CHAPTER 11

Displacement Method of Analysis: Slope-Deflection Equations

In this chapter we will briefly outline the basic ideas for analyzing structures using the displacement method of analysis. Once these concepts have been presented, we will develop the general equations of slope deflection and then use them to analyze statically indeterminate beams and frames.

11–1 Displacement Method of Analysis: General Procedures

All structures must satisfy equilibrium, load-displacement, and compatibility of displacements requirements in order to ensure their safety. It was stated in Sec. 10–1 that there are two different ways to satisfy these requirements when analyzing a statically indeterminate structure. The force method of analysis, discussed in the previous chapter, is based on identifying the unknown redundant forces and then satisfying the structure's compatibility equations. This is done by expressing the displacements in terms of the loads by using the load-displacement relations. The solution of the resultant equations yields the redundant reactions, and then the equilibrium equations are used to determine the remaining reactions on the structure.

The *displacement method* works the opposite way. It first requires satisfying equilibrium equations for the structure. To do this the unknown displacements are written in terms of the loads by using the load-displacement relations, then these equations are solved for the displacements. Once the displacements are obtained, the unknown loads are determined from the compatibility equations using the load-displacement relations. Every displacement method follows this general procedure. In this chapter, the procedure will be generalized to produce the slope-deflection equations. In Chapter 12, the moment-distribution method will be developed. This method sidesteps the calculation of the displacements and instead makes it possible to apply a series of converging corrections that allow direct calculation of the end moments. Finally, in Chapters 14, 15, and 16, we will illustrate how to apply this method using matrix analysis, making it suitable for use on a computer.

In the discussion that follows we will show how to identify the unknown displacements in a structure and we will develop some of the important load-displacement relations for beam and frame members. The results will be used in the next section and in later chapters as the basis for applying the displacement method of analysis.

Degrees of Freedom. When a structure is loaded, specified points on it, called *nodes*, will undergo unknown displacements. These displacements are referred to as the *degrees of freedom* for the structure, and in the displacement method of analysis it is important to specify these degrees of freedom since they become the unknowns when the method is applied. The number of these unknowns is referred to as the degree in which the structure is kinematically indeterminate.

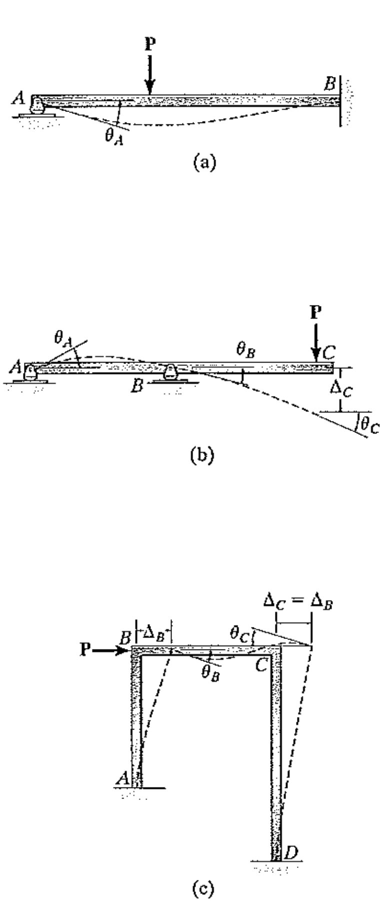


Fig. 11–1

To determine the kinematic indeterminacy we can imagine the structure to consist of a series of members connected to nodes, which are usually located at *joints, supports,* at the *ends* of a member, or where the members have a sudden *change in cross section.* In three dimensions, each node on a frame or beam can have at most three linear displacements and three rotational displacements; and in two dimensions, each node can have at most two linear displacements and one rotational displacement. Furthermore, nodal displacements may be restricted by the supports, or due to assumptions based on the behavior of the structure. For example, if the structure is a beam and only deformation due to bending is considered, then there can be no linear displacement along the axis of the beam since this displacement is caused by axial-force deformation.

To clarify these concepts we will consider some examples, beginning with the beam in Fig. 11–1*a*. Here any load **P** applied to the beam will cause node A only to rotate (neglecting axial deformation), while node B is completely restricted from moving. Hence the beam has only one unknown degree of freedom, θ_A, and is therefore kinematically indeterminate to the first degree. The beam in Fig. 11–1*b* has nodes at A, B, and C, and so has four degrees of freedom, designated by the rotational displacements θ_A, θ_B, θ_C, and the vertical displacement Δ_C. It is kinematically indeterminate to the fourth degree. Consider now the frame in Fig. 11–1*c*. Again, if we neglect axial deformation of the members, an arbitrary loading **P** applied to the frame can cause nodes B and C to rotate, and these nodes can be displaced horizontally by an *equal* amount. The frame therefore has three degrees of freedom, θ_B, θ_C, Δ_B, and thus it is kinematically indeterminate to the third degree.

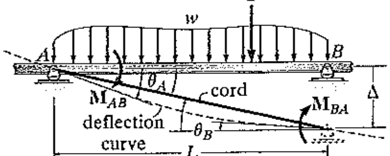
In summary, specifying the kinematic indeterminacy or the number of unconstrained degrees of freedom for the structure is a necessary first step when applying a displacement method of analysis. It identifies the number of unknowns in the problem, based on the assumptions made regarding the deformation behavior of the structure. Furthermore, once these nodal displacements are known, the deformation of the structural members can be completely specified, and the loadings within the members obtained.

11–2 Slope-Deflection Equations

As indicated previously, the method of consistent displacements studied in Chapter 10 is called a force method of analysis, because it requires writing equations that relate the unknown forces or moments in a structure. Unfortunately, its use is limited to structures which are *not* highly indeterminate. This is because much work is required to set up the compatibility equations, and furthermore each equation written involves *all the unknowns,* making it difficult to solve the resulting set of equations unless a computer is available. By comparison, the slope-deflection method is not as involved. As we shall see, it requires less work both to write the necessary equations for the solution of a problem and to solve these equations for the unknown displacements and associated internal loads. Also, the method can be easily programmed on a computer and used to analyze a wide range of indeterminate structures.

The slope-deflection method was originally developed by Heinrich Manderla and Otto Mohr for the purpose of studying secondary stresses in trusses. Later, in 1915, G. A. Maney developed a refined version of this technique and applied it to the analysis of indeterminate beams and framed structures.

General Case. The slope-deflection method is so named since it relates the unknown slopes and deflections to the applied load on a structure. In order to develop the general form of the slope-deflection equations, we will consider the typical span AB of a continuous beam as shown in Fig. 11–2, which is subjected to the arbitrary loading and has a constant EI. We wish to relate the beam's internal end moments M_{AB} and M_{BA} in terms of its three degrees of freedom, namely, its angular displacements θ_A and θ_B, and linear displacement Δ which could be caused by a relative settlement between the supports. Since we will be developing a formula, *moments* and *angular displacements* will be considered *positive* when they act *clockwise on the span*, as shown in Fig. 11–2. Furthermore, the *linear displacement* Δ is considered *positive* as shown, since this displacement causes the cord of the span and the span's cord angle ψ to rotate *clockwise.*



EI is constant
positive sign convention

Fig. 11–2

The slope-deflection equations can be obtained by using the principle of superposition by considering *separately* the moments developed at each support due to each of the displacements, θ_A, θ_B, and Δ, and then the loads.

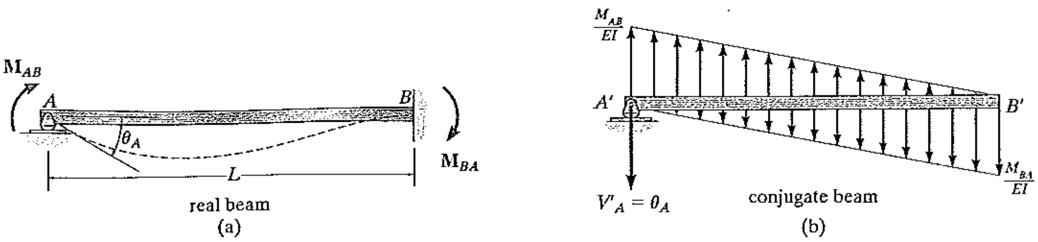


Fig. 11–3

Angular Displacement at A, θ_A. Consider node A of the member shown ir Fig. 11–3*a* to rotate θ_A while its far-end node B is *held fixed*. To determine th moment M_{AB} needed to cause this displacement, we will use the conjugate beam method. For this case the conjugate beam is shown in Fig. 11–3*b*. Notic that the end shear at A' acts downward on the beam, since θ_A is clockwise The deflection of the "real beam" in Fig. 11–3*a* is to be zero at A and B, anc therefore the corresponding sum of the *moments* at each end A' and B' o the conjugate beam must also be zero. This yields

$$\circlearrowright + \Sigma M_{A'} = 0; \qquad \left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{2L}{3} = 0$$

$$\circlearrowright + \Sigma M_{B'} = 0; \qquad \left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{2L}{3} + \theta_A L = 0$$

from which we obtain the following load-displacement relationships.

$$\boxed{M_{AB} = \frac{4EI}{L}\theta_A} \qquad (11\text{–}1$$

$$\boxed{M_{BA} = \frac{2EI}{L}\theta_A} \qquad (11\text{–}2$$

Angular Displacement at B, θ_B. In a similar manner, if end B of the bean rotates to its final position θ_B, while end A is *held fixed*, Fig. 11–4, we cai relate the applied moment M_{BA} to the angular displacement θ_B and th reaction moment M_{AB} at the wall. The results are

$$\boxed{M_{BA} = \frac{4EI}{L}\theta_B} \qquad (11\text{–}3$$

$$\boxed{M_{AB} = \frac{2EI}{L}\theta_B} \qquad (11\text{–}4$$

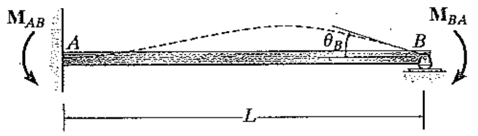


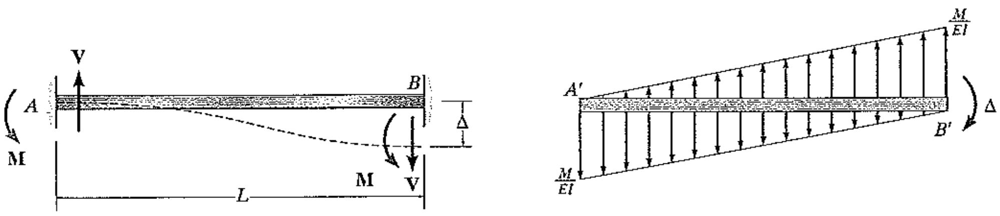
Fig. 11–4

Relative Linear Displacement, Δ. If the far node *B* of the member is displaced relative to *A*, so that the cord of the member rotates clockwise (positive displacement) and yet both ends do not rotate, then equal but opposite moment and shear reactions are developed in the member, Fig. 11–5*a*. As before, the moment **M** can be related to the displacement Δ using the conjugate-beam method. In this case, the conjugate beam, Fig. 11–5*b*, is free at both ends, since the real beam (member) is fixed supported. However, due to the *displacement* of the real beam at *B*, the *moment* at the end *B′* of the conjugate beam must have a magnitude of Δ as indicated.* Summing moments about *B′*, we have

$$\circlearrowright + \Sigma M_{B'} = 0; \qquad \left[\frac{1}{2}\frac{M}{EI}(L)\left(\frac{2}{3}L\right)\right] - \left[\frac{1}{2}\frac{M}{EI}(L)\left(\frac{1}{3}L\right)\right] - \Delta = 0$$

$$\boxed{M_{AB} = M_{BA} = M = \frac{-6EI}{L^2}\Delta} \qquad (11\text{–}5)$$

By our sign convention, this induced moment is negative since for equilibrium it acts counterclockwise on the member.



real beam
(a)

conjugate beam
(b)

Fig. 11–5

*The moment diagrams shown on the conjugate beam were determined by the method of superposition for a simply supported beam, as explained in Sec. 4–5.

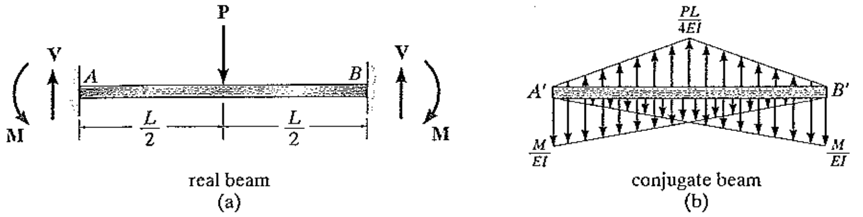


Fig. 11–6

Fixed-End Moments. In the previous cases we have considere relationships between the displacements and the necessary moments M_A and M_{BA} acting at nodes A and B, respectively. In general, however, th linear or angular displacements of the nodes are caused by loadings actin on the *span* of the member, not by moments acting at its nodes. In orde to develop the slope-deflection equations, we must transform these *spa loadings* into equivalent moments acting at the nodes and then use th load-displacement relationships just derived. This is done simply by findin the reaction moment that each load develops at the nodes. For exampl consider the fixed-supported member shown in Fig. 11–6*a*, which i subjected to a concentrated load **P** at its center. The conjugate beam fo this case is shown in Fig. 11–6*b*. Since we require the slope at each end t be zero,

$$+\uparrow \Sigma F_y = 0; \qquad \left[\frac{1}{2}\left(\frac{PL}{4EI}\right)L\right] - 2\left[\frac{1}{2}\left(\frac{M}{EI}\right)L\right] = 0$$

$$M = \frac{PL}{8}$$

This moment is called a *fixed-end moment* (FEM). Note that accordin to our sign convention, it is negative at node A (counterclockwise) an positive at node B (clockwise). For convenience in solving problem fixed-end moments have been calculated for other loadings and ar tabulated on the inside back cover of the book. Assuming these FEM have been computed for a specific problem (Fig. 11–7), we have

$$M_{AB} = (\text{FEM})_{AB} \qquad M_{BA} = (\text{FEM})_{BA} \qquad (11\text{–}6$$

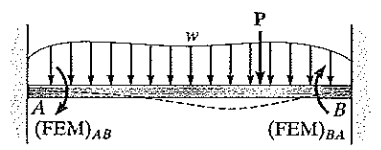


Fig. 11–7

Slope-Deflection Equation. If the end moments due to each displacement (Eqs. 11–1 through 11–5) and the loading (Eq. 11–6) are added together, the resultant moments at the ends can be written as

$$M_{AB} = 2E\left(\frac{I}{L}\right)\left[2\theta_A + \theta_B - 3\left(\frac{\Delta}{L}\right)\right] + (\text{FEM})_{AB}$$
$$M_{BA} = 2E\left(\frac{I}{L}\right)\left[2\theta_B + \theta_A - 3\left(\frac{\Delta}{L}\right)\right] + (\text{FEM})_{BA} \qquad (11\text{–}7)$$

Since these two equations are similar, the result can be expressed as a single equation. Referring to one end of the span as the near end (N) and the other end as the far end (F), and letting the *member stiffness* be represented as $k = I/L$, and the *span's cord rotation* as ψ (psi) $= \Delta/L$, we can write

$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N \qquad (11\text{–}8)$$

For Internal Span or End Span with Far End Fixed

where

M_N = internal moment in the near end of the span; this moment is *positive clockwise* when acting on the span.

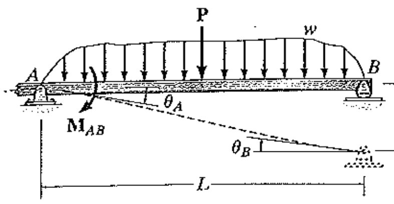
E, k = modulus of elasticity of material and span stiffness $k = I/L$.

θ_N, θ_F = near- and far-end slopes or angular displacements of the span at the supports; the angles are measured in *radians* and are *positive clockwise.*

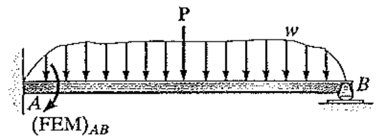
ψ = span rotation of its cord due to a linear displacement, that is, $\psi = \Delta/L$; this angle is measured in *radians* and is *positive clockwise.*

$(\text{FEM})_N$ = fixed-end moment at the near-end support; the moment is *positive clockwise* when acting on the span; refer to the table on the inside back cover for various loading conditions.

From the derivation Eq. 11–8 is both a compatibility and load-displacement relationship found by considering only the effects of bending and neglecting axial and shear deformations. It is referred to as the general *slope-deflection equation.* When used for the solution of problems, this equation is applied *twice* for each member span (AB); that is, application is from A to B and from B to A for span AB in Fig. 11–2.



(a)



(b)

Fig. 11–8

Pin-Supported End Span. Occasionally an end span of a beam or frame is supported by a pin or roller at its *far end,* Fig. 11–8*a*. When this occurs, the moment at the roller or pin must be zero; and provided the angular displacement θ_B at this support does not have to be determined, we can modify the general slope-deflection equation so that it has to be applied *only once* to the span rather than twice. To do this we will apply Eq. 11–8 or Eqs. 11–7 to each end of the beam in Fig. 11–8. This results in the following two equations:

$$\begin{aligned} M_N &= 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N \\ 0 &= 2Ek(2\theta_F + \theta_N - 3\psi) + 0 \end{aligned} \qquad (11\text{–}9)$$

Here the $(\text{FEM})_F$ is equal to zero since the far end is pinned, Fig. 11–8*b*. Furthermore, the $(\text{FEM})_N$ can be obtained, for example, using the table in the right-hand column on the inside back cover of this book. Multiplying the first equation by 2 and subtracting the second equation from it *eliminates* the unknown θ_F and yields

$$M_N = 3Ek(\theta_N - \psi) + (\text{FEM})_N \qquad (11\text{–}10)$$

Only for End Span with Far End Pinned or Roller Supported

Since the moment at the far end is zero, only *one* application of this equation is necessary for the end span. This simplifies the analysis since the general equation, Eq. 11–8, would require *two* applications for this span and therefore involve the (extra) unknown angular displacement θ_B (or θ_F) at the end support.

To summarize application of the slope-deflection equations, consider the continuous beam shown in Fig. 11–9 which has four degrees of freedom. Here Eq. 11–8 can be applied twice to each of the three spans, i.e., from A to B, B to A, B to C, C to B, C to D, and D to C. These equations would involve the four unknown rotations, θ_A, θ_B, θ_C, θ_D. Since the end moments at A and D are zero, however, it is not necessary to determine θ_A and θ_D. A shorter solution occurs if we apply Eq. 11–10 from B to A and C to D and then apply Eq. 11–8 from B to C and C to B. These four equations will involve only the unknown rotations θ_B and θ_C.

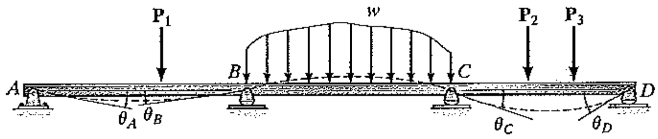


Fig. 11–9

11–3 Analysis of Beams

PROCEDURE FOR ANALYSIS

Degrees of Freedom. Label all the supports and joints (nodes) in order to identify the spans of the beam or frame between the nodes. By drawing the deflected shape of the structure, it will be possible to identify the number of degrees of freedom. Here each node can possibly have an angular displacement and a linear displacement. *Compatibility* at the nodes is maintained provided the members that are fixed connected to a node undergo the same displacements as the node. If these displacements are unknown, and in general they will be, then for convenience *assume* they act in the *positive direction* so as to cause *clockwise* rotation of a member or joint, Fig. 11–2.

Slope-Deflection Equations. The slope-deflection equations relate the unknown moments applied to the nodes to the displacements of the nodes for any span of the structure. If a load exists on the span, compute the FEMs using the table given on the inside back cover. Also, if a node has a linear displacement, Δ, compute $\psi = \Delta/L$ for the adjacent spans. Apply Eq. 11–8 to each end of the span, thereby generating *two* slope-deflection equations for each span. However, if a span at the *end* of a continuous beam or frame is pin supported, apply Eq. 11–10 only to the restrained end, thereby generating *one* slope-deflection equation for the span.

Equilibrium Equations. Write an equilibrium equation for each unknown degree of freedom for the structure. Each of these equations should be expressed in terms of unknown internal moments as specified by the slope-deflection equations. For beams and frames write the moment equation of equilibrium at each support, and for frames also write joint moment equations of equilibrium. If the frame sidesways or deflects horizontally, column shears should be related to the moments at the ends of the column. This is discussed in Sec. 11–5.

Substitute the slope-deflection equations into the equilibrium equations and solve for the unknown joint displacements. These results are then substituted into the slope-deflection equations to determine the internal moments at the ends of each member. If any of the results are *negative,* they indicate *counterclockwise* rotation; whereas *positive* moments and displacements are applied *clockwise.*

EXAMPLE 11-1

Draw the shear and moment diagrams for the beam shown in Fig. 11–10*a*. *EI* is constant.

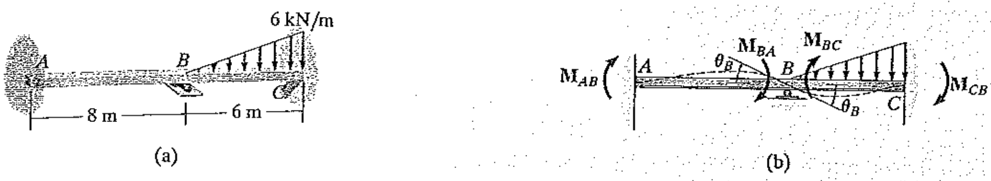


Fig. 11–10

Solution

Slope-Deflection Equations. Two spans must be considered in this problem. Since there is *no* span having the far end pinned or roller supported, Eq. 11–8 applies to the solution. Using the formulas for the FEMs tabulated for the triangular loading given on the inside back cover, we have

$$(\text{FEM})_{BC} = -\frac{wL^2}{30} = -\frac{6(6)^2}{30} = -7.2\ \text{kN}\cdot\text{m}$$

$$(\text{FEM})_{CB} = \frac{wL^2}{20} = \frac{6(6)^2}{20} = 10.8\ \text{kN}\cdot\text{m}$$

Note that $(\text{FEM})_{BC}$ is negative since it acts counterclockwise *on the beam* at B. Also, $(\text{FEM})_{AB} = (\text{FEM})_{BA} = 0$ since there is no load on span AB.

In order to identify the unknowns, the elastic curve for the beam is shown in Fig. 11–10*b*. As indicated, there are four unknown internal moments. Only the slope at B, θ_B, is unknown. Since A and C are fixed supports, $\theta_A = \theta_C = 0$. Also, since the supports do not settle, nor are they displaced up or down, $\psi_{AB} = \psi_{BC} = 0$. For span AB, considering A to be the near end and B to be the far end, we have

$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2E\left(\frac{I}{8}\right)[2(0) + \theta_B - 3(0)] + 0 = \frac{EI}{4}\theta_B \qquad (1)$$

Now, considering B to be the near end and A to be the far end, we have

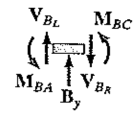
$$M_{BA} = 2E\left(\frac{I}{8}\right)[2\theta_B + 0 - 3(0)] + 0 = \frac{EI}{2}\theta_B \qquad (2)$$

In a similar manner, for span BC we have

$$M_{BC} = 2E\left(\frac{I}{6}\right)[2\theta_B + 0 - 3(0)] - 7.2 = \frac{2EI}{3}\theta_B - 7.2 \qquad (3)$$

$$M_{CB} = 2E\left(\frac{I}{6}\right)[2(0) + \theta_B - 3(0)] + 10.8 = \frac{EI}{3}\theta_B + 10.8 \qquad (4)$$

Equilibrium Equations. The above four equations contain five unknowns. The necessary fifth equation comes from the condition of moment equilibrium at support B. The free-body diagram of a segment of the beam at B is shown in Fig. 11–10c. Here $\mathbf{M}_{BA}$ and $\mathbf{M}_{BC}$ are assumed to act in the positive direction to be consistent with the slope-deflection equations.* The beam shears contribute negligible moment about B since the segment is of differential length. Thus,



(c)

$$\circlearrowright + \Sigma M_B = 0; \qquad M_{BA} + M_{BC} = 0 \qquad (5)$$

To solve, substitute Eqs. (2) and (3) into Eq. (5), which yields

$$\theta_B = \frac{6.17}{EI}$$

Resubstituting this value into Eqs. (1)–(4) yields

$$M_{AB} = 1.54 \text{ kN} \cdot \text{m}$$

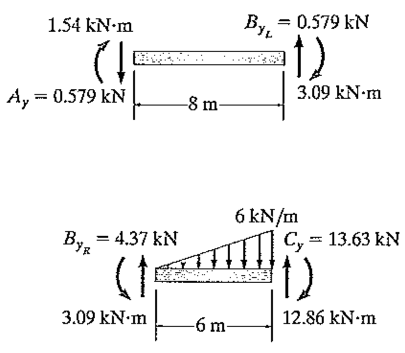
$$M_{BA} = 3.09 \text{ kN} \cdot \text{m}$$

$$M_{BC} = -3.09 \text{ kN} \cdot \text{m}$$

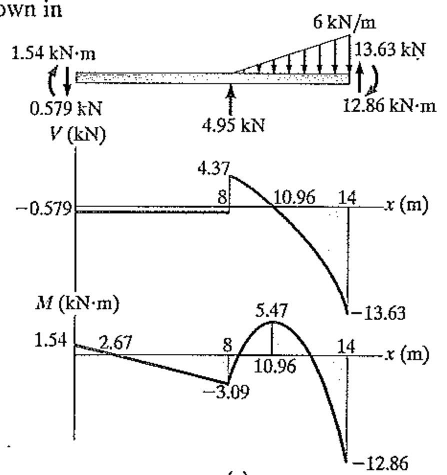
$$M_{CB} = 12.86 \text{ kN} \cdot \text{m}$$

The negative value for M_{BC} indicates that this moment acts counterclockwise on the beam, not clockwise as shown in Fig. 11–10b.

Using these results, the shears at the end spans are determined from the equilibrium equations, Fig. 11–10d. The free-body diagram of the entire beam and the shear and moment diagrams are shown in Fig. 11–10e.



(d)

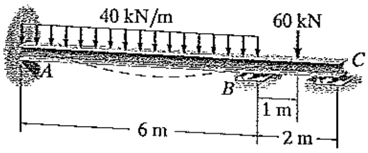


(e)

*Clockwise on the beam segment, but—by the principle of action, equal but opposite reaction—counterclockwise on the support.

EXAMPLE 11-2

Draw the shear and moment diagrams for the beam shown in Fig. 11–11*a*. *EI* is constant.



(a)

Fig. 11–11

Solution

Slope-Deflection Equations. Two spans must be considered in this problem. Equation 11–8 applies to span AB. We can use Eq. 11–10 for span BC since the *end C is on a roller*. Using the formulas for the FEMs tabulated on the inside back cover, we have

$$(\text{FEM})_{AB} = -\frac{wL^2}{12} = -\frac{1}{12}(40)(6)^2 = -120 \text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{BA} = \frac{wL^2}{12} = \frac{1}{12}(40)(6)^2 = 120 \text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{BC} = -\frac{3PL}{16} = -\frac{3(60)(2)}{16} = -22.5 \text{ kN}\cdot\text{m}$$

Note that $(\text{FEM})_{AB}$ and $(\text{FEM})_{BC}$ are negative since they act counterclockwise on the beam at A and B, respectively. Also, since the supports do not settle, $\psi_{AB} = \psi_{BC} = 0$. Applying Eq. 11–8 for span AB and realizing that $\theta_A = 0$, we have

$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2E\left(\frac{I}{6}\right)[2(0) + \theta_B - 3(0)] - 120$$

$$M_{AB} = 0.3333EI\theta_B - 120 \qquad (1)$$

$$M_{BA} = 2E\left(\frac{I}{6}\right)[2\theta_B + 0 - 3(0)] + 120$$

$$M_{BA} = 0.667EI\theta_B + 120 \qquad (2)$$

Applying Eq. 11–10 with B as the near end and C as the far end, we have

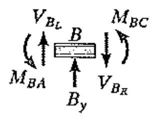
$$M_N = 3E\left(\frac{I}{L}\right)(\theta_N - \psi) + (\text{FEM})_N$$

$$M_{BC} = 3E\left(\frac{I}{2}\right)(\theta_B - 0) - 22.5$$

$$M_{BC} = 1.5EI\theta_B - 22.5 \qquad (3)$$

Remember that Eq. 11–10 is *not* applied from C (near end) to B (far end).

Equilibrium Equations. The above three equations contain four unknowns. The necessary fourth equation comes from the conditions of equilibrium at the support B. The free-body diagram is shown in Fig. 11–11*b*. We have



(b)

$$\circlearrowleft + \Sigma M_B = 0; \qquad M_{BA} + M_{BC} = 0 \qquad (4)$$

To solve, substitute Eqs. (2) and (3) into Eq. (4), which yields

$$\theta_B = -\frac{45}{EI}$$

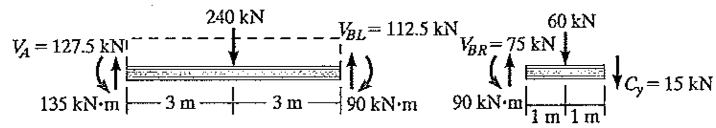
Since θ_B is negative (counterclockwise) the elastic curve for the beam has been correctly drawn in Fig. 11–11*a*. Substituting θ_B into Eqs. (1)–(3), we get

$$M_{AB} = -135 \text{ kN·m}$$

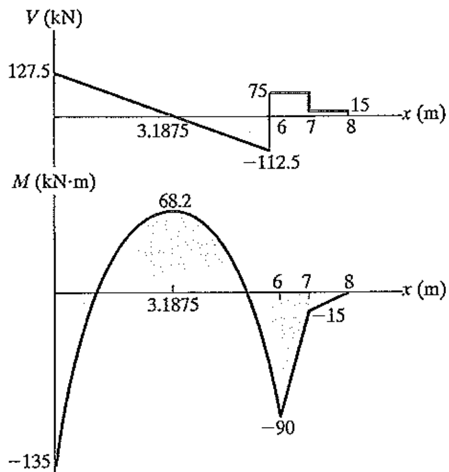
$$M_{BA} = 90 \text{ kN·m}$$

$$M_{BC} = -90 \text{ kN·m}$$

Using these data for the moments, the shear reactions at the ends of the beam spans have been determined in Fig. 11–11*c*. The shear and moment diagrams are plotted in Fig. 11–11*d*.



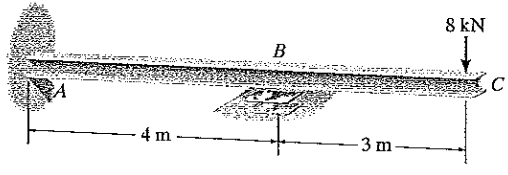
(c)



(d)

EXAMPLE 11-3

Determine the moment at A and B for the beam shown in Fig. 11–12a. The support at B is displaced (settles) 80 mm. Take $E = 200$ GPa, $I = 5(10^6)$ mm^4.

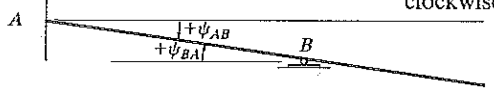


(a)

Fig. 11–12

Solution

Slope-Deflection Equations. Only one span (AB) must be considered in this problem since the moment $\mathbf{M}_{BC}$ due to the overhang can be calculated from statics. Since there is no loading on span AB, the FEMs are zero. As shown in Fig. 11–12b, the downward displacement (settlement) of B causes the cord for span AB to rotate clockwise. Thus,



(b)

$$\psi_{AB} = \psi_{BA} = \frac{0.08\text{ m}}{4} = 0.02\text{ rad}$$

The stiffness for AB is

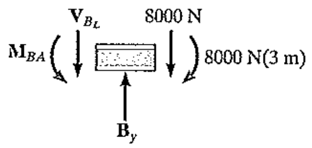
$$k = \frac{I}{L} = \frac{5(10^6)\text{ mm}^4(10^{-12})\text{ m}^4/\text{mm}^4}{4\text{ m}} = 1.25(10^{-6})\text{ m}^3$$

Applying the slope-deflection equation, Eq. 11–8, to span AB, with $\theta_A = 0$, we have

$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2(200(10^9)\text{ N/m}^2)[1.25(10^{-6})\text{ m}^3][2(0) + \theta_B - 3(0.02)] + 0 \quad (1)$$

$$M_{BA} = 2(200(10^9)\text{ N/m}^2)[1.25(10^{-6})\text{ m}^3][2\theta_B + 0 - 3(0.02)] + 0 \quad (2)$$



(c)

Equilibrium Equations. The free-body diagram of the beam at support B is shown in Fig. 11–12c. Moment equilibrium requires

$$\curvearrowright + \Sigma M_B = 0; \qquad M_{BA} - 8000\text{ N}(3\text{ m}) = 0$$

Substituting Eq. (2) into this equation yields

$$1(10^6)\theta_B - 30(10^3) = 24(10^3)$$

$$\theta_B = 0.054\text{ rad}$$

Thus, from Eqs. (1) and (2),

$$M_{AB} = -3.00\text{ kN}\cdot\text{m}$$

$$M_{BA} = 24.0\text{ kN}\cdot\text{m}$$

EXAMPLE 11-4

Determine the internal moments at the supports of the beam shown in Fig. 11–13*a*. The support at C is displaced (settles) 30 mm. Take $E = 200$ GPa, $I = 600(10^6)$ mm^4.

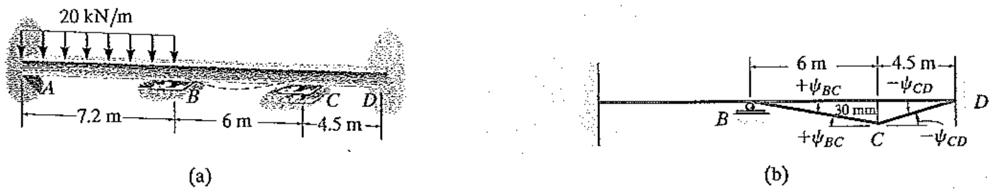


Fig. 11–13

Solution

Slope-Deflection Equations. Three spans must be considered in this problem. Equation 11–8 applies since the end supports A and D are fixed. Also, only span AB has FEMs.

$$(\text{FEM})_{AB} = -\frac{wL^2}{12} = -\frac{1}{12}(20)(7.2)^2 = -86.4 \text{ kN·m}$$

$$(\text{FEM})_{BA} = \frac{wL^2}{12} = \frac{1}{12}(20)(7.2)^2 = 86.4 \text{ kN·m}$$

As shown in Fig. 11–13*b*, the displacement (or settlement) of the support C causes ψ_{BC} to be positive, since the cord for span BC rotates clockwise, and ψ_{CD} to be negative, since the cord for span CD rotates counterclockwise. Hence,

$$\psi_{BC} = \frac{0.03 \text{ m}}{6 \text{ m}} = 0.005 \text{ rad} \quad \psi_{CD} = -\frac{0.03 \text{ m}}{4.5 \text{ m}} = -0.00667 \text{ rad}$$

Also, expressing the units for the stiffness in meters, we have

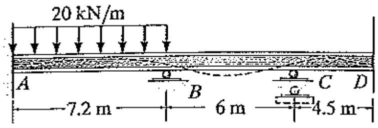
$$k_{AB} = \frac{600(10^6)(10^{-12})}{7.2} = 83.33(10^{-6}) \text{ m}^3$$

$$k_{BC} = \frac{600(10^6)(10^{-12})}{6} = 100(10^{-6}) \text{ m}^3$$

$$k_{CD} = \frac{600(10^6)(10^{-12})}{4.5} = 133.33(10^{-6}) \text{ m}^3$$

Noting that $\theta_A = \theta_D = 0$ since A and D are fixed supports, and applying the slope-deflection Eq. 11–8 twice to each span, we have

EXAMPLE 11-4 (Continued)



(a)

For span AB:

$$M_{AB} = 2[200(10^6)[83.33(10^{-6})]2(0) + \theta_B - 3(0)] - 86.4$$

$$M_{AB} = 33333.3\theta_B - 86.4 \qquad (1)$$

$$M_{BA} = 2[200(10^6)[83.33(10^{-6})]2\theta_B + 0 - 3(0)] + 86.4$$

$$M_{BA} = 66666.7\theta_B + 86.4 \qquad (2)$$

For span BC:

$$M_{BC} = 2[200(10^6)[100(10^{-6})]2\theta_B + \theta_C - 3(0.005)] + 0$$

$$M_{BC} = 80000\theta_B + 40000\theta_C - 600 \qquad (3)$$

$$M_{CB} = 2[200(10^6)[100(10^{-6})]2\theta_C + \theta_B - 3(0.005)] + 0$$

$$M_{CB} = 80000\theta_C + 40000\theta_B - 600 \qquad (4)$$

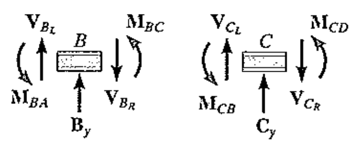
For span CD:

$$M_{CD} = 2[200(10^6)[133(10^{-6})]2\theta_C + 0 - 3(-0.00667)] + 0$$

$$M_{CD} = 106666.7\theta_C + 0 + 1066.7 \qquad (5)$$

$$M_{DC} = 2[200(10^6)[133.33(10^{-6})]2(0) + \theta_C - 3(-0.00667)] + 0$$

$$M_{DC} = 53333.3\theta_C + 1066.7 \qquad (6)$$



(c)

Equilibrium Equations. These six equations contain eight unknowns. Writing the moment equilibrium equations for the supports at B and C, Fig. 10–13c, we have

$$\circlearrowleft + \Sigma M_B = 0; \qquad M_{BA} + M_{BC} = 0 \qquad (7)$$

$$\circlearrowleft + \Sigma M_C = 0; \qquad M_{CB} + M_{CD} = 0 \qquad (8)$$

In order to solve, substitute Eqs. (2) and (3) into Eq. (7), and Eqs. (4) and (5) into Eq. (8). This yields

$$\theta_C + 3.667\theta_B = 0.01284$$

$$-\theta_C - 0.214\theta_B = 0.00250$$

Thus,

$$\theta_B = 0.00444 \text{ rad} \qquad \theta_C = -0.00345 \text{ rad}$$

The negative value for θ_C indicates counterclockwise rotation of the tangent at C, Fig. 11–13a. Substituting these values into Eqs. (1)–(6) yields

$M_{AB} = 61.6$ kN·m *Ans.*

$M_{BA} = 383$ kN·m *Ans.*

$M_{BC} = -383$ kN·m *Ans.*

$M_{CB} = -698$ kN·m *Ans.*

$M_{CD} = 698$ kN·m *Ans.*

$M_{DC} = 883$ kN·m *Ans.*

11–4 Analysis of Frames: No Sidesway

A frame will not sidesway, or be displaced to the left or right, provided it is properly restrained. Examples are shown in Fig. 11–14. Also, no sidesway will occur in an unrestrained frame provided it is symmetric with respect to both loading and geometry, as shown in Fig. 11–15. For both cases the term ψ in the slope-deflection equations is equal to zero, since bending does not cause the joints to have a linear displacement.

The following examples illustrate application of the slope-deflection equations using the procedure for analysis outlined in Sec. 11–3 for these types of frames.

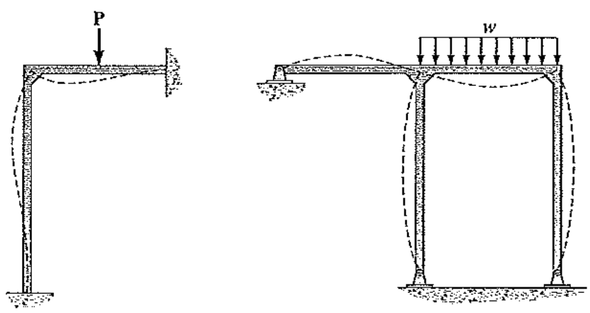


Fig. 11–14

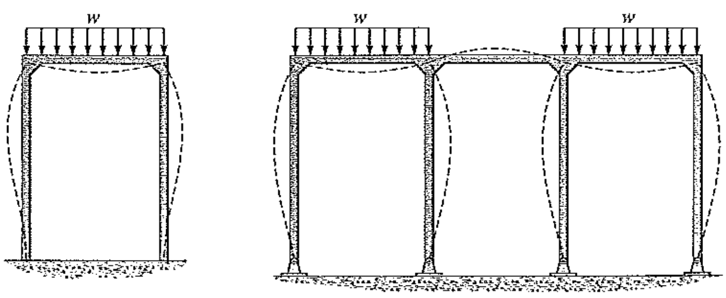
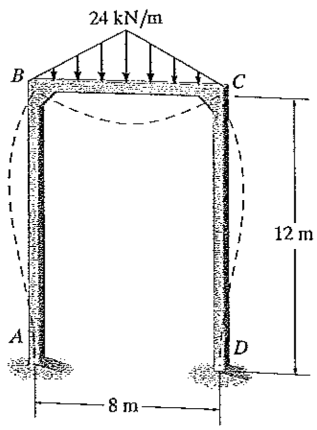


Fig. 11–15

EXAMPLE 11-5



(a)

Fig. 11-16

Determine the moments at each joint of the frame shown in Fig. 11–16*a*. *EI* is constant.

Solution

Slope-Deflection Equations. Three spans must be considered in this problem: *AB*, *BC*, and *CD*. Since the spans are fixed supported at *A* and *D*, Eq. 11–8 applies for the solution.

From the table on the inside back cover, the FEMs for *BC* are

$$(\text{FEM})_{BC} = -\frac{5wL^2}{96} = -\frac{5(24)(8)^2}{96} = -80 \text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{CB} = \frac{5wL^2}{96} = \frac{5(24)(8)^2}{96} = 80 \text{ kN}\cdot\text{m}$$

Note that $\theta_A = \theta_D = 0$ and $\psi_{AB} = \psi_{BC} = \psi_{CD} = 0$, since no sidesway will occur.

Applying Eq. 11–8, we have

$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2E\left(\frac{I}{12}\right)[2(0) + \theta_B - 3(0)] + 0$$

$$M_{AB} = 0.1667EI\theta_B \qquad (1)$$

$$M_{BA} = 2E\left(\frac{I}{12}\right)[2\theta_B + 0 - 3(0)] + 0$$

$$M_{BA} = 0.333EI\theta_B \qquad (2)$$

$$M_{BC} = 2E\left(\frac{I}{8}\right)[2\theta_B + \theta_C - 3(0)] - 80$$

$$M_{BC} = 0.5EI\theta_B + 0.25EI\theta_C - 80 \qquad (3)$$

$$M_{CB} = 2E\left(\frac{I}{8}\right)[2\theta_C + \theta_B - 3(0)] + 80$$

$$M_{CB} = 0.5EI\theta_C + 0.25EI\theta_B + 80 \qquad (4)$$

$$M_{CD} = 2E\left(\frac{I}{12}\right)[2\theta_C + 0 - 3(0)] + 0$$

$$M_{CD} = 0.333EI\theta_C \qquad (5)$$

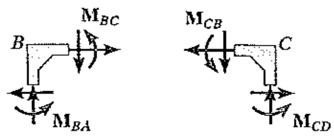
$$M_{DC} = 2E\left(\frac{I}{12}\right)[2(0) + \theta_C - 3(0)] + 0$$

$$M_{DC} = 0.1667EI\theta_C \qquad (6)$$

Equilibrium Equations. The preceding six equations contain eight unknowns. The remaining two equilibrium equations come from moment equilibrium at joints B and C, Fig. 11–16*b*. We have

$$M_{BA} + M_{BC} = 0 \quad (7)$$

$$M_{CB} + M_{CD} = 0 \quad (8)$$



(b)

To solve these eight equations, substitute Eqs. (2) and (3) into Eq. (7) and substitute Eqs. (4) and (5) into Eq. (8). We get

$$0.833EI\theta_B + 0.25EI\theta_C = 80$$

$$0.833EI\theta_C + 0.25EI\theta_B = -80$$

Solving simultaneously yields

$$\theta_B = -\theta_C = \frac{137.1}{EI}$$

which conforms with the way the frame deflects as shown in Fig. 11–16*a*. Substituting into Eqs. (1)–(6), we get

$$M_{AB} = 22.9 \text{ kN} \cdot \text{m} \qquad \textit{Ans.}$$

$$M_{BA} = 45.7 \text{ kN} \cdot \text{m} \qquad \textit{Ans.}$$

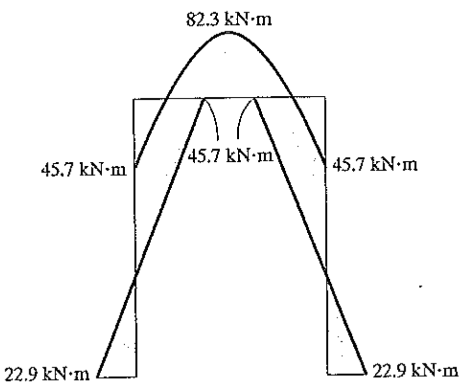
$$M_{BC} = -45.7 \text{ kN} \cdot \text{m} \qquad \textit{Ans.}$$

$$M_{CB} = 45.7 \text{ kN} \cdot \text{m} \qquad \textit{Ans.}$$

$$M_{CD} = -45.7 \text{ kN} \cdot \text{m} \qquad \textit{Ans.}$$

$$M_{DC} = -22.9 \text{ kN} \cdot \text{m} \qquad \textit{Ans.}$$

Using these results, the reactions at the ends of each member can be determined from the equations of equilibrium, and the moment diagram for the frame can be drawn, Fig. 11–16*c*.



(c)

EXAMPLE 11-6

Determine the internal moments at each joint of the frame shown in Fig. 11–17*a*. The moment of inertia for each member is given in the figure. Take $E = 200$ GPa.

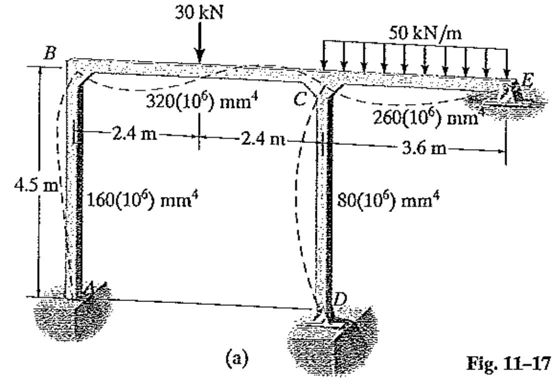


Fig. 11–17

Solution

Slope-Deflection Equations. Four spans must be considered in this problem. Equation 11–8 applies to spans AB and BC, and Eq. 11–10 will be applied to CD and CE, because the ends at D and E are pinned.

Computing the member stiffnesses, we have

$$k_{AB} = \frac{160(10^6)(10^{-12})}{4.5} = 35.56(10^{-6})\text{ m}^3 \qquad k_{CD} = \frac{80(10^6)(10^{-12})}{4.5} = 17.78(10^{-6})\text{ m}^3$$

$$k_{BC} = \frac{320(10^6)(10^{-12})}{4.8} = 66.67(10^{-6})\text{ m}^3 \qquad k_{CE} = \frac{260(10^6)(10^{-12})}{3.6} = 72.23(10^{-6})\text{ m}^3$$

The FEMs due to the loadings are

$$(\text{FEM})_{BC} = -\frac{PL}{8} = -\frac{30(4.8)}{8} = -18\text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{CB} = \frac{PL}{8} = \frac{30(4.8)}{8} = 18\text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{CE} = -\frac{wL^2}{8} = -\frac{50(3.6)^2}{8} = -81\text{ kN}\cdot\text{m}$$

Applying Eqs. 11–8 and 11–10 to the frame and noting that $\theta_A = 0$, $\psi_{AB} = \psi_{BC} = \psi_{CD} = \psi_{CE} = 0$ since no sidesway occurs, we have

$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2[200(10^6)](35.56)(10^{-6})[2(0) + \theta_B - 3(0)] + 0$$

$$M_{AB} = 14222.2\theta_B \qquad (1)$$

$$M_{BA} = 2[200(10^6)](35.56)(10^{-6})[2\theta_B + 0 - 3(0)] + 0$$

$$M_{BA} = 28444.4\theta_B \qquad (2)$$

$$M_{BC} = 2[200(10^6)](66.67)(10^{-6})[2\theta_B + \theta_C - 3(0)] - 18$$

$$M_{BC} = 53333.3\theta_B + 26666.7\theta_C - 18 \qquad (3)$$

$$M_{CB} = 2[200(10^6)](66.67)(10^{-6})[2\theta_C + \theta_B - 3(0)] + 18$$

$$M_{CB} = 26666.7\theta_B + 53333.3\theta_C + 18 \qquad (4)$$

$$M_N = 3Ek(\theta_N - \psi) + (\text{FEM})_N$$

$$M_{CD} = 3[200(10^6)](17.78)(10^{-6})[\theta_C - 0] + 0 \qquad (5)$$

$$M_{CD} = 10666.7\theta_C$$

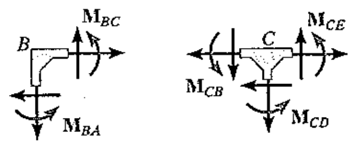
$$M_{CE} = 3[200(10^6)](72.22)(10^{-6})[\theta_C - 0] - 81$$

$$M_{CE} = 43333.3\theta_C - 81 \qquad (6)$$

Equations of Equilibrium. These six equations contain eight unknowns. Two moment equilibrium equations can be written for joints B and C, Fig. 11–17b. We have

$$M_{BA} + M_{BC} = 0 \qquad (7)$$

$$M_{CB} + M_{CD} + M_{CE} = 0 \qquad (8)$$



(b)

In order to solve, substitute Eqs. (2) and (3) into Eq. (7), and Eqs. (4)–(6) into Eq. (8). This gives

$$81777.7\theta_B + 26666.7\theta_C = 18$$

$$26666.7\theta_B + 107333.3\theta_C = 63$$

Solving these equations simultaneously yields

$$\theta_B = 3.124(10^{-5}) \text{ rad} \qquad \theta_C = 5.792(10^{-4}) \text{ rad}$$

These values, being clockwise, tend to distort the frame as shown in Fig. 11–17a. Substituting these values into Eqs. (1)–(6) and solving, we get

$M_{AB} = 0.444$ kN·m *Ans.*

$M_{BA} = 0.888$ kN·m *Ans.*

$M_{BC} = -0.888$ kN·m *Ans.*

$M_{CB} = 49.7$ kN·m *Ans.*

$M_{CD} = 6.18$ kN·m *Ans.*

$M_{CE} = -55.9$ kN·m *Ans.*

11–5 Analysis of Frames: Sidesway

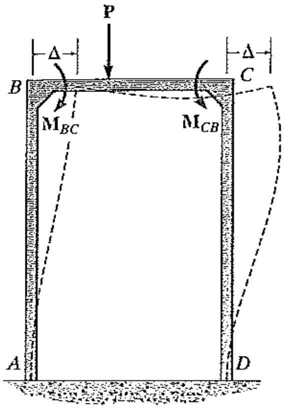
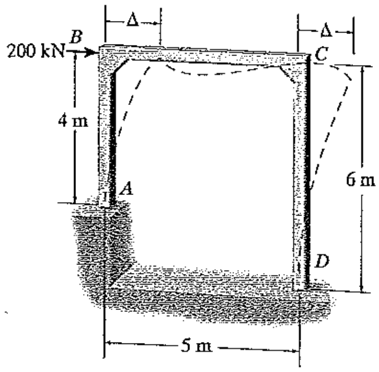


Fig. 11–18

A frame will sidesway, or be displaced to the side, when it or the loading acting on it is nonsymmetric. To illustrate this effect, consider the frame shown in Fig. 11–18. Here the loading **P** causes *unequal* moments $\mathbf{M}_{BC}$ and $\mathbf{M}_{CB}$ at the joints B and C, respectively. $\mathbf{M}_{BC}$ tends to displace joint B to the right, whereas $\mathbf{M}_{CB}$ tends to displace joint C to the left. Since $\mathbf{M}_{BC}$ is larger than $\mathbf{M}_{CB}$, the net result is a sidesway Δ of both joints B and C to the right, as shown in the figure.* When applying the slope-deflection equation to each column of this frame, we must therefore consider the column rotation ψ (since $\psi = \Delta/L$) as unknown in the equation. As a result an extra equilibrium equation must be included for the solution. In the previous sections it was shown that unknown *angular displacements* θ were related by joint *moment equilibrium equations*. In a similar manner, when unknown joint *linear displacements* Δ (or span rotations ψ) occur, we must write *force equilibrium equations* in order to obtain the complete solution. The unknowns in these equations, however, must only involve the internal *moments* acting at the ends of the columns, since the slope-deflection equations involve these moments. The technique for solving problems for frames with sidesway is best illustrated by examples.

EXAMPLE 11-7



(a)

Fig. 11–19

Determine the moments at each joint of the frame shown in Fig. 11–19*a*. EI is constant.

Solution

Slope-Deflection Equations. Since the ends A and D are fixed, Eq. 11–8 applies for all three spans of the frame. Sidesway occurs here since both the applied loading and the geometry of the frame are nonsymmetric. Here the load is applied directly to joint B and therefore no FEMs act at the joints. As shown in Fig. 11–19*a*, both joints B and C are assumed to be displaced an *equal amount* Δ. Consequently, $\psi_{AB} = \Delta/4$ and $\psi_{DC} = \Delta/6$. Both terms are positive since the cords of members AB and CD "rotate" clockwise. Relating ψ_{AB} to ψ_{DC}, we have $\psi_{AB} = (6/4)\psi_{DC}$. Applying Eq. 11–8 to the frame, we have

$$M_{AB} = 2E\left(\frac{I}{4}\right)\left[2(0) + \theta_B - 3\left(\frac{6}{4}\psi_{DC}\right)\right] + 0 = EI(0.5\theta_B - 2.25\psi_{DC}) \quad (1)$$

$$M_{BA} = 2E\left(\frac{I}{4}\right)\left[2\theta_B + 0 - 3\left(\frac{6}{4}\psi_{DC}\right)\right] + 0 = EI(1.0\theta_C - 2.25\psi_{DC}) \quad (2)$$

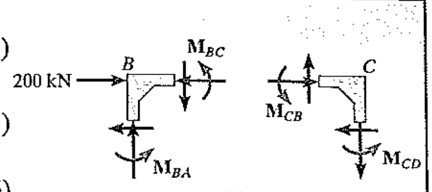
$$M_{BC} = 2E\left(\frac{I}{5}\right)[2\theta_B + \theta_C - 3(0)] + 0 = EI(0.8\theta_B + 0.4\theta_C) \quad (3)$$

*Recall that the deformation of all three members due to shear and axial force is neglected.

$$M_{CB} = 2E\left(\frac{I}{5}\right)[2\theta_C + \theta_B - 3(0)] + 0 = EI(0.8\theta_C + 0.4\theta_B) \quad (4)$$

$$M_{CD} = 2E\left(\frac{I}{6}\right)[2\theta_C + 0 - 3\psi_{DC}] + 0 = EI(0.667\theta_C - 1.0\psi_{DC}) \quad (5)$$

$$M_{DC} = 2E\left(\frac{I}{6}\right)[2(0) + \theta_C - 3\psi_{DC}] + 0 = EI(0.333\theta_C - 1.0\psi_{DC}) \quad (6)$$



(b)

Equations of Equilibrium. The six equations contain nine unknowns. Two moment equilibrium equations for joints B and C, Fig. 11–19*b*, can be written, namely,

$$M_{BA} + M_{BC} = 0 \quad (7)$$
$$M_{CB} + M_{CD} = 0 \quad (8)$$

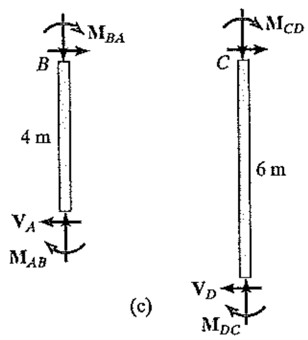
Since a horizontal displacement Δ occurs, we will consider summing forces on the *entire frame* in the x direction. This yields

$\xrightarrow{+} \Sigma F_x = 0; \qquad 200 - V_A - V_D = 0$

The horizontal reactions or column shears V_A and V_D can be related to the internal moments by considering the free-body diagram of each column separately, Fig. 11–19*c*. We have

$$\Sigma M_B = 0; \qquad V_A = -\frac{M_{AB} + M_{BA}}{4}$$

$$\Sigma M_C = 0; \qquad V_D = -\frac{M_{DC} + M_{CD}}{6}$$



(c)

Thus,

$$200 + \frac{M_{AB} + M_{BA}}{4} + \frac{M_{DC} + M_{CD}}{6} = 0 \quad (9)$$

In order to solve, substitute Eqs. (2) and (3) into Eq. (7), Eqs. (4) and (5) into Eq. (8), and Eqs. (1), (2), (5), (6) into Eq. (9). This yields

$$1.8\theta_B + 0.4\theta_C - 2.25\psi_{DC} = 0$$
$$0.4\theta_B + 1.467\theta_C - \psi_{DC} = 0$$
$$1.5\theta_B + 0.667\theta_C - 5.833\psi_{DC} = \frac{800}{EI}$$

Solving simultaneously, we have

$$EI\theta_B = 243.78 \qquad EI\theta_C = 75.66 \qquad EI\psi_{DC} = 208.48$$

Finally, using these results and solving Eqs. (1)–(6) yields

$M_{AB} = -347$ kN·m *Ans.*

$M_{BA} = -225$ kN·m *Ans.*

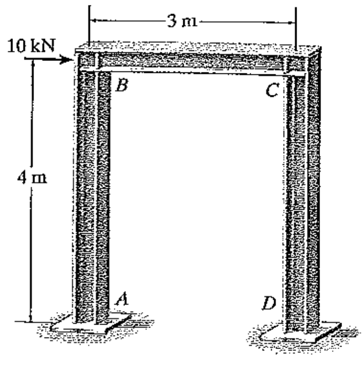
$M_{BC} = 225$ kN·m *Ans.*

$M_{CB} = 158$ kN·m *Ans.*

$M_{CD} = -158$ kN·m *Ans.*

$M_{DC} = -183$ kN·m *Ans.*

EXAMPLE 11-8



(a)

Fig. 11–20

Determine the moments at each joint of the frame shown in Fig. 11–20a. The supports at A and D are fixed and joint C is assumed pin connected. EI is constant for each member.

Solution

Slope-Deflection Equations. We will apply Eq. 11–8 to member AB since it is fixed connected at both ends. Equation 11–10 can be applied from B to C and from D to C since the pin at C supports zero moment. As shown by the deflection diagram, Fig. 11–20b, there is an unknown linear displacement Δ of the frame and unknown angular displacement θ_B at joint B.* Due to Δ, the cord members AB and CD rotate clockwise, $\psi = \psi_{AB} = \psi_{DC} = \Delta/4$. Realizing that $\theta_A = \theta_D = 0$ and that there are no FEMs for the members, we have

$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

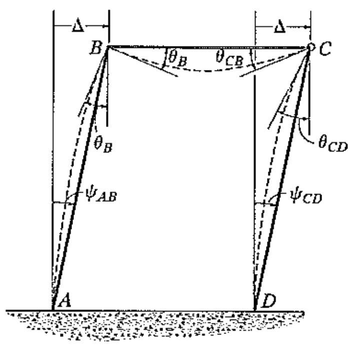
$$M_{AB} = 2E\left(\frac{I}{4}\right)[2(0) + \theta_B - 3\psi] + 0 \qquad (1)$$

$$M_{BA} = 2E\left(\frac{I}{4}\right)(2\theta_B + 0 - 3\psi) + 0 \qquad (2)$$

$$M_N = 3E\left(\frac{I}{L}\right)(\theta_N - \psi) + (\text{FEM})_N$$

$$M_{BC} = 3E\left(\frac{I}{3}\right)(\theta_B - 0) + 0 \qquad (3)$$

$$M_{DC} = 3E\left(\frac{I}{4}\right)(0 - \psi) + 0 \qquad (4)$$



(b)

Equilibrium Equations. Moment equilibrium of joint B, Fig. 11–20c, requires

$$M_{BA} + M_{BC} = 0 \qquad (5)$$

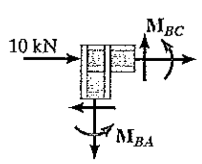
If forces are summed for the *entire frame* in the horizontal direction, we have

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 10 - V_A - V_D = 0 \qquad (6)$$

As shown on the free-body diagram of each column, Fig. 11–20d, we have

$$\Sigma M_B = 0; \qquad V_A = -\frac{M_{AB} + M_{BA}}{4}$$

$$\Sigma M_C = 0; \qquad V_D = -\frac{M_{DC}}{4}$$



(c)

*The angular displacements θ_{CB} and θ_{CD} at joint C (pin) are not included in the analysis since Eq. 11–10 is to be used.

Thus, from Eq. (6),

$$10 + \frac{M_{AB} + M_{BA}}{4} + \frac{M_{DC}}{4} = 0 \qquad (7)$$

Substituting the slope-deflection equations into Eqs. (5) and (7) and simplifying yields

$$\theta_B = \frac{3}{4}\psi$$

$$10 + \frac{EI}{4}\left(\frac{3}{2}\theta_B - \frac{15}{4}\psi\right) = 0$$

Thus,

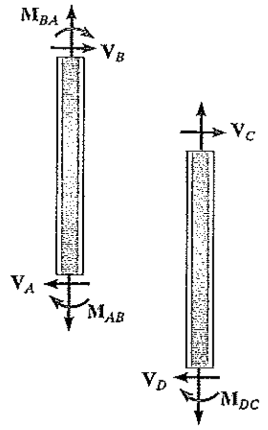
$$\theta_B = \frac{240}{21EI} \qquad \psi = \frac{320}{21EI}$$

Substituting these values into Eqs. (1)–(4), we have

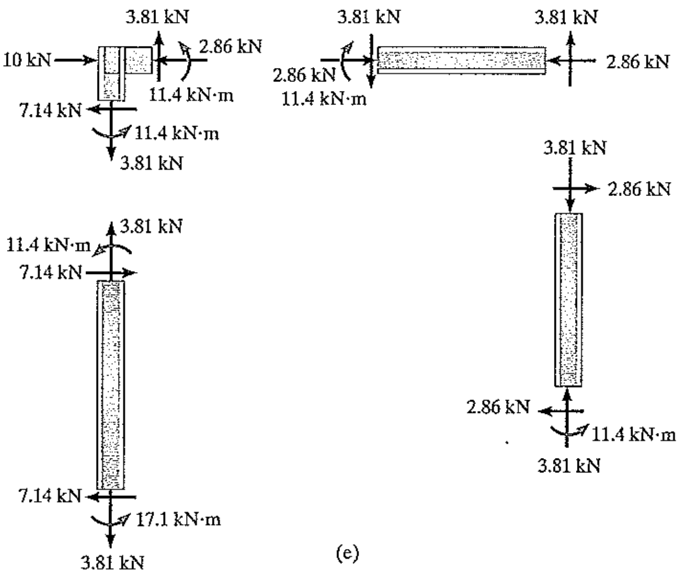
$M_{AB} = -17.1\ \text{kN}\cdot\text{m}$, $M_{BA} = -11.4\ \text{kN}\cdot\text{m}$ *Ans.*

$M_{BC} = 11.4\ \text{kN}\cdot\text{m}$, $M_{DC} = -11.4\ \text{kN}\cdot\text{m}$ *Ans.*

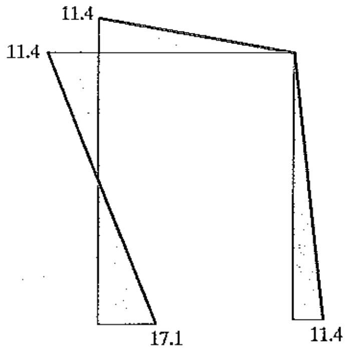
Using these results, the end reactions on each member can be determined from the equations of equilibrium, Fig. 11–20*e*. The moment diagram for the frame is shown in Fig. 11–20*f*.



(d)

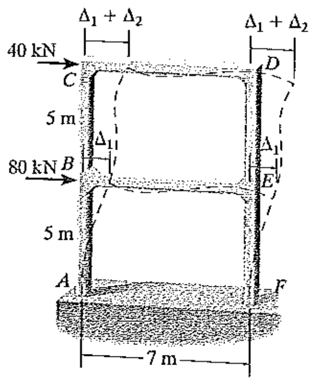


(e)



(f)

EXAMPLE 11-9



(a)

Fig. 11–21

Explain how the moments in each joint of the two-story frame shown in Fig. 11–21*a* are determined. *EI* is constant.

Solution

Slope-Deflection Equation. Since the supports at A and F are fixed, Eq. 11–8 applies for all six spans of the frame. No FEMs have to be calculated, since the applied loading acts at the joints. Here the loading displaces joints B and E an amount Δ_1, and C and D an amount $\Delta_1 + \Delta_2$. As a result, members AB and FE undergo rotations of $\psi_1 = \Delta_1/5$ and BC and ED undergo rotations of $\psi_2 = \Delta_2/5$.

Applying Eq. 11–8 to the frame yields

$$M_{AB} = 2E\left(\frac{I}{5}\right)[2(0) + \theta_B - 3\psi_1] + 0 \qquad (1)$$

$$M_{BA} = 2E\left(\frac{I}{5}\right)[2\theta_B + 0 - 3\psi_1] + 0 \qquad (2)$$

$$M_{BC} = 2E\left(\frac{I}{5}\right)[2\theta_B + \theta_C - 3\psi_2] + 0 \qquad (3)$$

$$M_{CB} = 2E\left(\frac{I}{5}\right)[2\theta_C + \theta_B - 3\psi_2] + 0 \qquad (4)$$

$$M_{CD} = 2E\left(\frac{I}{7}\right)[2\theta_C + \theta_D - 3(0)] + 0 \qquad (5)$$

$$M_{DC} = 2E\left(\frac{I}{7}\right)[2\theta_D + \theta_C - 3(0)] + 0 \qquad (6)$$

$$M_{BE} = 2E\left(\frac{I}{7}\right)[2\theta_B + \theta_E - 3(0)] + 0 \qquad (7)$$

$$M_{EB} = 2E\left(\frac{I}{7}\right)[2\theta_E + \theta_B - 3(0)] + 0 \qquad (8)$$

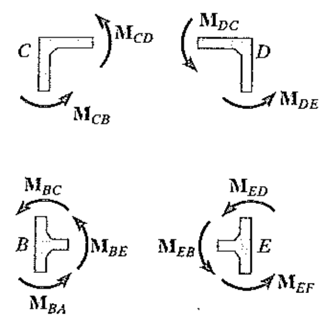
$$M_{ED} = 2E\left(\frac{I}{5}\right)[2\theta_E + \theta_D - 3\psi_2] + 0 \qquad (9)$$

$$M_{DE} = 2E\left(\frac{I}{5}\right)[2\theta_D + \theta_E - 3\psi_2] + 0 \qquad (10)$$

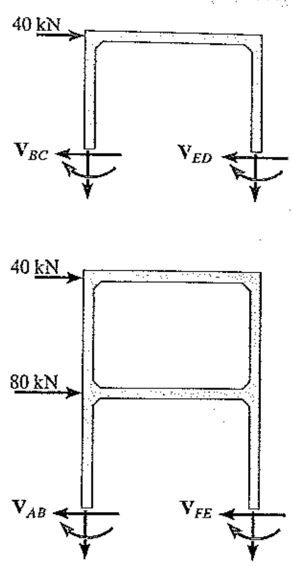
$$M_{FE} = 2E\left(\frac{I}{5}\right)[2(0) + \theta_E - 3\psi_1] + 0 \qquad (11)$$

$$M_{EF} = 2E\left(\frac{I}{5}\right)[2\theta_E + 0 - 3\psi_1] + 0 \qquad (12)$$

These 12 equations contain 18 unknowns.



(b)



(c)

Equilibrium Equations. Moment equilibrium of joints B, C, D, and E, Fig. 11–21b, requires

$$M_{BA} + M_{BE} + M_{BC} = 0 \quad (13)$$

$$M_{CB} + M_{CD} = 0 \quad (14)$$

$$M_{DC} + M_{DE} = 0 \quad (15)$$

$$M_{EF} + M_{EB} + M_{ED} = 0 \quad (16)$$

As in the preceding examples, the shear at the base of all the columns for any story must balance the applied horizontal loads, Fig. 11–21c. This yields

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 40 - V_{BC} - V_{ED} = 0$$

$$40 + \frac{M_{BC} + M_{CB}}{5} + \frac{M_{ED} + M_{DE}}{5} = 0 \quad (17)$$

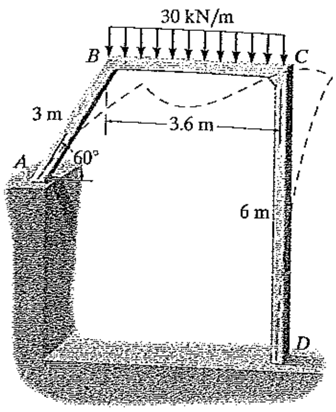
$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 40 + 80 - V_{AB} - V_{FE} = 0$$

$$120 + \frac{M_{AB} + M_{BA}}{5} + \frac{M_{EF} + M_{FE}}{5} = 0 \quad (18)$$

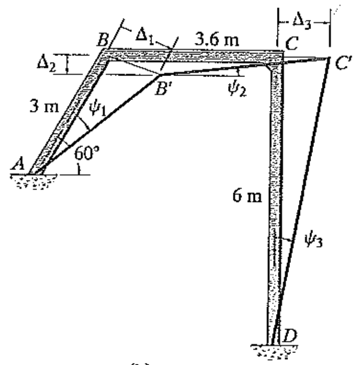
Solution requires substituting Eqs. (1)–(12) into Eqs. (13)–(18), which yields six equations having six unknowns, ψ_1, ψ_2, θ_B, θ_C, θ_D, and θ_E. These equations can then be solved simultaneously. The results are resubstituted into Eqs. (1)–(12), which yields the moments at the joints.

EXAMPLE 11-10

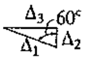
Determine the moments at each joint of the frame shown in Fig. 11–22*a*. EI is constant for each member.



(a)



(b)



(c)

Fig. 11–22

Solution

Slope-Deflection Equations. Equation 11–8 applies to each of the three spans. The FEMs are

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{30(3.6)^2}{12} = -32.4 \text{ kN·m}$$

$$(\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{30(3.6)^2}{12} = 32.4 \text{ kN·m}$$

The sloping member AB causes the frame to sidesway to the right as shown in Fig. 11–22*a*. As a result, joints B and C are subjected to both rotational *and* linear displacements. The linear displacements are shown in Fig. 11–22*b*, where B moves Δ_1 to B' and C moves Δ_3 to C'. These displacements cause the members' cords to rotate ψ_1, ψ_3 (clockwise) and $-\psi_2$ (counterclockwise) as shown.* Hence,

$$\psi_1 = \frac{\Delta_1}{3} \qquad \psi_2 = -\frac{\Delta_1}{3.6} \qquad \psi_3 = \frac{\Delta_3}{6}$$

As shown in Fig. 11–22*c*, the three displacements can be related. For example, $\Delta_2 = 0.5\Delta_1$ and $\Delta_3 = 0.866\Delta_1$. Thus, from the above equations we have

$$\psi_2 = -0.417\psi_1 \qquad \psi_3 = 0.433\psi_1$$

Using these results, the slope-deflection equations for the frame are

*Recall that distortions due to axial forces are neglected and the arc displacements BB' and CC' can be considered as straight lines, since ψ_1 and ψ_3 are actually very small.

$$M_{AB} = 2E\left(\frac{I}{3}\right)[2(0) + \theta_B - 3\psi_1] + 0 \quad (1)$$

$$M_{BA} = 2E\left(\frac{I}{3}\right)[2\theta_B + 0 - 3\psi_1] + 0 \quad (2)$$

$$M_{BC} = 2E\left(\frac{I}{3.6}\right)[2\theta_B + \theta_C - 3(-0.417\psi_1)] - 32.4 \quad (3)$$

$$M_{CB} = 2E\left(\frac{I}{3.6}\right)[2\theta_C + \theta_B - 3(-0.417\psi_1)] + 32.4 \quad (4)$$

$$M_{CD} = 2E\left(\frac{I}{6}\right)[2\theta_C + 0 - 3(0.433\psi_1)] + 0 \quad (5)$$

$$M_{DC} = 2E\left(\frac{I}{6}\right)[2(0) + \theta_C - 3(0.433\psi_1)] + 0 \quad (6)$$

These six equations contain nine unknowns.

Equations of Equilibrium. Moment equilibrium at joints B and C yields

$$M_{BA} + M_{BC} = 0 \quad (7)$$

$$M_{CD} + M_{CB} = 0 \quad (8)$$

The necessary third equilibrium equation can be obtained by summing moments about point O on the entire frame, Fig. 11–22*d*. This eliminates the unknown normal forces $\mathbf{N}_A$ and $\mathbf{N}_D$, and therefore

$\curvearrowleft + \Sigma M_O = 0;$

$$M_{AB} + M_{DC} - \left(\frac{M_{AB}+M_{BA}}{3}\right)(10.2) - \left(\frac{M_{DC}+M_{CD}}{6}\right)(12.24) - 108(1.8) = 0$$

$$-2.4M_{AB} - 3.4M_{BA} - 2.04M_{CD} - 1.04M_{DC} - 194.4 = 0 \quad (9)$$

Substituting Eqs. (2) and (3) into Eq. (7), Eqs. (4) and (5) into Eq. (8), and Eqs. (1), (2), (5), and (6) into Eq. (9) yields

$$0.733\theta_B + 0.167\theta_C - 0.392\psi_1 = \frac{9.72}{EI}$$

$$0.167\theta_B + 0.533\theta_C + 0.0784\psi_1 = -\frac{9.72}{EI}$$

$$-1.840\theta_B - 0.512\theta_C + 3.880\psi_1 = \frac{58.32}{EI}$$

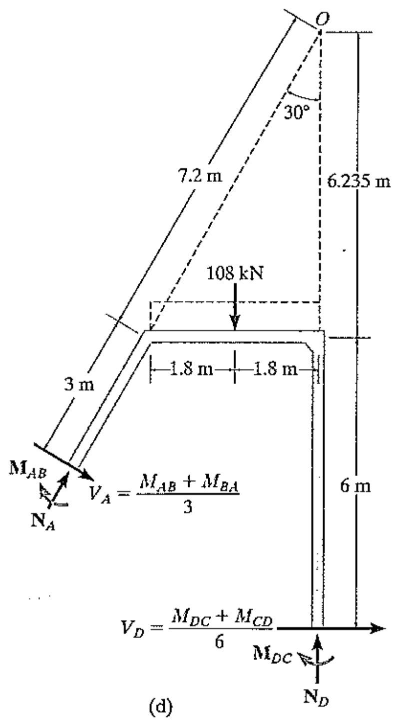
Solving these equations simultaneously yields

$$EI\theta_B = 35.51 \qquad EI\theta_C = -33.33 \qquad EI\psi_1 = 27.47$$

Substituting these values into Eqs. (1)–(6), we have

$M_{AB} = -31.3$ kN·m $\quad M_{BC} = 7.60$ kN·m $\quad M_{CD} = -34.2$ kN·m *Ans.*

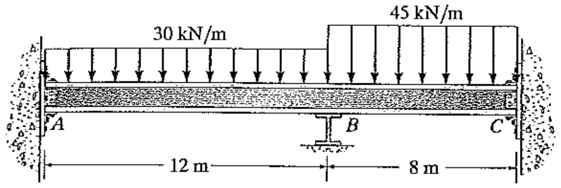
$M_{BA} = -7.60$ kN·m $\quad M_{CB} = 34.2$ kN·m $\quad M_{DC} = -23.0$ kN·m *Ans.*



(d)

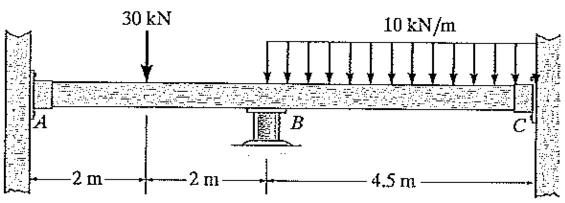
PROBLEMS

11–1. Determine the moments at A, B, and C. Assume the support at B is a roller and A and C are fixed. EI is constant.



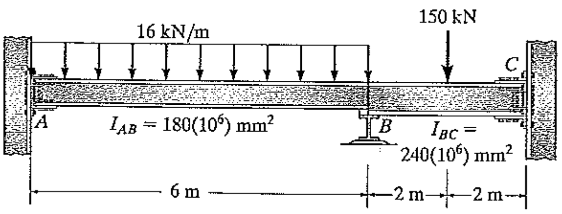
Prob. 11–1

11–2. Determine the moments at A, B, and C, then draw the moment diagram. EI is constant. Assume the support at B is a roller and A and C are fixed.



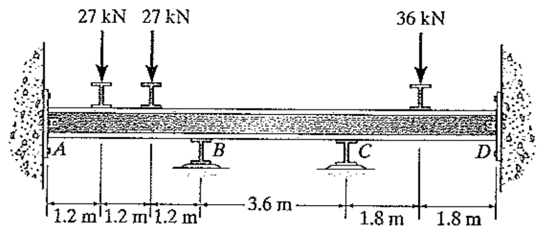
Prob. 11–2

11–3. Determine the moments at A, B, and C, then draw the moment diagram for the beam. The moment of inertia of each span is indicated in the figure. Assume the support at B is a roller and A and C are fixed. $E = 200$ GPa.



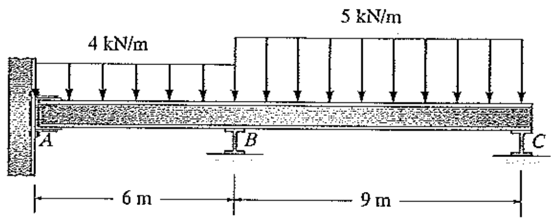
Prob. 11–3

***11–4.** Determine the reactions at the supports, then draw the moment diagram. Assume A and D are pins and B and C are rollers. The support at B settles 9 mm. Take $E = 200$ GPa and $I = 1800(10^6)$ mm^4.



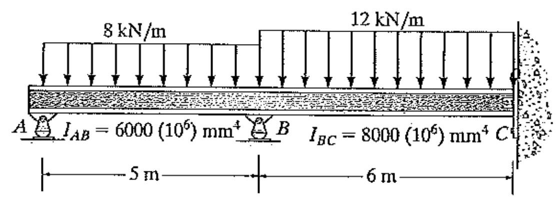
Prob. 11–4

11–5. Determine the moments at A, B, and C. The support at B settles 45 mm. $E = 200$ GPa and $I = 3200(10^6)$ mm^2. Assume the supports at B and C are rollers and A is fixed.



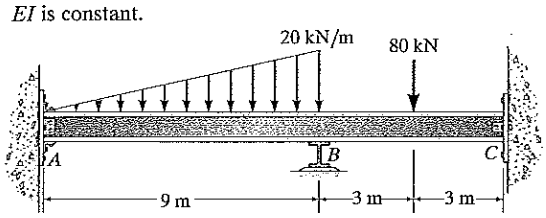
Prob. 11–5

11–6. Determine the internal moment in the beam at B, then draw the moment diagram. Assume A and B are rockers and C is a pin.



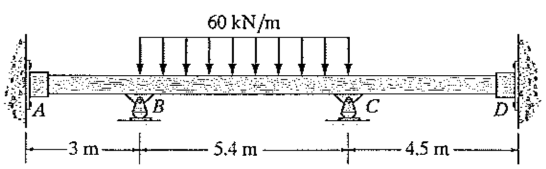
Prob. 11–6

11–7. Determine the moments acting at A and B. Assume A is fixed supported, B is a roller, and C is a pin. EI is constant.



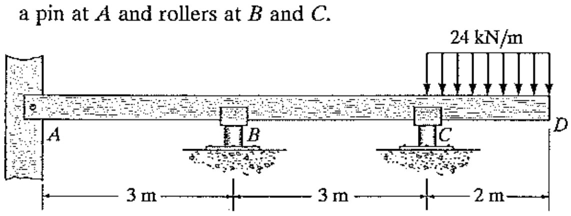
Prob. 11–7

***11–8.** Determine the moments at the supports, then draw the moment diagram. Assume A and D are fixed. EI is constant.



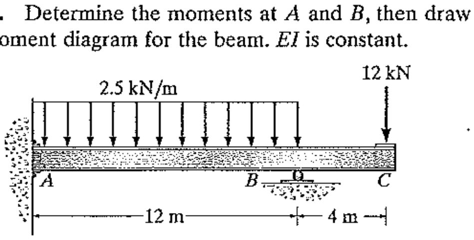
Prob. 11–8

11–9. Determine the moments at B and C of the overhanging beam, then draw the bending moment diagram. EI is constant. Assume the beam is supported by a pin at A and rollers at B and C.



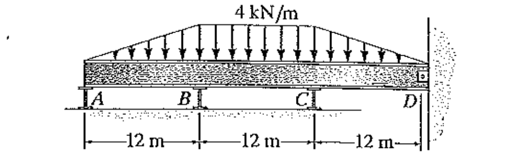
Prob. 11–9

11–10. Determine the moments at A and B, then draw the moment diagram for the beam. EI is constant.



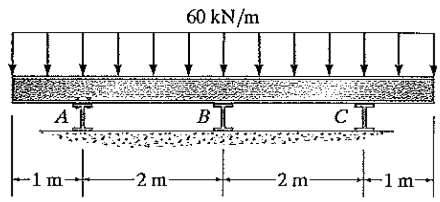
Prob. 11–10

11–11. Determine the moments at B and C, then draw the moment diagram. Assume A, B and C are rollers and D is pinned. EI is constant.



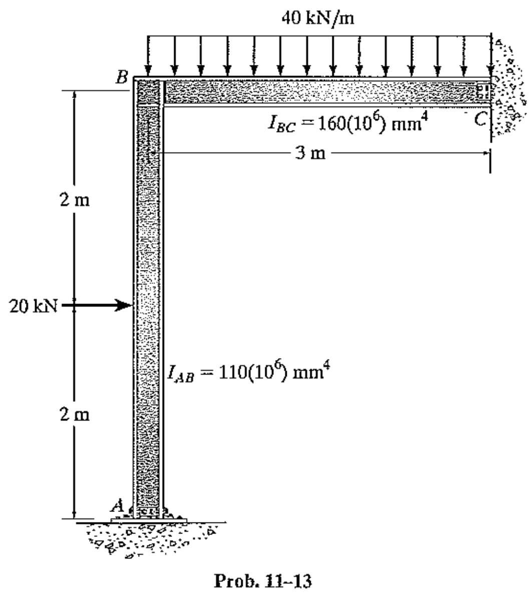
Prob. 11–11

***11–12.** Determine the internal moments at the supports A, B, and C, then draw the moment diagram. Assume A is pinned, and B and C are rollers. EI is constant.



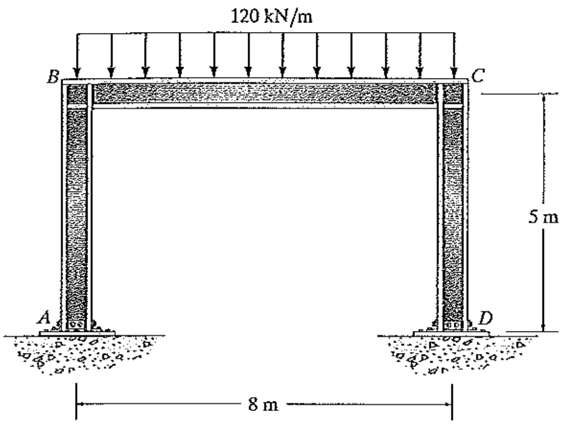
Prob. 11–12

11–13. Determine the moments at the ends of each member of the frame. Take $E = 200$ GPa. The moment of inertia of each member is listed in the figure. Assume the joint at B is fixed, C is pinned, and A is fixed.



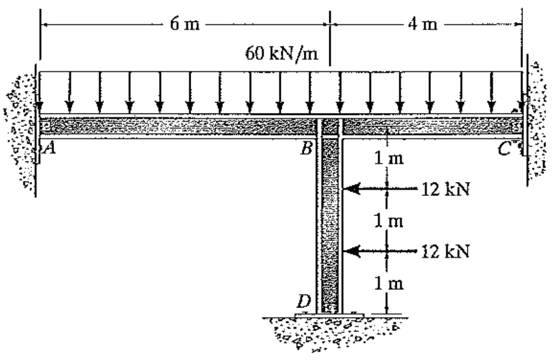
Prob. 11–13

11–14. Determine the reactions at A and D. Take EI to be the same for each member. Assume the supports at A and D are fixed.



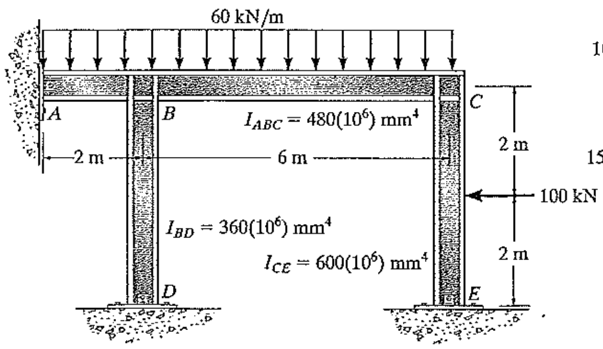
Prob. 11–14

***11–16.** Determine the moments acting at the ends of each member. Take $E = 200$ GPa, $I = 420(10^6)$ mm^4, and $I = 660(10^6)$ mm^4. Assume A and D are pin supported and C is fixed.



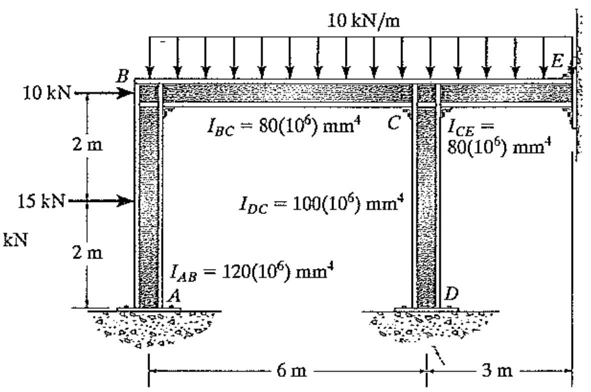
Prob. 11–16

11–15. Determine the internal moments acting at each joint. Assume A, D, and E are pinned and B and C are fixed joints. Take $E = 200$ GPa. The moment of inertia of each member is listed in the figure.



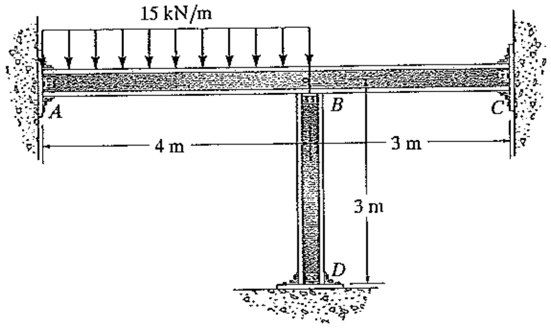
Prob. 11–15

11–17. Determine the moments at each joint of the frame, then draw the moment diagram for member BCE. Assume B, C, and E are fixed connected and A and D are pins.



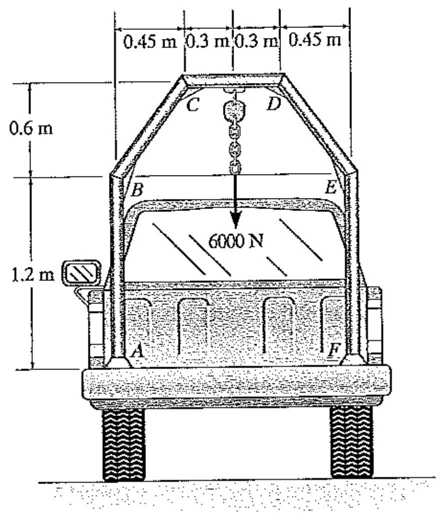
Prob. 11–17

11–18. When the 15 kN/m load is applied to the three-member frame the support at D settles 10 mm. Determine the moment acting at each of the fixed supports A, C, and D. The members are pin connected at B. $E = 200$ GPa, and $I = 800(10^6)$ mm^4.



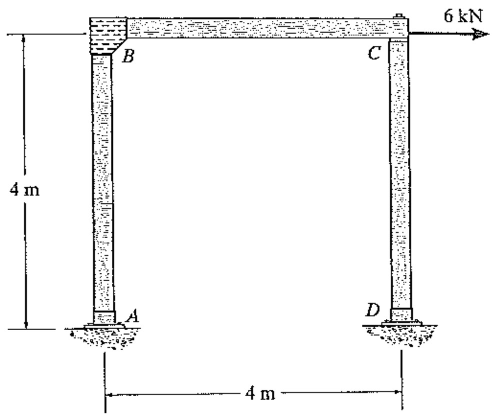
Prob. 11–18

11–19. The frame at the rear or the truck is made by welding pipe segments together. If the applied load is 6000 N, determine the moments at the fixed joints B, C, D, and E. Assume the supports at A and F are pinned. EI is constant.



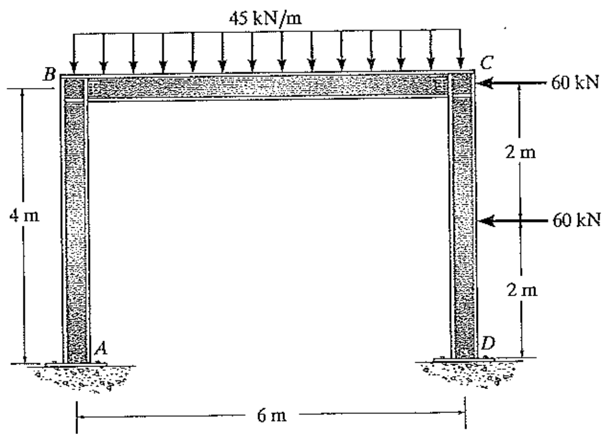
Prob. 11–19

***11–20.** The wood frame is subjected to the load of 6 kN. Determine the moments at the fixed joints A, B, and D. The joint at C is pinned. EI is constant.



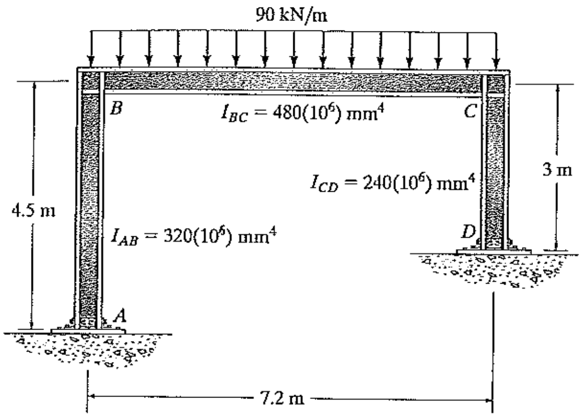
Prob. 11–20

11–21. Determine the moments at the ends of each member. Assume A and D are pins and B and C are fixed-connected joints EI is the same for all members.



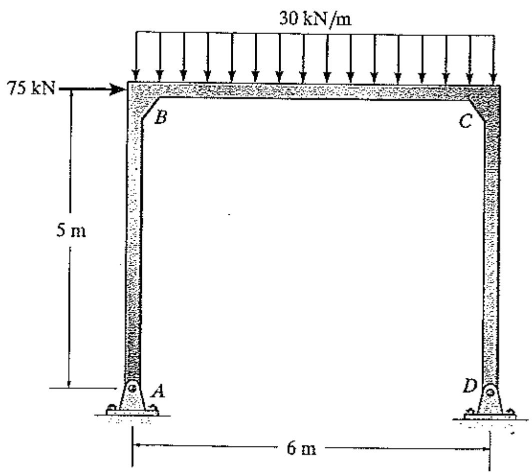
Prob. 11–21

11–22. Determine the moments acting at the ends of each member. Assume the supports at A and D are fixed. The moment of inertia of each member is indicated in the figure. $E = 200$ GPa.



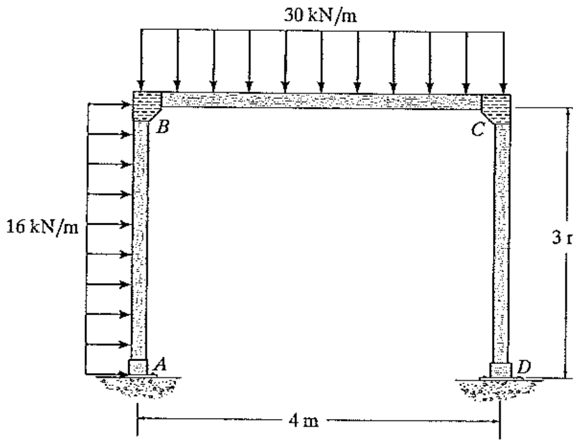
Prob. 11–22

11–23. Determine the moments acting at the ends of each member of the frame. EI is the same for all members.



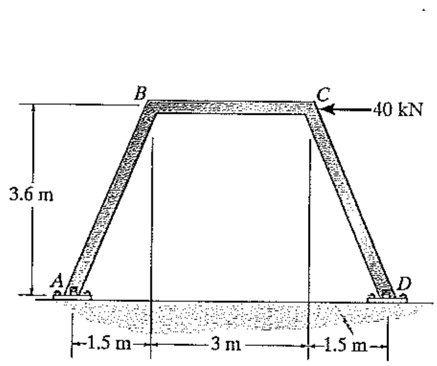
Prob. 11–23

***11–24.** Determine the moments acting at the ends of each member. EI is the same for all members. Assume all joints are fixed.



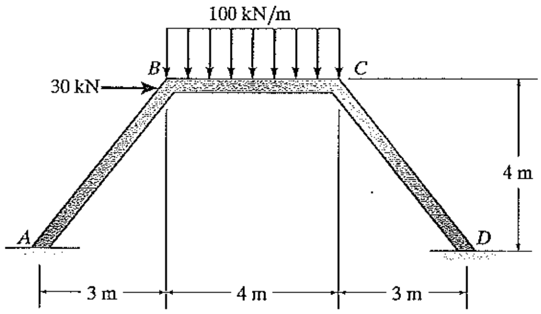
Prob. 11–24

11–25. Determine the moment at each joint of the battered-column frame. The supports at A and D are pins. EI is constant.



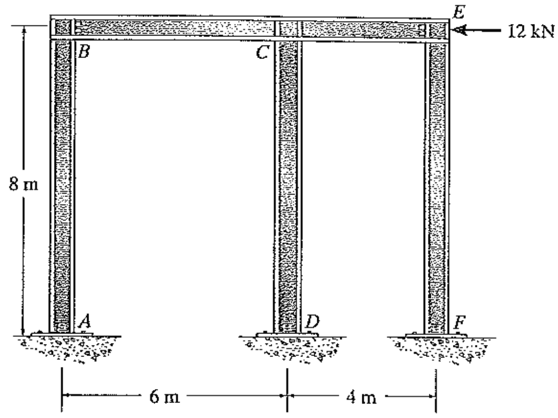
Prob. 11–25

11–26. Determine the moments acting at the supports A and D of the battered-column frame. Take $E = 200$ GPa, $I = 50(10^6)$ mm^4.



Prob. 11–26

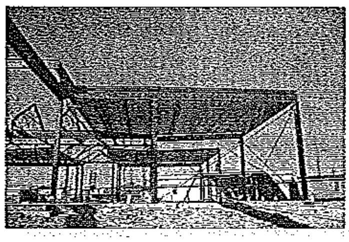
11–27. Wind loads are transmitted to the frame at joint E. If A, B, E, D, and F are all pin-connected and C is fixed connected, determine the moments at joint C and draw the bending moment diagrams for the girder BCE. EI is constant.

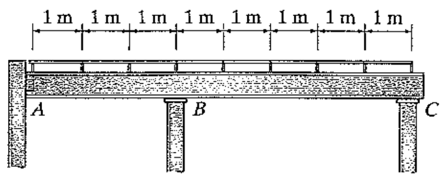


Prob. 11–27

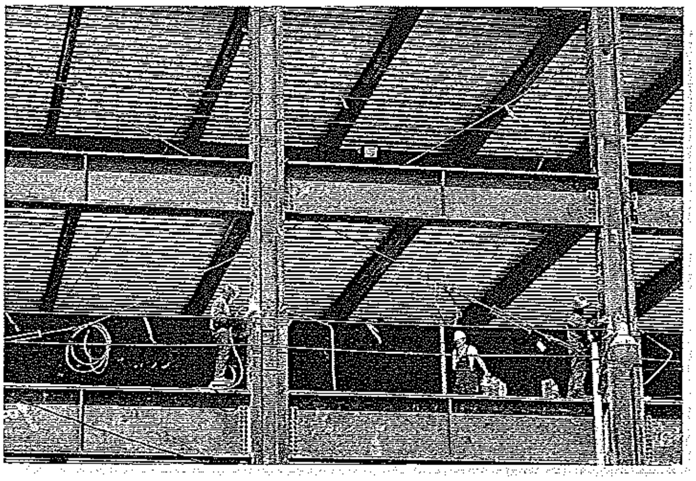
PROJECT PROBLEM

11–1P. The roof is supported by joists that rest on two girders. Each joist can be considered simply supported, and the front girder can be considered attached to the three columns by a pin at A and rollers at B and C. Assume the roof will be made from 75 mm thick cinder concrete, and each joist has a weight of 2.5 kN. According to code the roof will be subjected to a snow loading of 1.2 kN/m^2. The joists have a length of 8 m. Draw the shear and moment diagrams for the girder. Assume the supporting columns are rigid.





Prob. 11–1P



The girders on this building frame are statically indeterminate. The force analysis can be done using the method of moment distribution.