of footprints using the properties of glide reflections.



Tessellations

- Overview:** Participants explore the properties of triangles and parallel lines by tessellating a scalene triangle.
- Objective:** **TExES Mathematics Competencies**
III.012.C. The beginning teacher applies the properties of parallel and perpendicular lines to solve problems.
III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
III.014.B. The beginning teacher uses the properties of transformations and their compositions to solve problems.
III.014.C. The beginning teacher uses transformations to explore and describe reflectional, rotational, and translational symmetry.
- Geometry TEKS**
b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
c.2. The student uses properties of transformations and their compositions to make connections between mathematics and the real world in applications such as tessellations or fractals.
e.2.A. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties of parallel and perpendicular lines.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.
- Background:** Participants must understand translations and rotations.
- Materials:** centimeter ruler, patty paper, protractor, colored pencils, unlined 8.5 in. by 11 in. paper, 3 in. by 5 in. index card cut in half, 11 in. by 17 in. sheet of paper, colored markers, masking tape, easel paper
- New Terms:** tessellation

Procedures:

Look around the room for a tessellation. Be ready to explain what you know about tessellations. Possible answers are the tiles on the ceiling or floor. Explain that a *tessellation* is a pattern made up of one or more shapes, completely tiling the surface, with no gaps and no overlaps. The original figure is called the fundamental region.

According to the dictionary, tessellate means to form or arrange small squares in a checkered or mosaic pattern. The word tessellate is derived from the Ionic version of the Greek word tesseres, which, in English, means four. The first tessellations were made from square tiles (Alejandre, 2004).

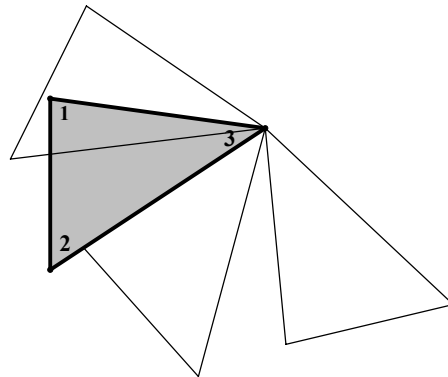
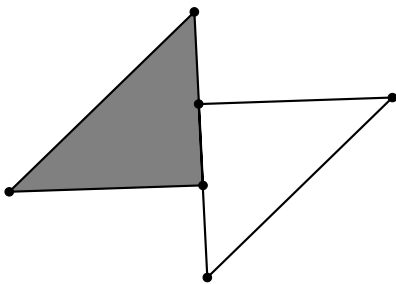
What properties are required in order for a figure to tessellate?

At each vertex, the angle measures must add to 360° ; if vertices meet at a point on a line, the angle measures must add to 180° .

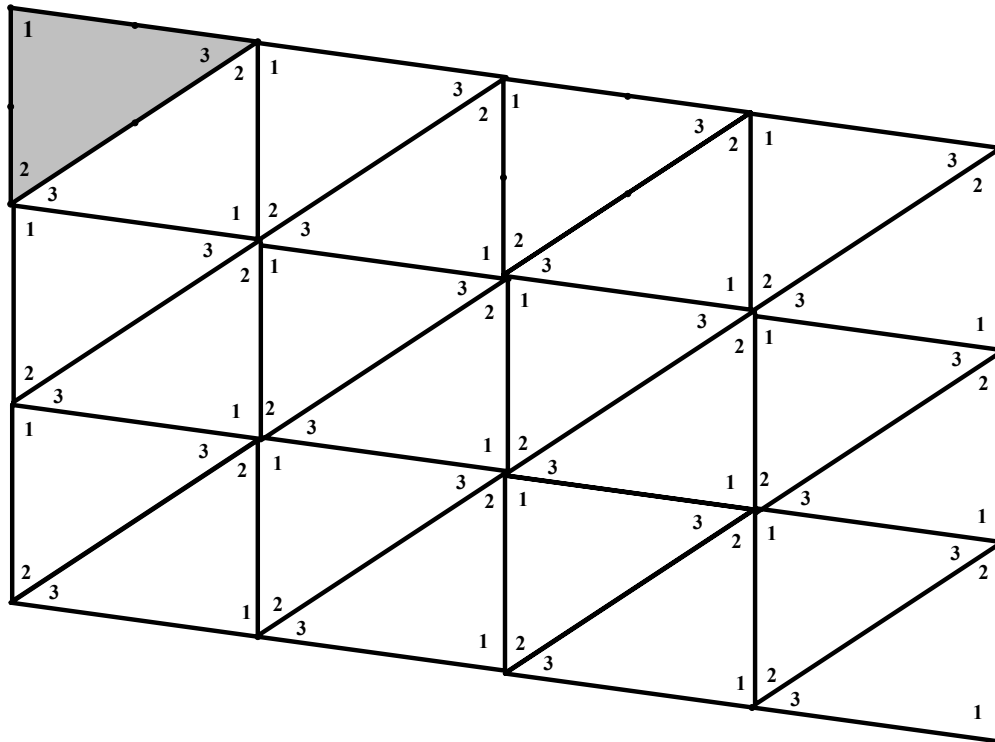
Participants create a tessellation using an acute scalene triangle or an obtuse scalene triangle but not a right scalene triangle. The triangle will be the fundamental region of the tessellation. Draw the triangle on a one-half index card. Cut out the triangle to use as a template. Shade each of the three angles of the triangle in a different color.

Create the tessellation using only rotations and translations, and not reflections. The tessellation may not have gaps or overlaps.

Two illustrations of non-acceptable placements follow.



One possible tessellation is shown:



When most participants have completed their tessellations, ask each group to discuss the triangle properties and the parallel line and angle properties which emerge from the tessellation. Each group records their properties on large easel paper with diagrams as needed. Then each group displays their poster on the wall.

Participants walk around the room as a group noting relationships they did not discover. Bring the participants back for a whole group discussion about tessellations. Using the overhead tessellation, individual group members summarize and explain one property their group discovered.

Make sure that the following relationships arise.

Triangle properties:

- The sum of the angles of a triangle is 180° . (Three angles of different colors that line up along a line are the same as the three colors for the three angles of the triangle.)
- The exterior angle of a triangle (made up of two colors) is equal to the sum of the measures of the two nonadjacent interior angles (the same two colors).

Parallel line properties created by transversal lines:

- Alternate interior angles are congruent (same color).
- Alternate exterior angles are congruent (same color).
- Interior angles on the same side of the transversal are supplementary (three colors altogether).

- Exterior angles on the same side of the transversal are supplementary (three colors altogether).
- Corresponding angles are congruent (same color).

Other relationships:

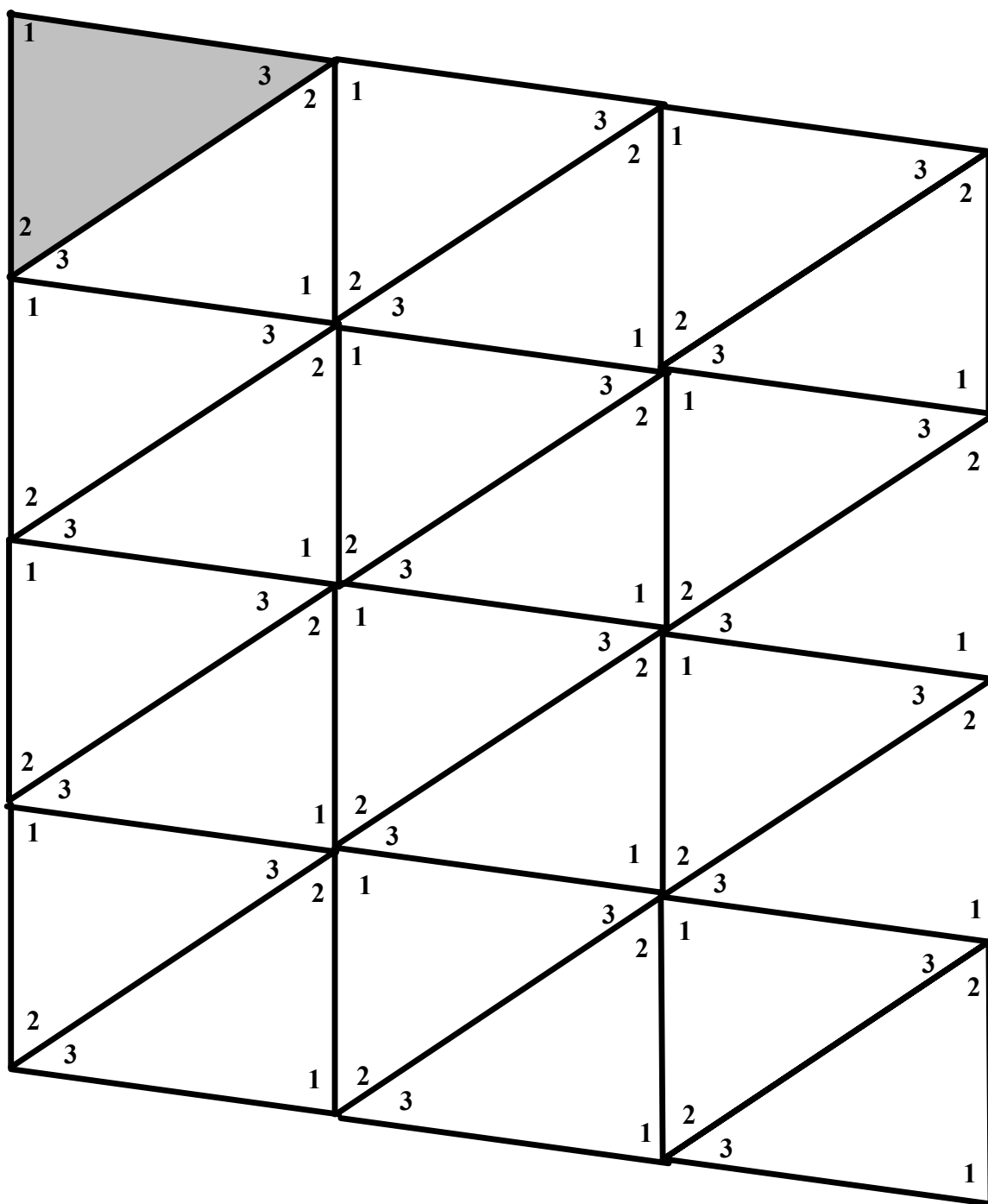
- Vertical angles are congruent (same color).
- The sum of the measures of the angles of a quadrilateral is 360° (two sets of the three colors).
- The sum of the measures of the angles of a convex polygon of n sides is $(n - 2)180^\circ$ ($n - 2$ sets of the three colors).
- The sum of the exterior angles of a polygon is 360° (two sets of the three colors).

A regular polygon has congruent sides and angles. Which regular polygons tessellate the plane?

The triangle, square, and hexagon are the only regular polygons that tessellate the plane. The measure of each angle of the equilateral triangle is 60° , and six 60° -angles tessellate around a point. The measure of each angle of a square is 90° , and four 90° -angles tessellate around a point. The measure of each angle of a hexagon is 120° , and three 120° -angles tessellate around a point.

Success with this activity indicates that participants are performing at the Descriptive Level since they apply properties of translations and rotation, and determine properties among parallel lines and triangles.

Tessellation



Do You See What I See?

Overview: After exploring two-dimensional transformations, participants explore transformations of three-dimensional objects by building, sketching the solids top, front, and side views then comparing the volume and surface area of the reflected solid to the original solid.

Objective: **TExES Mathematics Competencies**
III.011.B. The beginning teacher applies formulas for perimeter, area, surface area, and volume of geometric figures and shapes (e.g., polygons, pyramids, prisms, cylinders).
III.013.D. The beginning teacher computes the perimeter, area, and volume of figures and shapes created by subdividing and combining other figures and shapes (e.g., arc length, area of sectors).
III.013.F. The beginning teacher uses top, front, side, and corner views of three-dimensional shapes to create complete representations and solve problems.
III.014.C. The beginning teacher uses transformations to explore and describe reflectional, rotational, and translational symmetry.

Geometry TEKS

d.1.C. The student uses top, front, side, and corner views of three-dimensional objects to create accurate and complete representations and solve problems.
e.1.D. The student finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.

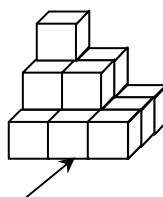
Background: Participants should be able to identify and describe reflections, rotations, and translations in a coordinate plane.

Materials: linking cubes, plastic mirrors, a small object such as a color tile for each participant

New Terms: surface area, volume

Procedures:

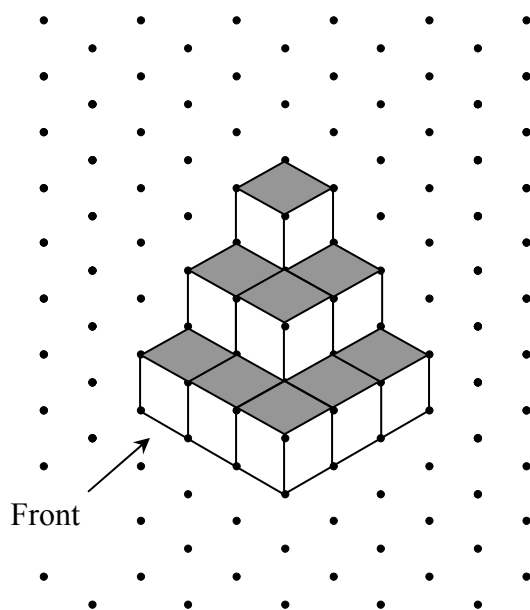
Participants should begin by constructing the following three-dimensional solid with linking cubes. The instructor should model the construction of the solid shown below and guide participants through their construction.



Front

Using linking cubes, construct a three-dimensional solid.

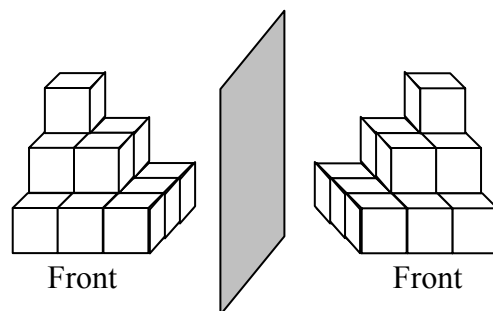
1. Sketch the solid on the isometric grid paper below.



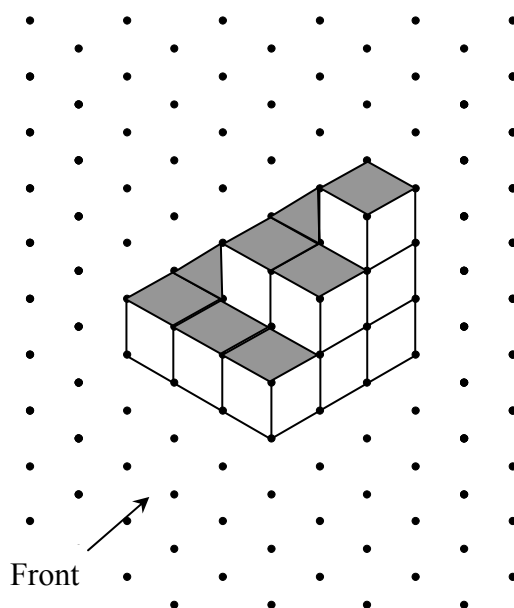
2. Sketch the top, front, and right views of the solid.

TOP View						FRONT View						RIGHT View					

3. Define the unit of *volume* as a single cube. Find the volume of this solid. Justify your answer.
The volume of the solid is 14 cubes, since the bottom layer has 9 cubes, the middle layer has 4 cubes, and the top layer has 1 cube.
4. Define *total surface area* as the total number of unit squares on the outer surface of the solid, including the base that rests on the table, regardless of its orientation. Define a unit of area as a single square. Find the total surface area of the solid. Justify your answer.
The surface area of the solid is 42 squares. The top layer has 5 squares, the middle layer has 11 squares, and the bottom layer has 26 squares.
5. Place a mirror parallel to one view of the solid. Using the mirror as a plane of reflection, predict what the reflected image of the solid would look like. Using the mirror as a plane of reflection, build the reflection of the solid.



Sketch the reflected solid on the isometric grid paper below.



6. Sketch the top, front, and right views of the reflected solid.

TOP View						FRONT View						RIGHT View					

7. What is the volume of the reflected solid? Justify your answer.
The volume of the solid is 14 cubes, since the bottom layer has 9 cubes, the middle layer has 4 cubes, and the top layer has 1 cube.

8. How does the volume of the reflected solid compare to the volume of the original solid?

The volume is the same as that of the original solid.

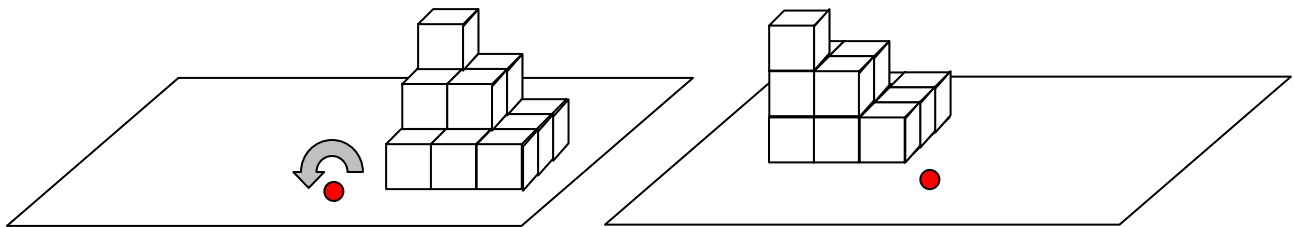
9. What is the surface area of the reflected solid? Justify your answer.

The surface area of the solid is 42 squares. The top layer has 5 squares, the middle layer has 11 squares, and the bottom layer has 26 squares.

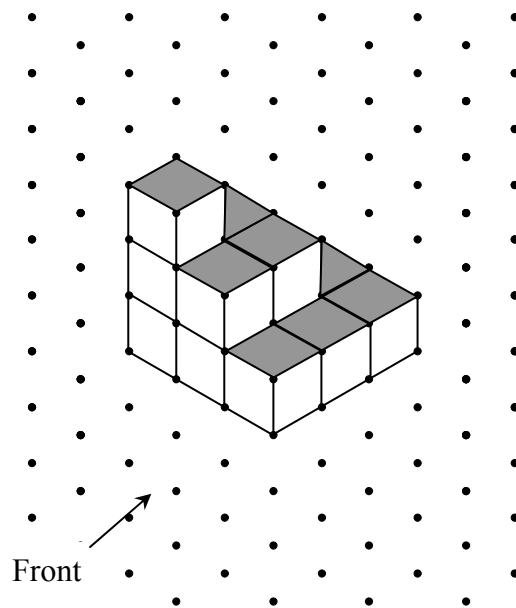
10. How does the surface area of the reflected solid compare to the surface area of the original solid?

The surface area is the same as the original solid.

11. Restore the solid to its original orientation. Place a marker about two inches from one corner of the solid. Using the marker as a point of rotation, rotate the solid 90° counter-clockwise in the plane of the table upon which you constructed the solid.



Sketch the rotated solid on the isometric grid paper below.



12. Sketch the top, front, and right views of the rotated solid.

TOP View						FRONT View						RIGHT View					

13. How does the volume and surface area of the rotated solid compare to the volume and surface area of the original solid? Why?

The volume and surface area are the same as the original solid, because rotation is an isometry; rotation preserves congruence.

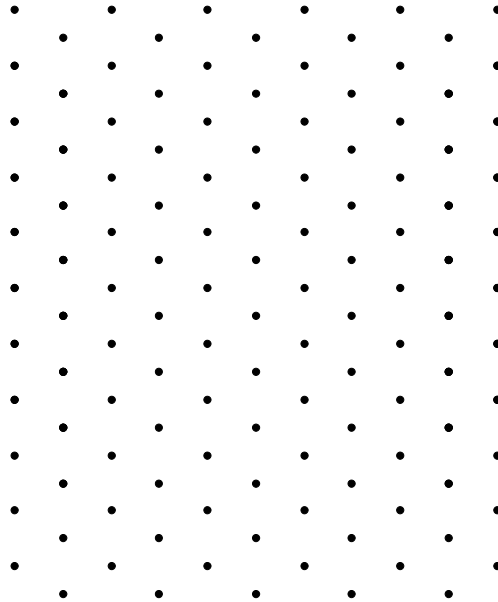
Remind participants to add the new terms total surface area and volume to their glossaries.

Participants begin this activity working at the Visual Level as they build and draw a copy of a three-dimensional solid in 1. The transition between the Visual Level and the Descriptive Level occurs in 2-3 as participants analyze the appearance of three-dimensional top, front, and right view and translate the observations to a two-dimensional representation. Participants are beginning to work at the Descriptive Level in 4-13 as they determine volume and surface area using known properties of the solid, reflect and rotate the solid, and compare the volume and surface area of the two figures. Beginning at the Visual Level, participants develop visual perception of solids. To be successful at this activity, the participant must be fluently working at the Visual Level of van Hiele's model.

Do You See What I See?

Using linking cubes, construct a three-dimensional solid.

1. Sketch the solid on the isometric grid paper below.

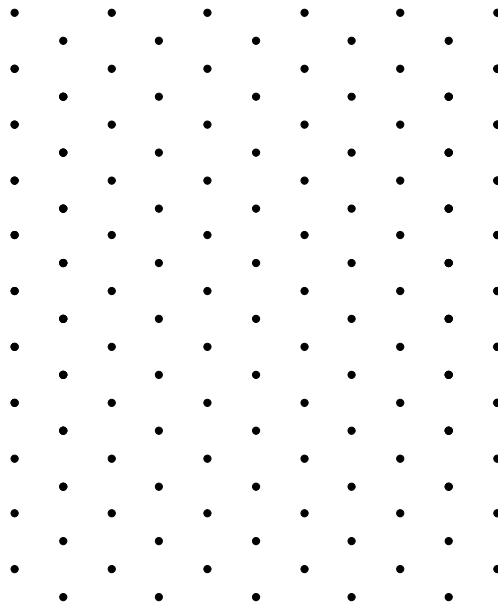


2. Sketch the top, front, and right views of the solid.

TOP View							FRONT View							RIGHT View						

3. Define the unit of volume as a single cube. Find the volume of this solid. Justify your answer.

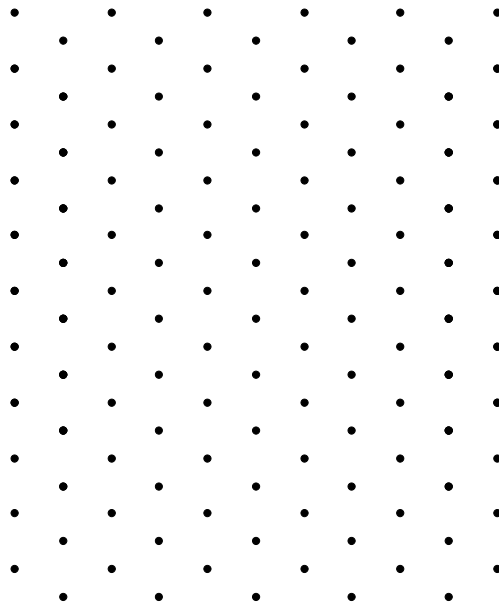
4. Define total surface area as the total number of unit squares on the outer surface of the solid, including the base that rests on the table, regardless of its orientation. Define a unit of area as a single square. Find the total surface area of this solid. Justify your answer.
5. Place a mirror parallel to one view of the solid. Using the mirror as a plane of reflection, predict what the reflected image of the solid would look like. Using the mirror as a plane of reflection, build the reflection of the solid. Sketch the reflected solid on the isometric grid paper below.



6. Sketch the top, front, and right views of the reflected solid.

TOP View						FRONT View						RIGHT View					

7. What is the volume of the reflected solid? Justify your answer.
8. How does the volume of the reflected solid compare to the volume of the original solid?
9. What is the surface area of the reflected solid? Justify your answer.
10. How does the surface area of the reflected solid compare to the surface area of the original solid?
11. Restore the solid to its original orientation. Place a marker about two inches from one corner of the solid. Using the marker as a point of rotation, rotate the solid 90° counter-clockwise in the plane of the table upon which you constructed the solid. Sketch the rotated solid on the isometric grid paper below.



12. Sketch the top, front, and right views of the rotated solid.

TOP View								FRONT View								RIGHT View							

13. How does the volume and surface area of the rotated solid compare to the volume and surface area of the original solid? Why?

References and Additional Resources

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