

Vectors Calculus

Mathematical Methods of Physics II

Department of Physics

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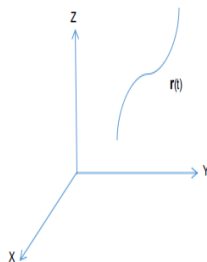
Time derivative of vectors

Let the path of moving bodies be represented by

$$\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

Then

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [r(t + \Delta t) - r(t)]$$



becomes the derivative.

This derivative is the velocity vector $\vec{V}$. If the velocity vector is velocity of fluid it defines a vector field because to each point in the region is associated a vector. Let $\vec{V}(x, y, z, t)$ be the velocity field for general unsteady three-dimensional fluid flow.

This vector field can be expanded in Cartesian coordinates as

$$\vec{V} = (u, v, w) = u(x, y, z, t)\mathbf{i} + v(x, y, z, t)\mathbf{j} + w(x, y, z, t)\mathbf{k}$$

Consider, for example, Newton's second law applied to our fluid particle,

$$\vec{F}_{particle} = m_{particle}\vec{a}_{particle}$$

$$\text{acceleration of a fluid particle: } \vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt}$$

However, at any instant in time t , the velocity of the particle is the same as the local value of the velocity field at the location $(x_{particle}(t), y_{particle}(t), z_{particle}(t))$ of the particle, since the fluid particle moves with the fluid by definition.

$$\begin{aligned}
 \vec{a}_{particle} &= \frac{d\vec{V}}{dt} = \frac{d\vec{V}(x_{particle}, y_{particle}, z_{particle}, t)}{dt} \\
 &= \frac{\partial \vec{V}}{\partial x_{particle}} \frac{dx_{particle}}{dt} + \frac{\partial \vec{V}}{\partial y_{particle}} \frac{dy_{particle}}{dt} + \frac{\partial \vec{V}}{\partial z_{particle}} \frac{dz_{particle}}{dt} + \frac{\partial \vec{V}}{\partial t} \frac{dt}{dt} \\
 &= u \frac{\partial \vec{V}}{\partial x} + v \frac{\partial \vec{V}}{\partial y} + w \frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}
 \end{aligned}$$

This can be written as $\vec{a}(x, y, z, t) = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla}) \vec{V}$

where $\vec{\nabla} = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$ is called the gradient operator.

At any instant in time t , the acceleration field must equal the acceleration of the fluid particle that happens to occupy the location (x, y, z) at that time t . Because the fluid particle is by definition accelerating with the fluid flow.

Example

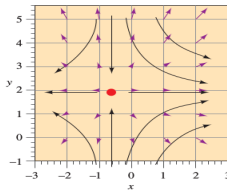
Consider a steady, incompressible, two-dimensional velocity field given by

$$\vec{V} = (0.5 + 0.8x)\mathbf{i} + (1.5 - 0.8y)\mathbf{j}$$

- ▶ Calculate the material acceleration at the point $(x = 2, y = 3)$.
- ▶ Sketch the material acceleration vectors in the domain between $x = -2$ to 2 and $y = 0$ to 5 .

$$a_x = \frac{du}{dt} + u \frac{du}{dx} = (0.4 + 0.64x)\mathbf{i}$$

$$a_y = \frac{dv}{dt} + v \frac{dv}{dy} = (-1.2 + 0.64y)\mathbf{j}$$



Gradient

Consider the vector operator $\vec{\nabla}$, if a scalar $\phi(x, y, z)$ has continuous first order partial derivative, the gradient of $\phi(x, y, z)$ is defined by

$$\begin{aligned}\vec{\nabla}\phi &= \left(i \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \phi \\ &= \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k\end{aligned}$$

Example

In physics gradient expresses the relation between a force field $\vec{F}(r)$ experienced by an object at r and the related potential $V(r)$,

$$\vec{F}(r) = -\vec{\nabla} V(r)$$

Compute the gradient of r^n and check if it is consistent with Coulomb's law.

We begin with writing $r = (x^2 + y^2 + z^2)^{1/2}$. Then,

$$\frac{\partial r}{\partial x} = \frac{x}{(x^2 + y^2 + z^2)^{1/2}} = \frac{x}{r}$$

similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, and $\frac{\partial r}{\partial z} = \frac{z}{r}$

- From these formulas we construct

$$\vec{\nabla} r = \frac{x}{r} \mathbf{e}_x + \frac{y}{r} \mathbf{e}_y + \frac{z}{r} \mathbf{e}_z = \frac{1}{r} (x \mathbf{e}_x + y \mathbf{e}_y + z \mathbf{e}_z) = \frac{\vec{r}}{r}$$

$$\text{since } \vec{\nabla} r = \frac{x}{r} \mathbf{e}_x + \frac{y}{r} \mathbf{e}_y + \frac{z}{r} \mathbf{e}_z = \hat{r}$$

this equation can be written as $\vec{\nabla} r = \hat{r}$

Continuing now to $\vec{\nabla} r^n$, we have

$$\vec{\nabla} r^n = n r^{n-1} \vec{\nabla} r = n r^{n-1} \hat{r}$$

- For the case $n = -1$ we get the result

$$\vec{F}(r) = -\vec{\nabla} r^{-1} = \frac{1}{r^2} \hat{r}$$

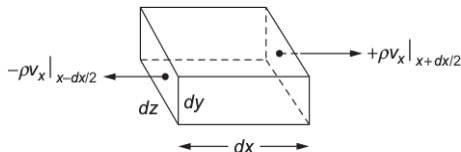
which agrees with Coulomb's law.

Divergence, $\vec{\nabla} \cdot$.

The divergence of a vector $\vec{A}$ is defined as the operation

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

To gain a clearer picture of the concept, let us suppose that a vector field $\vec{v}(r)$ represents the velocity of a fluid at the spatial points r , and that $\rho(r)$ represents the fluid density at r at a given time t . Then the direction and magnitude of the flow rate at any point will be given by the product $\rho(r)v(r)$.



Our objective is to calculate the net rate of change of the fluid density in a volume element at the point $\mathbf{r}$. To do so, we set up a parallelepiped of dimensions dx , dy , dz centered at $\mathbf{r}$ and with sides parallel to the xy , xz , and yz planes.

The density of fluid exiting the parallelepiped per unit time through the yz face located at $x - (dx/2)$ will be

$$\text{Flow at } x - \frac{dx}{2} : -\rho v_x|_{(x-\frac{dx}{2}, y, z)} dy dz$$

$$\text{and Flow at } x + \frac{dx}{2} : \rho v_x|_{(x+\frac{dx}{2}, y, z)} dy dz$$

Combining these, we have for both yz faces

$$\left(-\rho v_x|_{(x-\frac{dx}{2}, y, z)} + \rho v_x|_{(x+\frac{dx}{2}, y, z)}\right) dydz = \left(\frac{\partial \rho v_x}{\partial x}\right) dx dy dz$$

Finally, adding corresponding contributions from the other four faces of the parallelepiped, we reach

the net rate of change of the fluid density in a volume element

$$\begin{aligned} &= \left(\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right) dx dy dz \\ &\quad - \frac{\partial \rho}{\partial t} dx dy dz = \vec{\nabla} \cdot (\rho \vec{v}) dx dy dz \end{aligned}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon} \rho$$

where ϵ is the electric constant, and ρ is the total electric charge density.

The electric flux leaving a volume is proportional to the charge inside.

$$\vec{\nabla} \cdot \vec{B} = 0.$$

There are no magnetic monopoles; the total magnetic flux through a closed surface is zero.

Example

- Evaluate the divergence of central force field given by

$$f(r)\hat{r}$$

$$\text{Solution, } \frac{f(r)}{r} + \frac{df(r)}{dr}$$

Curl, $\vec{\nabla} \times$

Another possible operation with the vector operator $\vec{\nabla}$ is to take its cross product with a vector. For a vector $\vec{V}$ the curl is defined as

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

Maxwell's equations

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The voltage induced in a closed loop is proportional to the rate of change of the magnetic flux that the loop encloses.

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} \right)$$

The magnetic field induced around a closed loop is proportional to the electric current plus displacement current (rate of change of electric field) that the loop encloses.

Example

1. For a particle moving in a circular orbit

$$\vec{r} = e_x r \cos \omega t + e_y r \sin \omega t$$

- ▶ Evaluate $\vec{r} \times \vec{r}'$
 - ▶ Show that $\vec{r}'' + \omega^2 \vec{r} = 0$
2. Classically, orbital angular momentum is given by $\vec{L} = \vec{r} \times \vec{p}$, where $\vec{p}$ is the linear momentum. To go from classical mechanics to quantum mechanics, $\vec{p}$ is replaced by the operator $i\hbar \vec{\nabla}$. Evaluate Cartesian components of the quantum mechanical angular momentum operator.

Gradient, Divergence, and Curl in curvilinear coordinates

To express the del operator in other coordinates than the Cartesian coordinate system, we start with the vector $\vec{r}$,

$$\vec{r} = xi + yj + zk = f(u_1, u_2, u_3)i + g(u_1, u_2, u_3)j + h(u_1, u_2, u_3)k$$

where, u_1 , u_2 , and u_3 are coordinates of a curvilinear coordinate system. Then, the total differential $d\vec{r}$ is

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$

Writing $\frac{\partial \vec{r}}{\partial u_i}$ as $h_i \mathbf{e}_i$ where $h_i = \left| \frac{\partial \vec{r}}{\partial u_i} \right|$

The total differential changes to

$$d\vec{r} = h_1 du_1 \mathbf{e}_1 + h_2 du_2 \mathbf{e}_2 + h_3 du_3 \mathbf{e}_3$$

For a scalar function ϕ and a vector function

$\vec{A} = A_1 \mathbf{e}_1 + A_2 \mathbf{e}_2 + A_3 \mathbf{e}_3$ of orthogonal curvilinear coordinates u_1, u_2, u_3 we have

$$\vec{\nabla} \phi = \frac{1}{h_1} \frac{\partial \phi}{\partial u_1} \mathbf{e}_1 + \frac{1}{h_2} \frac{\partial \phi}{\partial u_2} \mathbf{e}_2 + \frac{1}{h_3} \frac{\partial \phi}{\partial u_3} \mathbf{e}_3$$

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_1 h_3 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right]$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \phi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial u_3} \right) \right]$$

For spherical coordinate system, $h_1 = 1$, $h_2 = r$, $h_3 = r \sin \theta$
 and for cylindrical coordinate system $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$.
 Because, $\vec{r} = xi + yj + zk$

$$\begin{aligned} x &= r \sin \theta \cos \phi & y &= r \sin \theta \sin \phi & z &= r \cos \theta \\ x &= \rho \cos \phi & y &= \rho \sin \phi & z &= z \end{aligned}$$

Example: Drive expressions for gradient, divergence, curl and Laplacian in spherical and cylindrical coordinates.

Conservative forces

A conservative force is a force with the property that the total work done in moving a particle between two points is independent of the taken path.

$\int_C \vec{F} \cdot d\vec{r}$ is independent of path

$\int_a^b \vec{F} \cdot d\vec{r} = \phi(b) - \phi(a)$ this implies that

$\vec{F} \cdot d\vec{r}$ is exact differential, say $d\phi$

$\vec{F} \cdot d\vec{r} = d\phi$ this in turn is the same as

$$F_x = \frac{\partial \phi}{\partial x}, F_y = \frac{\partial \phi}{\partial y}, F_z = \frac{\partial \phi}{\partial z}$$

$$\vec{F} = -\vec{\nabla} \phi$$

Since, $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$, we can say that $\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$. Similarly,
 $\frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}$ and $\frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}$
 This relations are summarized by a cross product,

$$\vec{\nabla} \times \vec{F} = 0$$

In general, for a conservative force field, the following properties are true.

1. The integral $\int_C \vec{F} \cdot d\vec{r}$ is path independent
2. $\oint_C \vec{F} \cdot d\vec{r} = 0$
3. $\vec{F}$ can be derived from a scalar potential, ϕ such that,
 $\vec{F} = -\vec{\nabla} \phi$
4. $\vec{\nabla} \times \vec{F} = 0$

Example

Determine whether the vector is conservative.

$$\vec{F} = x^2 \mathbf{e}_x + 2yze_y + y^2 \mathbf{e}_z$$

- ▶ The curl is zero.
- ▶ It can be derived from a potential,

$$\vec{F} = -\vec{\nabla}\phi, \quad \phi = -\left(\frac{x^3}{3} + y^2z\right)$$

Vector Identities

If $\vec{F}$ and $\vec{G}$ are vectors, ϕ and ψ are scalars then

$$\begin{aligned}\vec{\nabla}(\phi\psi) &= \phi\vec{\nabla}\psi + \psi\vec{\nabla}\phi \\ \vec{\nabla}(\vec{F} \cdot \vec{G}) &= \vec{F} \times (\vec{\nabla} \times \vec{G}) + \vec{G} \times (\vec{\nabla} \times \vec{F}) + (\vec{F} \cdot \vec{\nabla})\vec{G} + (\vec{G} \cdot \vec{\nabla})\vec{F} \\ \vec{\nabla} \cdot (\phi\vec{F}) &= \phi\vec{\nabla} \cdot \vec{F} + \vec{F} \cdot \vec{\nabla}\phi \\ \vec{\nabla} \cdot (\vec{\nabla}\phi) &= \nabla^2\phi \\ \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) &= 0 \\ \vec{\nabla} \cdot (\vec{F} \times \vec{G}) &= \vec{G} \cdot (\vec{\nabla} \times \vec{F}) - \vec{F} \cdot (\vec{\nabla} \times \vec{G}) \\ \vec{\nabla} \times (\phi\vec{F}) &= \phi\vec{\nabla} \times \vec{F} + \vec{\nabla}\phi \times \vec{F} \\ \vec{\nabla} \times (\vec{\nabla}\phi) &= 0 \\ \vec{\nabla} \times (\vec{\nabla} \times \vec{F}) &= \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \nabla^2\vec{F} \\ \vec{\nabla} \times (\vec{F} \times \vec{G}) &= \vec{F}(\vec{\nabla} \cdot \vec{G}) - \vec{G}(\vec{\nabla} \cdot \vec{F}) + (\vec{G} \cdot \vec{\nabla})\vec{F} - (\vec{F} \cdot \vec{\nabla})\vec{G}\end{aligned}$$