

Complex Variables

Mathematical Methods of Physics II

Department of Physics

May 24, 2019

Complex Variables

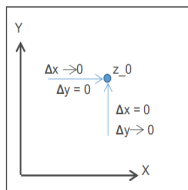
- ▶ If to each of a set of complex numbers, $z = x + iy$, there corresponds one or more values of a variable w , then w is called a function of the complex variable z , written $w = f(z)$.
- ▶ A function is single-valued if for each value of z there corresponds only one value of w ; otherwise it is multiple-valued or many-valued.
- ▶ In general we can write $w = f(z) = u(x, y) + iv(x, y)$, where u and v are real functions of x and y .
- ▶ Example 1. $w = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy = u + iv$ so that $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$. These are called the real and imaginary parts of $w = z^2$ respectively.

The derivative of $f(z)$, like that of a real function, is defined by

$$\lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{(z + \Delta z) - z} = \lim_{\Delta z \rightarrow 0} \frac{\Delta f}{\Delta z} = \frac{df}{dz}$$

Consider increments Δx and Δy of the variables x and y , respectively. Then, $\Delta z = \Delta x + i\Delta y$. Also, writing f as $u + iv$, $\Delta f = \Delta u + i\Delta v$, so that $\frac{\Delta f}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$

Let us take the limit indicated by two different approaches



$$\lim_{\Delta z \rightarrow 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$\text{and, } \lim_{\Delta z \rightarrow 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta y \rightarrow 0} \frac{\Delta f}{\Delta y} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

For the derivative $\frac{df}{dz}$ to exist the above relations must be identical.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \text{ and, } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- ▶ These are the famous Cauchy-Riemann conditions.
- ▶ If $f(z)$ is differentiable and single valued in a region of the complex plane, it is said to be an analytic function in that region.
- ▶ If $f'(z)$ doesn't exist at $z = z_0$, then z_0 is labeled a singular point.
- ▶ Furthermore, the real and imaginary parts of analytic functions satisfy Laplace's equation in 2D.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Example

1. Show that



$$f(z) = z^2 \text{ is analytic.}$$

And



$$f(z) = z^* \text{ is non analytic}$$

2. For

$$f(z) = \frac{1+z}{1-z}$$

- ▶ Find $\frac{df}{dz}$
- ▶ Determine where $f(z)$ is non analytic.

Integration

The integral of a complex variable over a path in the complex plane (known as a contour) may be defined by

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n f(\zeta_j)(z_j - z_{j-1}) = \int_{z_0}^{z'_0} f(z) dz = \int_C f(z) dz$$

where ζ is a point on the curve between z_j and z_{j-1} .

This can be reduced to the complex sum of real integrals.

$$\begin{aligned} \int_{z_1}^{z_2} f(z) dz &= \int_{x_1, y_1}^{x_2, y_2} (u + iv)(dx + idy) \\ &= \int_{x_1, y_1}^{x_2, y_2} (u dx - v dy) + i \int_{x_1, y_1}^{x_2, y_2} (u dy + v dx) \end{aligned}$$

Example

Evaluate

$$\int_{1+i}^{2+4i} z^2 dz$$

- ▶ along the parabola $x = t$, $y = t^2$ where $1 \leq t \leq 2$,
- ▶ along the straight line joining $1 + i$ and $2 + 4i$,
- ▶ along straight lines from $1 + i$ to $2 + i$ and then to $2 + 4i$.

Solution

$$\int_{1+i}^{2+4i} z^2 dz = \left. \frac{z^3}{3} \right|_{1+i}^{2+4i} = -\frac{86}{3} - 6i$$

Cauchy's integral theorem

If $f(z)$ is an analytic function at all points of a simply connected region in the complex plane and if C is a closed contour within that region, then

$$\oint_C f(z) dz = 0$$

Example: Evaluate $\oint_C z^n dz$

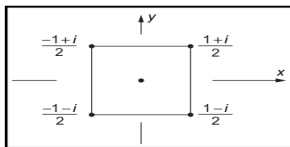
- ▶ On a circular contour
- ▶ On a square contour with vertices $\pm \frac{1}{2} \pm \frac{1}{2}i$

Solution

A circle is parametrized as $z = re^{i\theta}$ and $dz = ire^{i\theta} d\theta$

$$\begin{aligned}
 \oint_C z^n dz &= \oint_C (re^{i\theta})^n i r e^{i\theta} d\theta \\
 &= i r^{n+1} \left[\frac{e^{i(n+1)\theta}}{i(n+1)} \right]_0^{2\pi} = 0 \\
 &\quad \text{for } n \neq -1
 \end{aligned}$$

However, for $n = -1$ $\oint_C z^n dz = \oint_C \frac{dz}{z} = i \oint_C d\theta = 2\pi i$



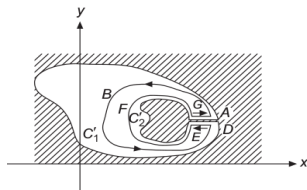
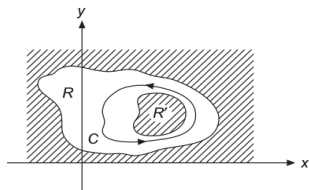
Integrating along the four segments (for particular value of n) and adding them we get $\oint_C z^n dz = 0$

for $n = -1$, using

$$\begin{aligned}
 \int \frac{dx}{x^2+a^2} &= \frac{1}{a} \arctan \frac{x}{a}, \\
 \oint_C z^{-1} dz &= 2\pi i
 \end{aligned}$$

Multiply Connected Regions

Cauchy's integral theorem is not valid for the contour C



we can construct a contour C' for which the theorem holds.

$$\text{Since } \int_G^A f(z)dz = \int_E^D f(z)dz$$

$$\oint_{ABD} f(z)dz = \oint_{GFE} f(z)dz$$

Or traversing in the same (counterclockwise) direction,

$$\oint_{C'_1} f(z)dz = \oint_{C'_2} f(z)dz$$

The integral of an analytic function over a closed path has a value that remains unchanged over all possible continuous deformations of the contour within the region of analyticity.

Cauchy's Integral Formula

If $f(z)$ is analytic on a closed contour C and within the interior region bounded by C ,

$$\frac{1}{2\pi i} \oint_C \frac{f(z)dz}{z - z_0} = \begin{cases} f(z_0), & z_0 \text{ within the contour} \\ 0, & z_0 \text{ outside the contour} \end{cases}$$

Derivatives

Cauchy's integral formula may be used to obtain an expression for the derivative of $f(z)$. Differentiating

$$\frac{1}{2\pi i} \oint_C \frac{f(z)dz}{z - z_0} = f(z_0)$$

with respect to z_0 , we get

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z - z_0)^{n+1}}$$

Example

1. Prove that

$$\oint_C \frac{dz}{(z-a)^n} = \begin{cases} 2\pi i & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

where C is a simple closed curve bounding a region having $z = a$ as interior point.

2. If $f(z)$ is analytic inside and on a simple closed curve C , and a is any point within C , prove that

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{z-a}$$

3. Evaluate $\oint_C \frac{\sin^2 z dz}{(z-a)^4}$
4. Evaluate $\int \frac{dz}{4z^2-1}$ over a unit circle.

Solution

1. Let C_1 be a circle of radius r having center at $z = a$

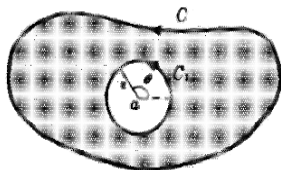
$$\oint_C \frac{dz}{(z-a)^n} = \oint_{C_1} \frac{dz}{(z-a)^n}$$

To evaluate the last integral, we use $z - a = re^{i\theta}$

2.

$$\oint_C \frac{f(z)dz}{z-a} = \oint_{C_1} \frac{f(z)dz}{z-a}$$

use $z - a = re^{i\theta}$, and $\lim_{r \rightarrow 0}$



and $dz = ire^{i\theta} d\theta$ this verifies the relation

$i \int_0^{2\pi} f(a + re^{i\theta}) d\theta$ and gives the desired result.

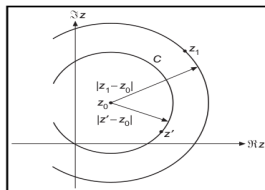
3. Using derivatives we get,
 $-\frac{8\pi i}{3} \cos a \sin a$

4. $\frac{1}{4z^2-1} = \frac{1}{4} \left(\frac{1}{z-1/2} - \frac{1}{z+1/2} \right),$

Taylor Series

Suppose we are trying to expand $f(z)$ about $z = z_0$ and we have $z = z_1$ as the nearest point for which $f(z)$ is not analytic. We construct a circle C centered at $z = z_0$ with radius less than $|z_1 - z_0|$. Since z_1 is assumed to be the nearest point at which $f(z)$ is not analytic, $f(z)$ is necessarily analytic on and within C . From Cauchy integral formula

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(z') dz'}{z' - z} \\ &= \frac{1}{2\pi i} \oint_C \frac{f(z') dz'}{(z' - z_0) - (z - z_0)} \\ &= \frac{1}{2\pi i} \oint_C \frac{f(z') dz'}{(z' - z_0) \left(1 - \frac{z - z_0}{z' - z_0}\right)} \end{aligned}$$



Here z' is a point on the contour C and z is any point interior to C .

With the help of the identity

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$$

for a point z interior to C , $|z - z_0| < |z' - z_0|$, we get

$$f(z) = \frac{1}{2\pi i} \oint_C \sum_{n=0}^{\infty} \frac{(z - z_0)^n f(z') dz'}{(z' - z_0)^{n+1}}$$

Interchanging summation and integration,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_C \frac{f(z') dz'}{(z' - z_0)^{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \Rightarrow \text{Taylor series} \end{aligned}$$

In view of the definition of z_1 , we can say that: The Taylor series of a function $f(z)$ about any interior point z_0 of a region in which $f(z)$ is analytic is a unique expansion that will have a radius of convergence equal to the distance from z_0 to the singularity of $f(z)$ closest to z_0 , meaning that the Taylor series will converge within this circle of convergence. The Taylor series may or may not converge at individual points on the circle of convergence.

Example: Develop the Taylor expansion of $\ln(1 + z)$.

We know $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$ here with $z_0 = 0$. And $\frac{\ln(z+1)^{(n)}|_0}{n!} = \frac{(-1)^{n-1}}{n}$ for $n \geq 1$. Therefore, the Taylor expansion of $\ln(z + 1)$ is

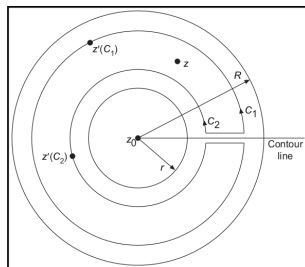
$$\ln(z + 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$$

Laurent Series

We frequently encounter functions that are analytic in an annular region, say, between circles of inner radius r and outer radius R about a point z_0

From Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z') dz'}{z' - z} - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z') dz'}{z' - z}$$



Noting that for C_1 , $|z' - z_0| > |z - z_0|$, while for C_2 , $|z' - z_0| < |z - z_0|$, we find

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} \frac{f(z') dz'}{(z' - z_0)^{n+1}} \\ + \frac{1}{2\pi i} \sum_{n=1}^{\infty} (z - z_0)^{-n} \oint_{C_2} (z' - z_0)^{n-1} f(z') dz'$$

These two series are combined into one series, known as a Laurent series, of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z') dz'}{(z' - z_0)^{n+1}}$$

C may be any contour within the annular region

Example

Make Laurent expansion of

$$f(z) = [z(z - 1)]^{-1}$$

about $z = 0$.

Solution

$f(z)$ is analytic in the annular region between,
 $z' = 0$ and $|z| = 1$

$$[z(z - 1)]^{-1} = \frac{-1}{z(1 - z)} = -\frac{1}{z} - \frac{1}{1 - z} = -\sum_{n=-1}^{\infty} z^n$$

Singularities

- ▶ We define a point z_0 as an isolated singular point of the function $f(z)$ if $f(z)$ is not analytic at $z = z_0$ but is analytic at all neighboring points.
- ▶ If $f(z) = \frac{\phi(z)}{(z-a)^n}$, where $\phi(z)$ is analytic everywhere in a region including $z = a$, and if n is a positive integer, then $f(z)$ has an isolated singularity at $z = a$ which is called a pole of order n .

Residue Theorem

If the Laurent expansion of a function,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

is integrated term by term by using a closed contour that encircles one isolated singular point z_0 once in a counterclockwise sense, we obtain,

$$a_n \oint (z - z_0)^n dz = 0 \quad n \neq -1$$

However, for $n = -1$,

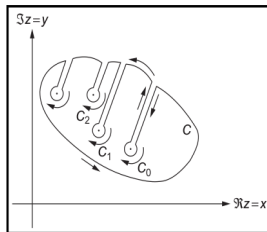
$$a_{-1} \oint (z - z_0)^{-1} dz = 2\pi i a_{-1}$$

Therefore,

$$\oint f(z) dz = 2\pi i a_{-1}$$

The constant a_{-1} , the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion, is called the residue of $f(z)$ at $z = z_0$.

Now consider the evaluation of the integral, over a closed contour C , of a function that has isolated singularities at points $z_1, z_2, \dots$



$$\oint_C f(z) dz + \oint_{C_1} f(z) dz + \oint_{C_2} f(z) dz + \oint_{C_3} f(z) dz + \dots = 0$$

The integrals C_i about the individual isolated singularities have the values

$$\oint_{C_i} f(z) dz = 2\pi i a_{-1,i}$$

Thus, we have

$$\oint_C f(z) dz = 2\pi i (\text{sum of the enclosed residues})$$

If $f(z)$ has a simple pole at $z = z_0$, then, with a_n the coefficients in the expansion of $f(z)$,

$$(z - z_0)f(z) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2$$

taking the limit as $z \rightarrow z_0$ we get:

$$a_{-1} = \lim_{z \rightarrow z_0} ((z - z_0)f(z))$$

If there is a pole order $n > 1$ at $z - z_0$, then $(z - z_0)^n f(z)$ must have the expansion

$$(z - z_0)^n f(z) = a_{-n} + \dots + a_{-1}(z - z_0)^{n-1} + a_0(z - z_0)^n + \dots$$

We see that a_{-1} is the coefficient of $(z - z_0)^{n-1}$ in the Taylor expansion of $(z - z_0)^n f(z)$, and therefore we can identify it as satisfying

$$a_{-1} = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \left[\frac{d^{n-1}}{dz^{n-1}} ((z - z_0)^n f(z)) \right]$$

Examples

1. Compute the residues of

1.1

$$\frac{1}{4z+1} \quad \text{at } z = -\frac{1}{4}, \quad \frac{1}{4}$$

1.2

$$\frac{1}{\sin z} \quad \text{at } z = 0, \quad 1$$

2. Integrate

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos \theta}, \quad \frac{2\pi}{\sqrt{3}}$$

Evaluation Of Definite Integrals

1. Trigonometric Integrals, Range $(0, 2\pi)$

Consider here integrals of the form

$$I = \int_0^{2\pi} f(\sin \theta, \cos \theta) d\theta, \quad \text{where } f \text{ is analytic.}$$

We make change of variable to $z = e^{i\theta}$ and $dz = ie^{i\theta} d\theta$ And making the substitution $d\theta = -i \frac{dz}{z}$, $\sin \theta = \frac{z - z^{-1}}{2i}$ and $\cos \theta = \frac{z + z^{-1}}{2}$, the integral becomes

$$I = -i \oint f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \frac{dz}{z}$$

with the path of integration the unit circle. By residue theorem,

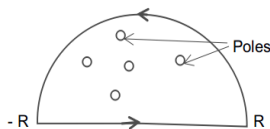
$$I = (-i)2\pi i \sum \text{residues within the unit circle}$$

2. Integrals, Range $-\infty$ to ∞

Consider definite integrals of the form

$$I = \int_{-\infty}^{\infty} f(x) dx$$

Consider a contour closed by a large semicircle in the upper half-plane.



$$\oint f(z) dz = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx + \lim_{R \rightarrow \infty} \int_C f(z) dz$$

But $\lim_{R \rightarrow \infty} \int_C f(z) dz = 0$, where C is the arc of the semicircle. So,

$$\int_{-\infty}^{\infty} f(x) dx = \oint f(z) dz = 2\pi i \sum \text{residues (upper half-plane)}$$

3. Integrals with Complex Exponentials

Consider the definite integral

$$I = \int_{-\infty}^{\infty} f(x)e^{iax} dx$$

Since the complex integral over the arc is zero, we get

$$\int_{-\infty}^{\infty} f(x)e^{iax} dx = 2\pi i \sum \text{residues of } e^{iaz} f(z) \text{ (upper half-plane)}$$

Examples

1. Evaluate the definite integral

$$I = \int_0^{2\pi} \frac{d\theta}{1 + a \cos \theta}, \quad |a| < 1$$

answer $\int_0^{2\pi} \frac{d\theta}{1 + a \cos \theta} = \frac{2\pi}{\sqrt{1 - a^2}}, \quad |a| < 1.$

2. Integrate

$$I = \int_0^{2\pi} \frac{\cos 2\theta d\theta}{5 - 4 \cos \theta}$$

answer $I = \frac{\pi}{6}$

Evaluate

3.

$$\text{answer } \int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$$

4.

$$I = \int_0^{\infty} \frac{\cos x}{x^2 + 1} dx$$
$$\text{answer } I = \frac{\pi}{2e}$$