Chapter 4: Analytic Geometry

The main topics of study in analytic geometry are straight lines and conic sections. Accordingly, by the end of this chapter you must

- be able to derive basic equations that are representing straight lines, circles, parabolas, ellipses, and hyperbola.
- know the main (important) properties of each of these five geometric objects.
- be able to identify equations of straight lines, circles, parabolas, ellipses, hyperbolas and sketch their graphs.

More specific objectives are given in each section.

The major part of this chapter is **conic sections**. Conic sections are **circles**, **parabolas, ellipse**s and **hyperbolas**. They are called conic sections because they are generated when a plane cuts a right circular double cone. Depending on how the plane cuts the cone the intersection forms a curve called a circle, an ellipse, a parabola or a hyperbola (See, Figure 4.1).

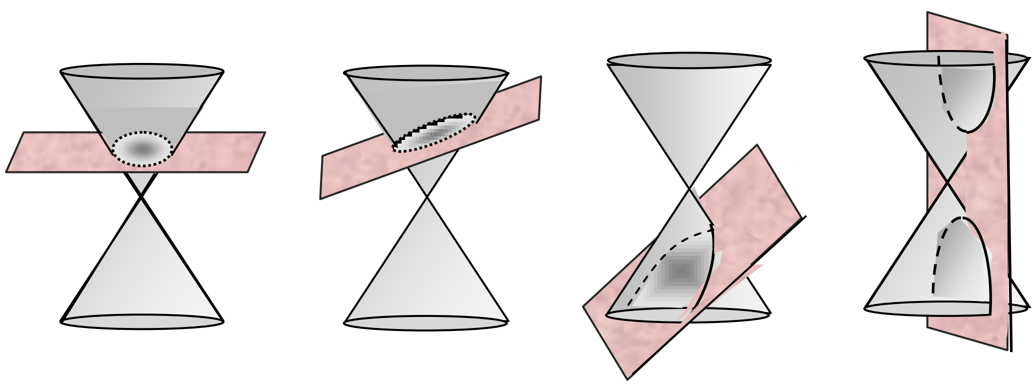


Figure 4.1: (a) circle (b) ellipse (c) parabola (d) hyperbola

We will see that a conic section is described by a second degree equation in x and y of the form

$$\mathrm{A}x^2 + \mathrm{C}y^2 + \mathrm{D}x + \mathrm{E}y + \mathrm{F} = 0,$$

when A, C, D, E and F are constant real numbers. In the analysis of such equations we will frequently need the method of completing the square. Recall that completing the square is the method of converting an equation of the form

$x^2 + ax = b$ to $(x+h)^2 = c$ (*Can you establish the relationships between* a,b *and* h,c ?)

To do this :- *Add* $\left(\frac{a}{2}\right)^2$ *to both sides of the former equation*.

- *Then complete the square of the resulting expression to get the later form.*

Here recall that: $x^2 + 2ax + a^2 = (x+a)^2$ and $x^2 - 2ax + a^2 = (x-a)^2$.

4.1 Distance Formula and Equation of Lines

By the end of this section, you should

- be able to find the distance between two points in the coordinate plane.
- be able to find the coordinates of a point that divides a line segment in a given ratio.
- know different forms of basic equations of a line
- be able to find equation of a line and draw the line.
- know when two lines are parallel.
- know when two lines are perpendicular.
- be able to find the distance between a point and a line in the coordinate plane.

4.1.1 Distance between two points and division of segments

If P and Q are two points on the coordinate plane, then PQ represents the line segment joining P and Q and $d(\text{P},\text{Q})$ or $|\text{PQ}|$ represents the distance between P and Q.

Recall that the distance between points a and b on a number line is $|a - b| = |b - a|$. Thus, the distance between two points P(x_1, y_1) and R(x_2, y_1) on a horizontal line must be $|x_2 - x_1|$ and the distance between Q(x_2, y_2) and R(x_2, y_1) on a vertical line must be $|y_2 - y_1|$. (See Figure 4.2).

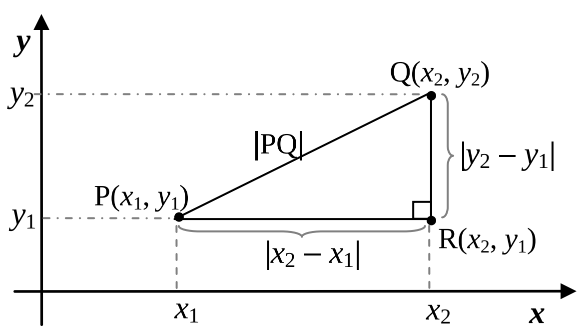


Figure 4.2

To find distance $|\text{PQ}|$ between any two points P(x_1, y_1) and Q(x_2, y_2), we note that triangle PRQ in Figure 4.2 is a right triangle, and so by Pythagorean Theorem we get:

$$|\text{PQ}|^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 \iff |\text{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Therefore, we have the following:

Distance Formula: The distance between the points P(x_1, y_1) and Q(x_2, y_2) is

$$|\text{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note that, from the distance formula, the distance between the origin O(0,0) and a point P(x, y) is

$$|OP| = \sqrt{x^2 + y^2}$$

Example 4.1: (i) The distance between O(0,0) and P(3,4) is

$$|OP| = \sqrt{3^2 + 4^2} = 5.$$

(ii) The distance between P(1,2) and Q(3,6) is

$$|PQ| = \sqrt{(3-1)^2 + (6-2)^2} = \sqrt{20}.$$

(iii) The distance between P(–1,2) and Q(5,–6) is

$$|PQ| = \sqrt{(5+1)^2 + (-6-2)^2} = 10.$$

Division point of a line segment: Given two distinct points P(x_1, y_1) and Q(x_2, y_2) in the coordinate plane, we want to find the coordinates (x_0, y_0) of the point R that lies on the segment PQ and divides the segment in the ratio r_1 to r_2; that is

$$\frac{|PR|}{|RQ|} = \frac{r_1}{r_2},$$

where r_1 and r_2 are given positive numbers. (See Figure 4.3).

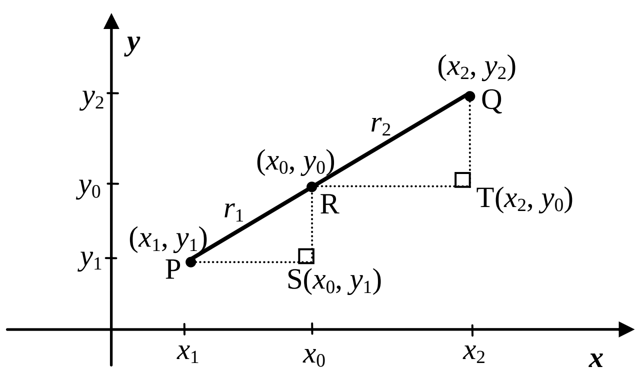


Figure 4.3

To determine (x_0, y_0), we construct two right triangles ΔPSR and ΔRTQ as in Figure 4.3. We then have |PS| = $x_0 - x_1$, |SR| = $y_0 - y_1$, |RT| = $x_2 - x_0$, and |TQ| = $y_2 - y_0$. Now since ΔPSR is similar to ΔRTQ, we have that

$$\frac{x_0 - x_1}{x_2 - x_0} = \frac{r_1}{r_2} \quad \text{and} \quad \frac{y_0 - y_1}{y_2 - y_0} = \frac{r_1}{r_2}$$

or $r_2(x_0 - x_1) = r_1(x_2 - x_0)$ and $r_2(y_0 - y_1) = r_1(y_2 - y_0)$.

Solving for x_0 and y_0, we obtain $x_0 = \dfrac{x_1 r_2 + x_2 r_1}{r_1 + r_2}$ and $y_0 = \dfrac{y_1 r_2 + y_2 r_1}{r_1 + r_2}$

Therefore, we have shown the following.

Theorem 4.1: Let P(x_1, y_1) and Q(x_2, y_2) be distinct points in the coordinate plane. If R(x_0, y_0) is a point on the line segment PQ that divides the segment in the ratio |PR| **:** |RQ| = r_1**:** r_2, then the coordinates of R is given by

$$(x_0, y_0) = \left(\frac{x_1r_2 + x_2r_1}{r_1 + r_2}, \ \frac{y_1r_2 + y_2r_1}{r_1 + r_2}\right)$$

In particular, the midpoint of PQ is given by $\left(\frac{x_1 + x_2}{2}, \ \frac{y_1 + y_2}{2}\right)$

Example 4.2: Given P(–3, 3) and Q(7,8),

(i) find the coordinates of the point R on the line segment PQ such that |PR| **:** |RQ| = 2 **:** 3.

(ii) find the coordinates of the midpoint of PQ.

Solution: (i) Obviously R(x_0,y_0) is given by

$$(x_0, y_0) = \left(\frac{-3\times3+7\times2}{2+3}, \ \frac{3\times3+8\times2}{2+3}\right) = (1, 5)$$

(ii) The coordinates of the midpoint is $\left(\frac{-3+7}{2}, \ \frac{3+8}{2}\right) = (2, \ 11/2)$.

Exercise 4.1.1

1. Find the distance between the following pair of points.
   (a) (–1, 0) and (3, 0). (d) The origin and $(-\sqrt{3}, \sqrt{6})$.
   (b) (1, –2) and (1, 4). (e) (a, a) and (–a, –a)
   (c) (–2, 3) and (2, 0) (f) (a, b) and (–a, –b)
2. If the vertices of ΔABC are A(1,1), B(4,5) and C(7, 1), find the perimeter of the triangle.
3. Let P = (–3,0) and Q be a point on the positive ***y***-axis. Find the coordinates of Q if |PQ| =5.
4. Suppose the endpoints of a line segment AB are A(–1,1) and B(5, 10). Find the coordinates of point P and Q if
   (a) P is the midpoint of AB.
   (b) P divides AB in the ratio 2:3 (That is, |AP|: |PB| = 2:3).
   (c) Q divides AB in the ratio 3:2.
   (d) P and Q trisect AB (i.e., divide it into three equal parts).
5. Let M(–1,3) be the midpoint of a line segment PQ. If the coordinates of P is (–5, –7), then what is the coordinates of Q?
6. Let A(a, 0), B(0,b) and O(0,0) be the vertices of a right triangle. Show that the midpoint of AB is equidistant from the vertices of the triangle

4.1.2 Equations of lines

An equation of a line ***l*** is an equation which must be satisfied by the coordinates (x, y) of every point on the line. A line can be vertical, horizontal or oblique. The equation of a vertical line that intersects the x-axis at $(a, 0)$ is $x=a$ because the x-coordinate of every point on the line is a. Similarly, the equation of a horizontal line that intersects the y-axis at $(0, b)$ is $y=b$ because the y-coordinate of every point on the line is b.

An oblique line is a straight line which is neither vertical nor horizontal. To find equation of an oblique line we use its slope which is the measure of the steepness of the line. In particular, the slope of a line is defined as follows.

Definition 4.1. The **slope** of a non-vertical line that passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined. Note that the slope of horizontal line is 0.

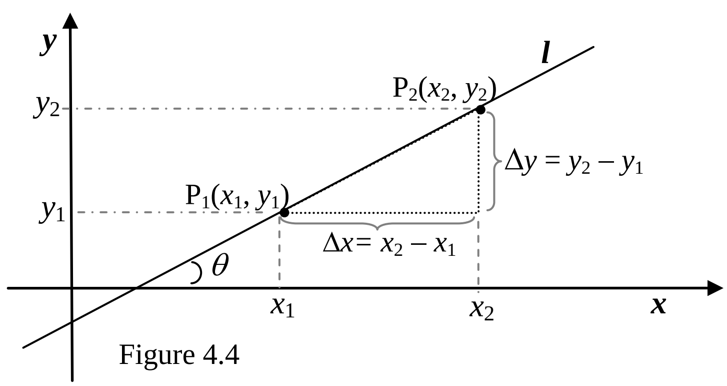


Figure 4.4

Thus the slope of a line ***l*** is the ratio of the change in y, Δy, to the change in x, Δx (see Figure 4.4). Hence, slope is the rate of change of y with respect x. The slope depends also on the angle of inclination of the line. Note that the angle of inclination θ is the angle between x-axis and the line (measured counterclockwise from the direction of positive x-axis to the line). Observe that

$$\tan\theta = \frac{\Delta y}{\Delta x}$$

Therefore, if θ is the angle of inclination of a line, then its slope is $m = \tan\theta$.

Now let us find an equation of the line that passes through a point $P_1(x_1, y_1)$ and has slope m. A point $P(x, y)$ with $x \neq x_1$ lies on this line if and only if the slope of the line though P_1 and P is m; that is

$$\frac{y - y_1}{x - x_1} = m$$

This leads to the following equation of the line:

$$y - y_1 = m(x - x_1)$$ (called point-slope form of equation of a line).

In general, depending on the given information, you can show that the equations of oblique lines can be obtained using the following formulas.

Given Information	**Formula for Equation of the Line**
Slope m and its y-intercept $(0,b)$	Slope-Intercept-Form: $y = mx + b$
Slope m and a point (x_1, y_1) on $\boldsymbol{l}$	Point-Slope-Form: $y - y_1 = m(x - x_1)$ Or $y = m(x - x_1) + y_1$
Two points (x_1, y_1) and (x_2, y_2) on $\boldsymbol{l}$	Two-Point Form: $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$
x-intercept $(a,0)$ and y-intercept $(0,b)$	Intercept Form: $\dfrac{x}{a} + \dfrac{y}{b} = 1$

Example 4.3: Find an equation of the line $\boldsymbol{l}$ if

(i) the line passes through $(3, -2)$ and its angle of inclination is 135°.

(ii) the line passes through the points $(1, 2)$ and $(4, -2)$

Solution: (i) The slope of $\boldsymbol{l}$ is $m = \tan(135°) = -1$; and it passes through point $(3, -2)$. Thus, using the point-slope form with $x_1 = 3$ and $y_1 = -2$, we obtain the equation of the line as $y - (-2) = -1(x - 3)$ which simplifies to $y = -x + 1$.

(ii) Given the line passes through $(1, 2)$ and $(4, -2)$, the slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-2}{4-1} = \frac{-4}{3}.$$

So, using the point-slope form with $x_1 = 1$ and $y_1 = 2$, we obtain the equation of the line as $y - 2 = \frac{-4}{3}(x - 1)$ which simplifies to $4x + 3y = 10$.

(Note that it is possible to use the two-point form to find the equation of this line)

General Form: In general, the equation of a straight line can be written as

$$ax + by + c = 0,$$

for constants a, b, c with a and b not both zero. Indeed, if $a=0$ the line is a horizontal line given by $y = -c/b$, if $b=0$ the line is a vertical line given by $x = -c/a$, and if both $a, b \neq 0$ it is the oblique line given by $y = -(a/b)x - c/b$ with slope $m = -a/b$ and y-intercept $-c/b$.

Parallel and Perpendicular lines: slopes can be used to check whether lines are parallel, perpendicular or not. In particular, let $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ be non-vertical lines with slope m_1 and m_2, respectively. Then,

. (i) $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are parallel, denoted by $\boldsymbol{l}_1 \parallel \boldsymbol{l}_2$, iff $m_1 = m_2$.

(ii) $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are perpendicular, denoted by $\boldsymbol{l}_1 \perp \boldsymbol{l}_2$, iff $m_1 m_2 = -1$ (or $m_2 = -\frac{1}{m_1}$)

Moreover, if $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are both vertical lines then they are parallel. However, if one of them is horizontal and the other is vertical, then they are perpendicular.

Example 4.4: Find an equation of the line through the point (3,2) that is parallel to the line $2x + 3y + 5 = 0$.

Solution: The given line can be written in the form $y = \frac{-2}{3}x - \frac{5}{3}$ which is the slope-intercept form; that is, it has slope $m = -2/3$. So, as parallel lines have the same slope, the required line has slope $-2/3$. Therefore, its equation in point-slope form is $y - 2 = \frac{-2}{3}(x - 3)$ which can be simplified to $2x + 3y = 12$.

Example 4.5: Show that the lines $2x + 3y + 5 = 0$ and $3x - 2y - 4 = 0$ are perpendicular.

Solution: The equations can be written as $y = \frac{-2}{3}x - \frac{5}{3}$ and $y = \frac{3}{2}x - 2$ from which we can see that $m_1 = -2/3$ and $m_2 = 3/2$. Since $m_1 m_2 = -1$, the lines are perpendicular.

Exercise 4.1.2

1. Find the slope and equation of the line determined by the following pair of points. Also find the y- and x- intercepts, if any, and draw each line.

   (a) (0, 2) and (3, 2)
   (b) (2, 0) and (2, 3)
   (c) The origin and (1,0)
   (d) The origin and (–1, 0)
   (e) The origin and (1,2)
   (f) The origin and (1,–3)
   (g) (1,2) and (3, 4)
   (h) (–2, –3), (2, 5)
   (i) (–1, 3) and (1, 6)
   (j) (–3, –2) and (2, –2)
   (k) (0, 3) and (3, 0)
   (l) (–1, 0) and (0, 2)

2. Find the slope and equation of the line whose angle of inclination is θ and passes through the point P, if

   (a) $\theta = \frac{1}{4}\pi$, P = (1,1).
   (b) $\theta = \frac{1}{4}\pi$, P = (0,1).
   (c) $\theta = \frac{3}{4}\pi$, P = (0,1).
   (d) $\theta = 0$, P = (0, 1).
   (e) $\theta = \frac{1}{3}\pi$, P = (1, 3).
   (f) $\theta = \frac{1}{3}\pi$, P = (1,–3).

3. Find the $\boldsymbol{x}$-and $\boldsymbol{y}$-intercepts and slope of the line given by $\frac{x}{2} - \frac{y}{3} = 1$, and draw the line.

4. Draw the triangle with vertices A(−2,4), B(1,−1) and C(6,2) and find the following.
   (a) Equations of the sides.
   (b) Equations of the medians.
   (c) Equations of the perpendicular bisectors of the sides.
   (d) Equation of the lines through the vertices parallel to the opposite sides.

5. Find the equation of the line that passes through (2, −1) and perpendicular to $3x + 4y = 6$.

6. Suppose ℓ_1 and ℓ_2 are perpendicular lines intersecting at (−1, 2). If the angle of inclination of ℓ_1 is 45°, then find an equation of ℓ_2.

7. Determine which of the following pair of lines are parallel, perpendicular or neither.
   (a) $2x - y + 1 = 0$ and $2x + 4y = 3$
   (b) $3x - 6y + 1 = 0$ and $x - 2y = 3$
   (c) $2x + 5y + 3 = 0$ and $5x + 3y + 2 = 0$
   (d) $y = 3x + 2$ and $3x + y = 2$
   (e) $2x - 3y = 5$ and $3x + 2y - 3 = 0$
   (f) $\frac{x}{3} + \frac{y}{2} = 1$ and $2x + 3y - 6 = 0$

8. Let L_1 be the line passing through P(a, b) and Q(b, a) such that $a \neq b$. Find an equation of the line L_2 in terms of a and b if
   (a) L_2 passes through P and perpendicular to L_1.
   (b) L_2 passes through (a, a) and parallel to L_1.

9. Let L_1 and L_2 be given by $2x + 3y - 4 = 0$ and $x + 3y - 5 = 0$, respectively. A third line L_3 is perpendicular to L_1. Find an equation of L_3 if the three lines intersect at the same point.

10. Determine the value(s) of k for which the line

$$(k+2)x + (k^2 - 9)y + 3k^2 - 8k + 5 = 0$$

   (a) is parallel to the x-axis.
   (b) is parallel to the y-axis.
   (c) passes through the origin
   (d) passes through the point (1,1).

   In each case write the equation of the line.

11. Determine the values of a and b for which the two lines $ax - 2y = 1$ and $6x - 4y = b$
   (a) have exactly one intersection point.
   (b) are distinct parallel lines.
   (c) coincide.
   (d) are perpendicular.

4.1.3 Distance between a point and a line

Suppose a line ***l*** and a point P(x,y) not on the line are given. The distance from P to ***l***, d(P, ***l***), is defined as the perpendicular distance between P and ***l*** . That is,

d(P, ***l***) = |PQ|, where Q is the point on ***l*** such that PQ⊥ ***l***. (See Figure 4.5)

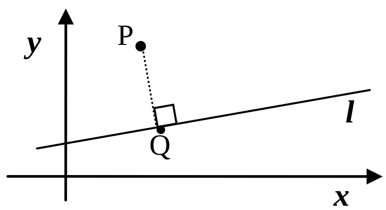


Figure 4.5: |PQ| = d(P, ***l***)

If P is on ***l***, then d(P, ***l***) = 0. Moreover, given a point P(h,k) observe that

(i) if the line ***l*** is a horizontal line $y=b$, then d(P, ***l***) = $|k-b|$.

(ii) if the line ***l*** is a vertical line $x=a$, then d(P, ***l***) = $|h-a|$

In general, however, to find the distance between a point P(x_0, y_0) and an arbitrary line ***l*** given by $ax + by + c = 0$, we have to first get a point Q on ***l*** such that PQ⊥ ***l*** and then compute |PQ|. This yields the formula given in the following Theorem.

Theorem 4.2: The distance between a point P(x_0, y_0) and a line L: $ax + by + c = 0$ is given by

$$d(\mathrm{P}, L) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

In particular, if we take $(x_0,y_0)=(0,0)$ in this formula, we obtain the distance between the origin O(0,0) and a line L : $ax + by + c = 0$ which is given by

$$d(\mathrm{O}, L) = \frac{|c|}{\sqrt{a^2 + b^2}}$$

Example 4.6: Show that the origin and P(6,4) are equidistant from the line L: $y = -(3/2)x + 13/2$.

Solution: By equidistant we mean equal distance. So, we need to show d(O, L) = d(P, L).

To use the above formula, we first write the equation of the line L in the general form which is $3x + 2y - 13 = 0$. Thus, $a = 3$, $b = 2$ and $c = -13$.

$$\Rightarrow \quad d(\mathrm{O}, L) = \frac{|c|}{\sqrt{a^2 + b^2}} = \frac{|-13|}{\sqrt{9+4}} = \frac{13}{\sqrt{13}}$$

and $$d(\mathrm{P}, L) = \frac{|3\times 6 + 2\times 4 - 13|}{\sqrt{3^2 + 2^2}} = \frac{13}{\sqrt{13}}$$

Therefore, d(O, L) = d(P, L) = $13/\sqrt{13}$.

Thus, O(0,0) and P(6,4) are equidistant from the given line L.

Exercise 4.1.3

1. Find the distance between the line L given by $y = 2x + 3$ and each of the following points.
   (a) The origin (b) (2, 3) (c) (1, 5) (d) (−1, −1)
2. Suppose L is the line through (1, 2) and (3, 2). What is the distance between L and
   (a) The origin (b) (2, −3) (c) $(a, 0)$ (d) (a, b) (e) $(a, 2)$
3. Suppose L is the vertical line that crosses the $\boldsymbol{x}$-axis at (5, 0). Find $d(P, L)$, when P is
   (a) The origin (b) (2, −4) (c) $(0, b)$ (d) $(5, b)$ (e) (a, b)
4. Suppose L is the line that passes through (0, −3) and (4, 0). Find the distance between L and each of the following points.
   (a) The origin (b) (1, 4) (c) (−1, 0) (d) (8, 3)
   (e) (0, 1) (f) (4, −2) (g) (1, −9/4) (h) (7, −4)
5. The vertices of ΔABC are given below. Find the length of the side BC, the height of the altitude from vertex A to BC, and the area of the triangle when its vertices are
   (a) A(3, 4), B(2, 1), and C(6, 1).
   (b) A(3, 4), B(1, 1), and C(5, 2).
6. Consider the quadrilateral whose vertices are A(1,2), B(2,6), C(6,8) and D(5,4). Then,
   (a) Show that the quadrilateral is a parallelogram.
   (b) How long is the side AD?
   (c) What is the height of the altitude of the quadrilateral from vertex A to the side AD.
   (d) Determine the area of the quadrilateral.

4.2 Circles

By the end of this section, you should

- know the geometric definition of a circle.
- be able to identify whether a given point is on, inside or outside a circle.
- be able to construct equation of a circle.
- be able to identify equations that represent circles
- be able to find the center and radius of a circle and sketch its graph if its equation is given.
- be able to identify whether a given circle and a line intersect at two points, one points or never intersect at all.
- know the properties of a tangent line to a circle.
- be able to find equation of a tangent line to a circle.

4.2.1 Definition of a Circle

Definition 4.2. A circle is the locus of points (set of points) in a plane each of which is equidistant from a fixed point in the plane. The fixed point is called the **center** of the circle and the constant distance is called its **radius**.

Definition 4.2 is illustrated by Figure 4.6 in which the center of the circle is denoted by "C" and its radius is denoted by *r*.

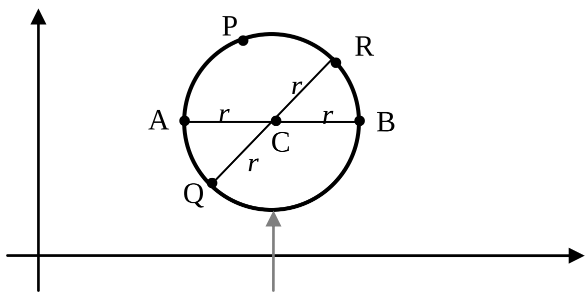


Figure 4.6. Circle with center C, radius *r*

Observe that a circle is symmetric with respect its center. Based on the definition, a point P is on the circle if and only if its distance to C is *r*, that is $|CP| = r$. A point in the plane is said to be inside the circle if its distance to the center C is less than *r*. Similarly, a point in the plane is said to be outside the circle if its distance to C is greater than *r*. Moreover, a chord of the circle is a line segment whose endpoints are on the circle. A diameter is a chord of the circle through the center C. Consequently, C is the midpoint of a diameter and the length of a diameter is $2r$. For example, AB and QR are diameters of the circle in Figure 4.6.

Example 4.7: Consider a circle of radius 5 whose center is at C(2,1). Determine whether each of the following points is on the circle, inside the circle or outside the circle:

$P_1(5, 5)$, $P_2(4, 5)$, $P_3(-2, 5)$, $P_4(-1, -2)$, $P_5(2,-4)$, $P_6(7, 0)$.

Solution: The distance between a given point $P(x,y)$ and the center C(2,1) is given by

$|PC| = \sqrt{(x-2)^2 + (y-1)^2}$ or $|PC|^2 = (x-2)^2 + (y-1)^2$. We need to compare $|PC|$ with the radius 5. Note that $|PC| = 5 \Leftrightarrow |PC|^2 = 25$, $|PC| < 5 \Leftrightarrow |PC|^2 < 25$,

and $|PC| > 5 \Leftrightarrow |PC|^2 > 25$.

Thus, P is on the circle if $|PC|^2 = 25$, inside the circle if $|PC|^2 < 25$ and outside the circle if $|PC|^2 > 25$. So, we can use the square distance to answer the question. Thus, as

$|P_1C|^2 = (5-2)^2+(5-1)^2 = 25$, $|P_2C|^2 = (4-2)^2+(5-1)^2 = 20$ and $|P_3C|^2 = (-2-2)^2+(5-1)^2 = 32$,

P_1 is on the circle, P_2 is inside the circle, and P_3 is outside the circle. Similarly, you can show that P_4 is inside the circle, P_5 is on the circle, and P_6 is outside the circle.

Exercise 4.2.1

1. Suppose the center of a circle is C(1,–2) and P(7, 6) is a point on the circle. What is the radius of the circle?
2. Let A(1, 2) and B(5, –2) are endpoints of a diameter of a circle. Find the center and radius of the circle.
3. Consider a circle whose center is the origin and radius is $\sqrt{5}$. Determine whether or not the circle contains the following point.
   (a) (1, 2) (b) (0,0) (c) $(0,-\sqrt{5})$ (d) (3/2, 3/2)
   (e) (5, 0) (f) (–1, –2), (g) $(\sqrt{3},\sqrt{2})$ (h) (5/2, 5/2)
4. Consider a circle of radius 5 whose center is at C(–3,4). Determine whether each of the following points is on the circle, inside the circle or outside the circle:
   (a) (0, 9) (b) (0,0) (c) (1,6) (d) (1, 0)
   (e) (–7, 1) (f) (–1, –1), (g) (2,4) (h) (5/2, 5/2)

4.2.2 Equation of a Circle

We now construct an equation that the coordinates (x,y) of the points on the circle should satisfy. So, let P(x,y) be any point on a circle of radius r and center C(h,k) (see, Figure 4.7). Then, the definition of a circle requires that

$$|CP| = r$$

$$\Rightarrow \quad \sqrt{(x-h)^2+(y-k)^2} = r$$

or

$$(x-h)^2+(y-k)^2 = r^2$$

(Standard equation of a circle with center (h,k) and radius r.)

In particular, if the center is at origin, i.e., (h,k) = (0,0), the equation is

$$x^2 + y^2 = r^2$$

(Standard Equation of a circle of radius r centered at origin)

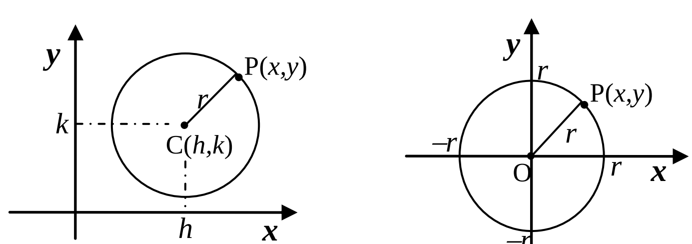


Figure 4.7 circles (a) center at C(h,k) (b) center at origin

Example 4.8: Find an equation of the circle with radius 4 and center (–2, 1).

Solution: Using the standard equation of a circle in which the center $(h, k) = (-2, 1)$ and radius $r = 4$ we obtain the equation

$$(x+2)^2 + (y-1)^2 = 16.$$

Example 4.9: Find the equation of a circle with endpoints of a diameter at P(–2, 0) and Q(4, 2).

Solution: The center of the circle C(*h*,*k*) is the mid-point of the diameter. Hence,

$(h,k) = \left(\frac{-2+4}{2}, \frac{0+2}{2}\right) = (1,1)$. Also, for its radius r, $r^2 = |CP|^2 = (1+2)^2 + (1-0)^2 = 10$.

Thus, the equation of the circle is $(x-h)^2 + (y-k)^2 = r^2$. That is,

$$(x-1)^2 + (y-1)^2 = 10.$$

Example 4.10: Suppose P(–2,4) and Q(5,3) are points on a circle whose center is on *x*-axis. Find the equation of the circle.

Solution: We need to obtain the center C and radius *r* of the circle to construct its equation. As the center is on *x*-axis, its second coordinate is 0. Therefore, let C(*h*,0) be the center of the circle. Note that $|PC|^2 = |QC|^2 = r^2$ as both P and Q are on the circle. So, from the first equality we get $(-2-h)^2 + 4^2 = (5-h)^2 + 3^2$. Solving this for *h* we get $h=1$. Hence, the center is at C(1, 0) and $r^2 = |QC|^2 = (5-1)^2 + 3^2 = 25$. Therefore, the equation of the circle is

$(x-1)^2 + y^2 = 25$.

Example 4.11: Determine whether the given equation represents a circle. If it does, identify its center and radius and sketch its graph.

(a) $x^2 + y^2 + 2x - 6y + 7 = 0$

(b) $x^2 + y^2 + 2x - 6y + 10 = 0$

(c) $x^2 + y^2 + 2x - 6y + 11 = 0$

Solution: We need to rewrite each equation in standard form to identify its center and radius. We do this by completing the square on the x-terms and y-terms of the equation as follows:

(a) $(x^2 + 2x) + (y^2 - 6y) = -7.$ (Grouping x-terms and y-terms)

$\Leftrightarrow (x^2 + 2x + 1^2) + (y^2 - 6y + 3^2) = -7 + 1 + 9.$ (Adding 1^2 and 3^2 to both sides)

$\Leftrightarrow (x+1)^2 + (y-3)^2 = 3.$

Comparing this with the standard equation of circle this is equation of a circle with center $(h, k) = (-1, 3)$ and radius $r = \sqrt{3}$. The graph of the circle is sketched in Figure 4.8

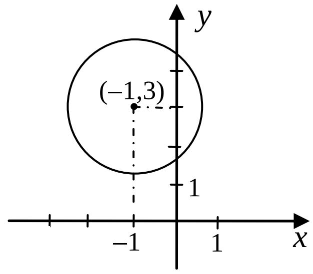


Figure 4. 8

(b) Following the same steps as in (a), you can see that $x^2 + y^2 + 2x - 6y + 10 = 0$ is equivalent to $(x+1)^2 + (y-3)^2 = 0$.

This is satisfied by the point (–1, 3) only. The locus of this equation is considered as a point-circle, circle of zero radius (sometimes called degenerated circle).

(c) Again following the same steps as in (a), you can see that $x^2 + y^2 + 2x - 6y + 11 = 0$ is equivalent to $(x+1)^2 + (y-3)^2 = -1$.

Note that this does not represent a circle; in fact it has no locus at all (Why?).

Remark: Consider an equation of the form

$$x^2 + y^2 + Dx + Ey + F = 0.$$

By completing the square you can show the following:

- If $D^2 + E^2 - 4F > 0$, then the equation represents a circle with center $\left(-\frac{D}{2}, -\frac{E}{2}\right)$ and radius $r = \frac{1}{2}\sqrt{D^2+E^2-4F}$.
- If $D^2 + E^2 - 4F = 0$, then the equation is satisfied by the point $\left(-\frac{D}{2}, -\frac{E}{2}\right)$ only. In this case the locus of the equation is called **point-circle** (circle of zero radius).
- If $D^2 + E^2 - 4F < 0$, then the equation has no locus.

Exercise 4.2.2

1. Determine whether each of the following points is inside, outside or on the circle with equation $x^2 + y^2 = 5$.

   (a) (–1, 2), (b) (3/2, 2) (c) $(0, -\sqrt{5})$ (d) (–1, 3/2)

2. Find an equation of the circle whose endpoints of a diameter are (0, –3) and (3, 3).
3. Determine an equation of a circle whose center is on *y*-axis and radius is 2.
4. Find an equation of the circle passing through (1, 0) and (0, 1) which has its center on the line $2x + 2y = 5$.
5. Find the value(s) of *k* for which the equation $2x^2 + 2y^2 + 6x - 4y + k = 0$ represent a circle.

6. An equation of a circle is $x^2 + y^2 - 6y + k = 0$. If the radius of the circle is 2, then what is the coordinates of its center?
7. Find equation of the circle passing through (0,0), (4, 0) and (2, 2).
8. Find equation of the circle inscribed in the triangle with vertices (–7, –10), (–7, 15), and (5,–1).
9. In each of the following, check whether or not the given equation represents a circle. If the equation represents a circle, then identify its center and the length of its diameter.

   (a) $x^2 + y^2 - 18x + 24y = 0$

   (b) $x^2 + y^2 - 2x + 4y + 5 = 0$

   (c) $x^2 + y^2 - 4x - 2y + 11 = 0$

   (d) $5x^2 + 5y^2 + 125x + 60y - 100 = 0$

   (e) $36x^2 + 36y^2 + 12x + 24y - 139 = 0$

   (f) $3x^2 + 3y^2 + 2x + 4y + 6 = 0$
10. Show that $x^2 + y^2 + Dx + Ey + F = 0$ represents a circle of positive radius iff $D^2 + E^2 - 4F > 0$.

4.2.3 Intersection of a circle with a line and tangent line to a circle

The number of intersection points of a given line and a circle is at most two; that is, either no intersection point, or only one intersection point, or two intersection points. For instance, in Figure 4.9, the line $\boldsymbol{l}_1$ has no intersection with the circle, $\boldsymbol{l}_2$ has two intersection points with the circle, namely, Q_1 and Q_2, and $\boldsymbol{l}_3$ has only one intersection point with the circle, namely, P.

A line which intersects a circle at one and only one point is called a **tangent line** to the circle. In this case, the intersection point is called the **point of tangency**. Thus, $\boldsymbol{l}_3$ a tangent line to the circle in Figure 4.9 and P is the point of tangency.

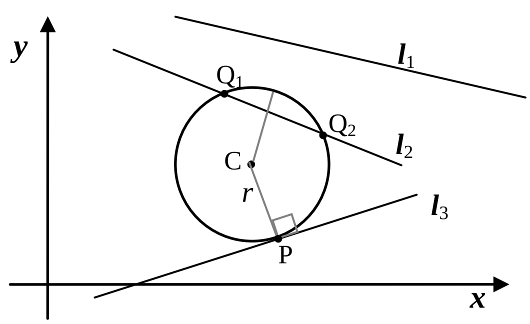


Figure 4.9: Intersection of a line and circle

In Figure 4.9, observe that every point on $\boldsymbol{l}_1$ are outside of the circle. Hence, $d(C,Q) > r$ for every point Q on $\boldsymbol{l}_1$. Consequently, $d(C, \boldsymbol{l}_1) > r$. On the other hand, there is a point on $\boldsymbol{l}_2$ which is inside the circle. Hence, $d(C, \boldsymbol{l}_2) < r$.

For the tangent line $\boldsymbol{l}_3$, the point of tangency P is on the circle implies that |CP| = r and P is the point on $\boldsymbol{l}_3$ closest to C. Therefore, $d(C, \boldsymbol{l}_3) = |CP| = r$. This shows also that $CP \perp \boldsymbol{l}_3$.

In general, given a circle of radius r with center C(h,k) and a line $\boldsymbol{l}$, by computing the distance $d(C, \boldsymbol{l})$ between C and $\boldsymbol{l}$ we can conclude the following.

(i). If $d(C, \boldsymbol{l}) > r$, then the line does not intersect with the circle.

(ii) If $d(C, \boldsymbol{l}) < r$, then the line is a secant of the circle; that is, they have two intersection points.

(iii) If $d(C, \boldsymbol{l}) = r$, then $\boldsymbol{l}$ is a tangent line to the circle. The point of tangency is the point P on the line (and on the circle) such that $CP \perp \boldsymbol{l}$. This means the product of the slopes of $\boldsymbol{l}$ and CP must be −1.

Example 4.12 Write the equation of the circle tangent to the ***x***-axis at (6,0) whose center is on the line $x - 2y = 0$.

Solution: The circle in the question is as in Figure 4.10.

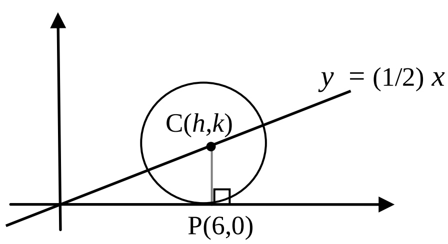


Figure 4.10

Let C(h, k) be the center of the circle. (h, k) is on the line $y = (1/2)x \Rightarrow k = (1/2)h$; and the circle is tangent to x-axis at P(6,0) ⇒ CP should be perpendicular to the x-axis.

$\Rightarrow h = 6 \Rightarrow k = 3$ and the radius is $r = |CP| = k - 0 = 3$.

Hence, the circle is centered at (6, 3) with radius $r = 3$. Therefore, the equation of the circle is $(x-6)^2 + (y-3)^2 = 9$.

Example 4.13 Suppose the line $y=x$ is tangent to a circle at point P(2,2). If the center of the circle is on the ***x***-axis, then what is the equation of the circle?

Solution: The circle in the question is as in Figure 4.11.

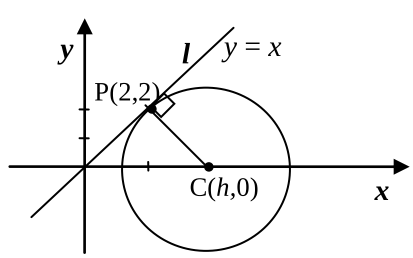


Figure 4.11

Let the center of the circle be C(h,0). We need to find h. The slope of the line $\boldsymbol{l}$: y=x is 1 and $\boldsymbol{l}$ is perpendicular to CP. Hence the slope of CP is –1.

So, the slope of CP = $\frac{2-0}{2-h} = -1 \quad \Rightarrow h-2=2$ or $h=4$.

$\Rightarrow$ The center of the circle is C(4,0); and $r^2 = |CP|^2 = (2-0)^2 + (2-4)^2 = 4+4 = 8$.

Therefore, the equation of the circle is $(x-4)^2 + y^2 = r^2 = 8$.

Exercise 4.2.3

1. Find the equation of the line tangent to the circle with the center at (–1, 1) and point of tangency at (–1, 3).
2. The center of a circle is on the line $y = 2x$ and the line $x=1$ is tangent to the circle at (1, 6). Find the center and radius of the circle.
3. Suppose two lines $y = x$ and $y = x - 4$ are tangent to a circle at (2, 2) and (4, 0), respectively. Find equation of the circle.
4. Find an equation of the line tangent to the circle $x^2 + y^2 - 2x + 2y = 2$ at (1,1).
5. Find equation of the line through $(\sqrt{32}, 0)$ and tangent to the circle with equation $x^2 + y^2 = 16$.
6. Suppose P(1,2) and Q(3, 0) are the endpoints of a diameter of a circle and L is the line tangent to the circle at Q.
   (a) Show that R(5, 2) is on L.
   (b) Find the area of ΔPQR, when R is the point given in (a).

4.3 Parabolas

By the end of this section, you should

- know the geometric definition of a parabola.
- know the meaning of vertex, focus, directrix, and axis of a parabola.
- be able to find equation of a parabola whose axis is horizontal or vertical.
- be able to identify equations representing parabolas.
- be able to find the vertex, focus, and directrix of a parabola and sketch the parabola.

4.3.1 Definition of a Parabola

Definition 4.3: Let L be a fixed line and F be a fixed point not on the line, both lying on the plane. A **parabola** is a set of points equidistant from L and F. The line L is called the **directrix** and the fixed point F is called the **focus** of the parabola.

This definition is illustrated by Figure 4.12.

- Note that the point halfway between the focus F and directrix L is on the parabola; it is called the **vertex,** denoted by V.
- |VF| is called the **focal length**.
- The line through F perpendicular to the directrix is called the **axis** of the parabola. It is the line of symmetry for the parabola.
- The chord BB′ through F perpendicular to the axis is called **latus rectum**.
- The length of the latus rectum, i.e, |BB′|, is called **focal width**.

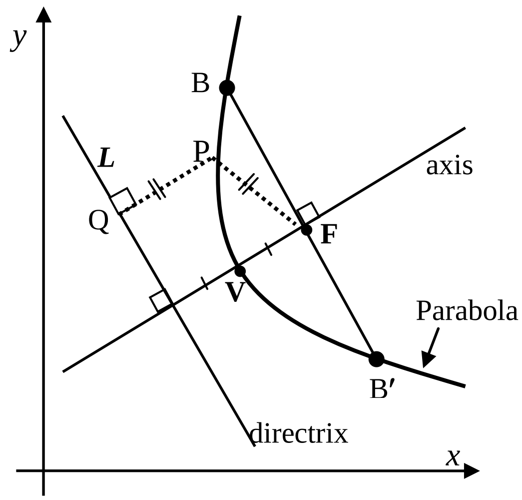


Figure 4.12: Parabola, $d(\text{P},L) = |\text{PF}|$

Letting $|\text{VF}| = p$, you may show that $|\text{BB}'| = 4p$; i.e., focal width is 4 times focal length.
If $P(x,y)$ is any point on the parabola, then by the definition, the distance of P from the directrix is equal to the distance between P and the focus F. This is used to determine an equation of a parabola. To do this, we consider first the cases when the axis of the parabola is parallel to one of the coordinate axes.

Exercise 4.3.1

Use the definition of parabola and the given information to answer or solve each of the following problems.

1. Suppose the focal length of a parabola is p, for some $p > 0$. Then, show that the focal width (length of the latus rectum) of the parabola is $4p$.
2. Suppose the vertex of a parabola is the origin and its focus is F(0,1). Then,
   (a) What is the focal length of the parabola.
   (b) Find the equations of the axis and directrix of the parabola.
   (c) Find the endpoints of the latus rectum of the parabola.
   (d) Determine whether each of the following point is on the parabola or not.
   (i) (4, 4) (ii) (2, 2) (iii) (−4, 4) (iv) (4, −4) (v) (1, 1/4)

(Note: By the definition, a point is on the parabola iff its distances from the focus and from the directrix are equal.)

3. Suppose the vertex of a parabola is V(0, 1) and its directrix is the line $x = -2$. Then,
   (a) Find the equation of the axis of the parabola.
   (b) Find the focus of the parabola.
   (c) Find the length and endpoints of the latus rectum of the parabola.
   (d) Determine whether each of the following point is on the parabola or not.
   (i) (1, 0) (ii) (3, 0) (iii) (8, 9) (iv) (8, –7) (v) (8, 8)

4.3.2 Equation of Parabolas

I: Equation of a parabola whose axis is parallel to the *y*-axis:

A parabola whose axis is parallel to ***y***-axis is called **vertical parabola**. A vertical parabola is either open upward (as in Figure 4.13 (a)) or open downward (as in Figure 4.13 (b)).

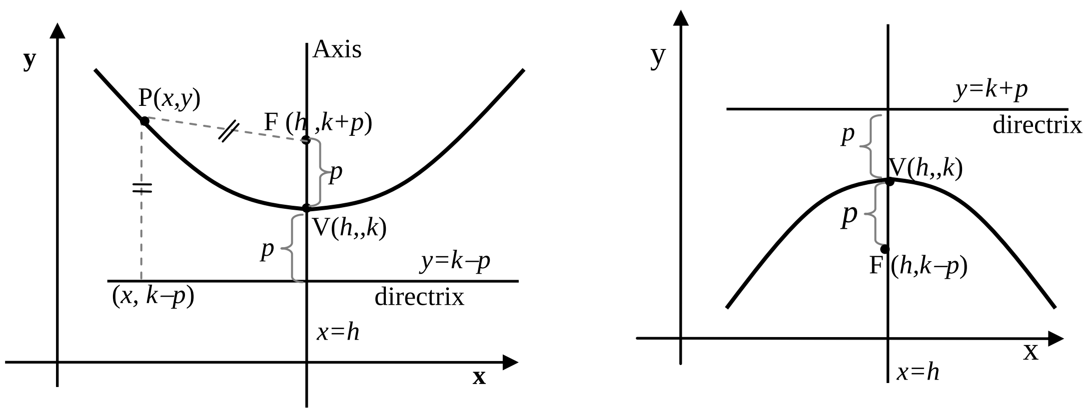


Figure 4.13: (a) parabola open upward (b) parabola open downward

Let p be the distance from vertex V(h,k) to the focus F of the parabola, i.e.,|VF| = p. Then, by the definition, F is located p units above V if the parabola opens upward and it is located p units below V if the parabola opens downward as indicated on Figure 4.13(a) and (b), respectively. To determine the desired equation, we first consider the case when the parabola opens upward.

Therefore, considering a vertical parabola with vertex V(h,k) that opens upward (Figure 4.13a), its focus is at F(h, $k+p$). $\Rightarrow$ The equation of its directrix is $y = k-p$.

Then, for any point P(x,y) on the parabola, |PF| is equal to the distance between P and the directrix if and only if

$$\sqrt{(x-h)^2+(y-k-p)^2} = y-k+p$$

Upon simplification, this becomes

$$(x-h)^2 = 4p(y-k)$$

called standard equation of a vertical parabola, vertex (h, k), focal length p, open upward.

In particular, if the vertex of a vertical parabola is at origin, i.e., $(h, k) = (0,0)$ and opens upward, then its equation is

$$x^2 = 4py$$ (In this case, its focus is at $F(0, p)$, and its directrix is $y = -p$)

If a vertical parabola with vertex $V(h, k)$ opens downward, then its directrix is above the parabola and its focus lies below the vertex (see Figure 4.13(b). In this case, the focus is at $F(h, k-p)$, and its directrix is given by $y = k+p$. Moreover, following the same steps as above , the equation of this parabola becomes

$$(x-h)^2 = -4p(y-k)$$ (Standard equation of a vertical parabola, open downward, vertex (h, k), and focal length p.)

In particular, if the vertex of a vertical parabola is at origin, i.e., $(h, k) = (0,0)$ and opens downward, then its equation is

$$x^2 = -4py$$ (In this case, its focus is at $F(0,-p)$, and its directrix is $y = p$)

Example 4.14: Find the vertex, focal length, focus and directrix of the parabola $y = x^2$.

Solution: The given equation, $x^2 = y$, is the standard equation of the parabola with vertex at origin (0,0) and $4p = 1 \Rightarrow$ its focal length is $p = 1/4$. Since the parabola opens upward, its focus is p units above its vertex $\Rightarrow$ its focus is at $F(0,1/4)$; and its directrix is horizontal line p units below its vertex $\Rightarrow$ its directrix is $y = -1/4$. You may sketch this parabola.

Example 4.15: If a parabola opens upward and the endpoints of its latus rectum are at $A(-4, 1)$ and $B(2, 1)$, then find the equation of the parabola, its directrix and sketch it.

Solution: Since the focus F of the parabola is at the midpoint of its latus rectum AB, we have

$F = \left(\frac{-4+2}{2}, \frac{1+1}{2}\right) = (-1,1)$, and focal width $4p = |AB| = 2-(-4) = 6 \Rightarrow$ focal length $p = 3/2$.

Moreover, as the parabola opens upward its vertex is p units below its focus. That is,

$V(h, k) = (-1, 1-3/2) = (-1, -1/2)$. Therefore, the equation of the parabola is

$$(x+1)^2 = 6\left(y+\tfrac{1}{2}\right).$$

And its directrix is horizontal line p units below its vertex, which is $y = -1/2 - 3/2 = -2$.
The parabola is sketch in the Figure 4.14 .

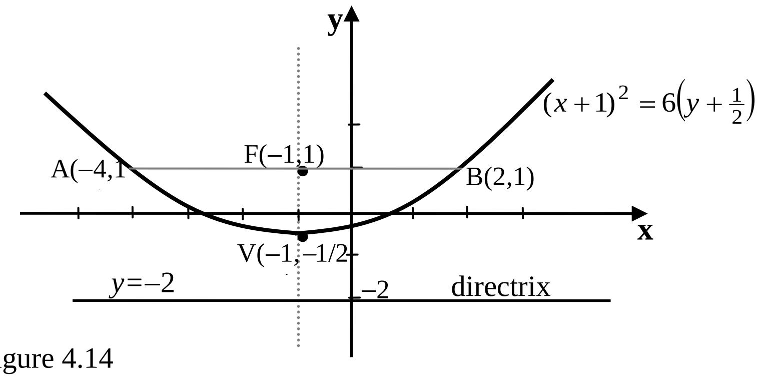


Figure 4.14

II: Equation of a parabola whose axis is parallel to the *x*-axis.

A parabola whose axis is parallel to ***x***-axis is called **horizontal parabola**. Such parabola opens either to the right or to the left as shown in Figure 4.15 (a) and (b), respectively.

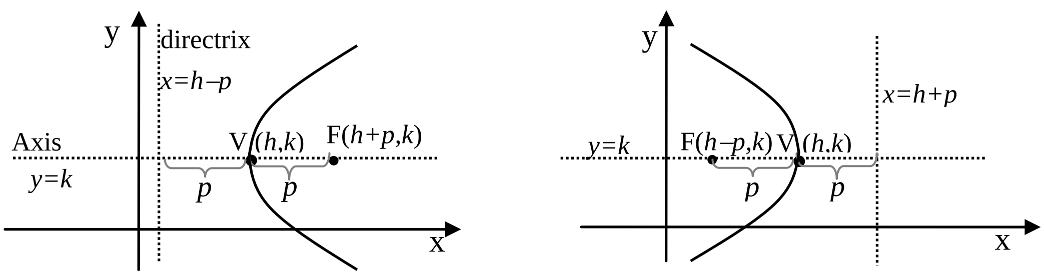


Figure 4.15: (a) Parabola open to the right (b) Parabola open to the left

The equations of such parabolas can be obtained by interchanging the role of *x* and *y* in the equations of the parabolas discussed previously. These equations are stated below. In both cases, let the vertex of the parabola be at V(*h,k*).

- If a horizontal parabola **opens to the right (**as in Fig.4.15(a)), then its focus is to the right of V at F(*h+p, k*), its directrix is $x = h-p$, and its equation is

$$(y-k)^2 = 4p(x-h)$$

- If a parabola **opens to the left (**as in Figure 4.15 (b)), then its focus is to the left of V at F(*h−p,k*), its directrix is $x=h+p$, and its equation is:

$$(y-k)^2 = -4p(x-h)$$

If the vertices of these parabolas are at the origin (0,0), then you can obtain their corresponding equations by setting *h*=0 and *k*=0 in the above equations.

Example 4.16: Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch its graph.

Solution: The equation is $y^2 = -10x$; and comparing this with the above equation, it is an equation of a parabola whose vertex is at (0,0), axis of symmetry is the x-axis, open to the left and 4*p*=10, i.e., *p*=5/2. Thus, the focus is F=(–5/2,0) and its directrix is *x* =5/2. Its graph is sketched in Figure 4.16.

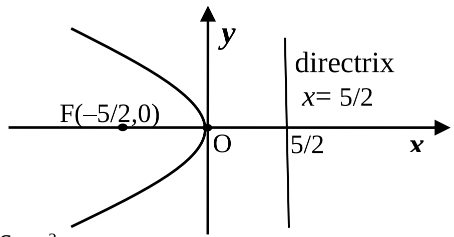


Figure 4.16: $y^2 + 10x = 0$

Example 4.17: Find the focus and directrix of the parabola $y^2 + 4y + 8x - 4 = 0$ and sketch it.

Solution: The eaquation is $y^2 + 4y = -8x + 4$. (Now complete the square of *y*-terms)

$\Rightarrow y^2 + 4y + \mathbf{2^2} = -8x + 4 + \mathbf{4}$

$\Rightarrow (y + 2)^2 = -8x + 8$

$\Rightarrow (y + 2)^2 = -8(x - 1)$

This is equation of a parabola with vertex at (*h*, *k*)=(1,–2), open to the left and focal length *p*, where 4*p*=8 $\Rightarrow p$=2. Therefore, its focus is
F= (*h*–*p*, *k*) = (–1, –2), and directrix *x*=*h*+*p* =3. The parabola is sketched in Figure 4.17.

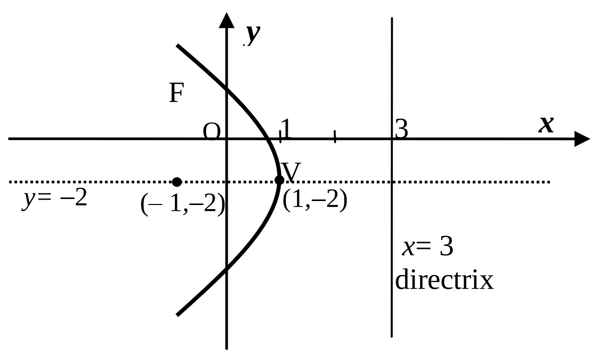


Figure 4.17: $y^2 + 4y + 8x - 4 = 0$

Remark:- An equation given as: $Ax^2 + Dx + Ey + F = 0$

or $Cy^2 + Dx + Ey + F = 0$

may represent a parabola whose axis is parallel to the ***y***-axis or parallel to the ***x***-axis, respectively. The vertex, focal length and focus for such parabolas can be identified after converting the equations into one of the standard forms by completing the square.

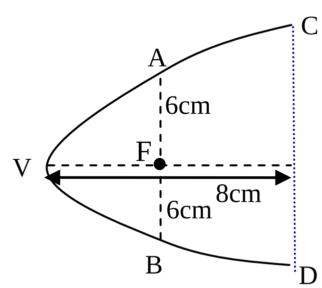
Exercise 4.3.2

For questions 1 to 8, find an equation of the parabola with the given properties and sketch its graph.

1. Focus (0, 1) and directrix $y = -1$.
2. Focus (–1, 2) and directrix $y = -2$.
3. Focus (3/2, 0) and directrix $x = -3/2$.
4. Focus (–1,–2) and directrix $x = 0$.
5. Vertex (3, 2) and Focus (3, 3).
6. Vertex (5, –2) and Focus (–5, –2).
7. Vertex (1, 0) and directrix $x = -2$.
8. Vertex (0, 2) and directrix $y = 4$.

For questions 9 to 17 find the vertex, focus and directrix of the parabola and sketch it.

9. $y = 2x^2$
10. $8x^2 = -y$
11. $4x - y^2 = 0$
12. $x + y^2 = 0$
13. $x - 1 = (y + 2)^2$
14. $(x + 2)^2 = 8y - 24$
15. $y^2 + 8x + 6y + 25 = 0$
16. $y^2 - 2y - 4x + 9 = 0$
17. $-4x^2 + 4x - \frac{1}{2}y = 1$
18. Find an equation of the parabola that has a vertical axis, its vertex at (1, 0) and passing through (0,1).
19. The vertex and endpoints of the latus rectum of the parabola $x^2 = 36y$ forms a triangle. Find the area of the triangle.
20. P(4, 6) is a point on a parabola whose focus is at (0, 2) and directrix is parallel to x-axis.
    (a) Find an equation of the parabola, its vertex and directrix.
    (b) Determine the distance from P to the directrix.
21. An iron wire bent in the shape of a parabola has latus rectum of length 60cm. What is its focal length?
22. A cross-section of a parabolic reflector is shown in the figure below. A bulb is located at the focus and the opening at the focus, AB, is 12 cm. What is the diameter of the opening, CD, 8 cm from the vertex?



4.4 Ellipses

By the end of this section, you should

- know the geometric definition of an ellipse.
- know the meaning of the center, vertices, foci, major axis and minor axis of an ellipse.
- be able to find equation of an ellipse whose major axis is horizontal or vertical.
- be able to identify equations representing ellipses.
- be able to find the center, foci and vertices of an ellipse and sketch the ellipse.

4.4.1 Definition of an Ellipse

Definition 4.4: Let F and F′ be two fixed points in the plane. An **ellipse** is the locus or set of all points in the plane such that the sum of the distances from each point to F and F′ is constant. That is, a point P is on the ellipse if and only if $|PF| + |PF'| = \text{constant}$. (See Figure 4.18).

The two fixed points, F and F′, are called **foci** (singular- **focus**) of the ellipse.

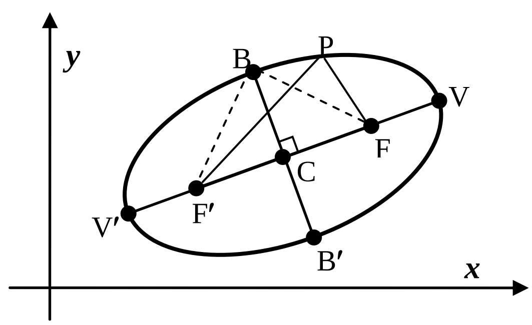


Figure 4.18: Ellipse: $|PF| + |PF'| = \text{constant}$

Note also the following terminologies and relationships about ellipse.

- The midpoint C between the foci F′ and F is called the **center** of the ellipse.
- The longest diameter (longest chord) V′V through F′ and F is called the **major axis** of the ellipse; and the chord BB′ through C which is perpendicular to V′V is called **minor axis**.
- The endpoints of the major axis, V′ and V, are called the **vertices** of the ellipse.
- From the definition, $|V'F'| + |V'F| = |VF'| + |VF| \Rightarrow |V'F'| = |VF| \Rightarrow |CV'| = |CV|$. Hence, C is the midpoint of V′V. We denote the length of the major axis by $2a$. That is, $|CV| = a$.
  $\Rightarrow |VF'| + |VF| = |V'V| = 2a.$
  $\Rightarrow |PF'| + |PF| = 2a,$ for any point P on the ellipse.
- We let $|BC| = b$. (You can show that C is the midpoint of BB′. So, $|B'C| = b$.)
- The distance from the center C to a focus F (or F′) is denoted by c, i.e., $|CF| = c = |CF'|$.

- Now, since $|BF'| + |BF| = 2a$ and BC is a perpendicular bisector of F′F, we obtain that $|BF'| = |BF| = a$. Hence, using the Pythagoras Theorem on ΔBCF, we obtain

$$b^2 + c^2 = a^2 \quad \text{or} \quad b^2 = a^2 - c^2.$$

(**Note:** $a \ge b$. If $a=b$, the ellipse would be a circle with radius $r = a = b$).

- The ratio of the distance between the two foci to the length of the major axis is called the **eccentricity** of the ellipse, and denoted by e. That is,

$$e = \frac{|F'F|}{|V'V|} = \frac{c}{a}. \qquad (\text{Note that } 0 < e < 1 \text{ because } 0 < c < a)$$

Exercise 4.4.1

Use the definition of ellipse and the given information to answer or solve each of the following problems.

1. Suppose F′ and F are the foci of an ellipse and B′ and B are the endpoints of the minor axis of the ellipse, as in Figure 4.18. Then, show that each of the followings hold.
   (a) ΔBF′F is isosceles triangle.
   (b) The quadrilateral BF′B′F is a rhombus.
   (c) FF′ is perpendicular bisector of BB′ ; and also BB′ is perpendicular bisector of FF′.
   (d) If the length of the major axis is $2a$, length of minor axis is $|BB'| = 2b$, and $|F'F| = 2c$, for some positive a, b, c, then
   (i) $|BF| = a$ (ii) $a^2 = b^2 + c^2$
2. Suppose the vertices of an ellipse are $(\pm 2, 0)$ and its foci are $(\pm 1, 0)$.
   (a) Where is the center of the ellipse?
   (b) Find the endpoints of its minor axis.
   (c) Find the lengths of the major and minor axes.
   (d) Determine whether each of the following points is on the ellipse or not.
   (i) $(1, 3/2)$ (ii) $(3/2, -1)$ (iii) $(-1, 3/2)$ (iv) $(-1, -3/2)$ (v) $(1, 1)$
   (Note: By the definition, a point is on the ellipse iff the sum of its distances to the two foci is $2a$)
3. Suppose the endpoints of the major axis of an ellipse are $(0, \pm 2)$ and the end points of its minor axis are $(\pm 1, 0)$.
   (a) Where is the center of the ellipse?
   (b) Find the coordinates of the foci.
   (c) Determine whether each of the following points is on the ellipse or not.
   (i) $(1/2, \sqrt{3})$ (ii) $(\sqrt{2}, 1)$ (iii) $(-1/2, -\sqrt{3})$ (iv) $(\sqrt{3}/2, 1)$
4. Suppose the endpoints of the minor axis of an ellipse are $(1, \pm 3)$ and its eccentricity is 0.8. Find the coordinates of (a) the center, (b) the foci , (c) the vertices of the ellipse.

4.4.2 Equation of an Ellipse

In order to obtain the simplest equation for an ellipse, we place the ellipse at standard position. An ellipse is said to be at standard position when its center is at the origin and its major axis lies on either the ***x***-axis or **y**-axis.

I. Equation of an ellipse at standard position:

There are two possible situations, namely, when the major axis lies on *x*-axis (called horizontal ellipse) and when the major axis lies on *y*-axis (called vertical ellipse). We first consider a horizontal ellipse as in Figure 4.19

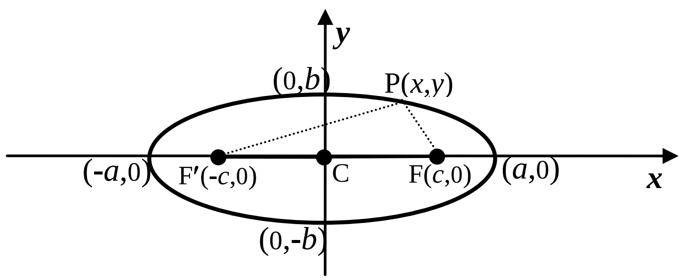


Figure 4.19: Horizontal ellipse at standard position

Let the center of the ellipse be at the origin, C(0,0) and foci at F′(−*c*,0), F (*c*,0) and vertices at (−*a*,0) and (*a*,0) (see Figure 4.19). Then, a point P(*x*,*y*) is on the ellipse iff

$$|PF'| + |PF| = 2a .$$

That is, $\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$

or $\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$

Squaring both sides we get

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

which simplifies to $a\sqrt{(x+c)^2 + y^2} = a^2 + cx$

Again squaring both sides, we get $a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + c^2x^2$

which becomes $(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$

Now recalling that $b^2 = a^2 - c^2$ and dividing both sides by a^2b^2, the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(Equation of horizontal ellipse at **standard position,** vertices ($\pm a$, 0), foci ($\pm c$, 0), where $c^2 = a^2 - b^2$)

For a vertical ellipse at standard position, the same procedure gives the equation

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

(Equation of vertical ellipse at **standard position,** vertices (0, $\pm a$), foci (0, $\pm c$), where $c^2 = a^2 - b^2$)

Note: Notice that here, for vertical ellipse, the larger denominator a^2 is under y^2.

Example 4.18: Locate the vertices and foci of $16x^2 + 9y^2 = 144$ and sketch its graph.

Solution: Dividing both sides of the equation by 144, we get:

$$\frac{x^2}{9}+\frac{y^2}{16}=1 \quad \text{or} \quad \frac{x^2}{3^2}+\frac{y^2}{4^2}=1.$$

This is equation of a vertical ellipse at standard position with $a=4$, so vertices at $(0, \pm 4)$, and $b=3$; i.e., endpoints of the minor axis at $(\pm 3, 0)$. Since $c^2 = a^2-b^2 = 7 \Rightarrow c = \sqrt{7}$, the foci are $\left(0, \pm\sqrt{7}\right)$. The graph is sketched in Figure 4.20.

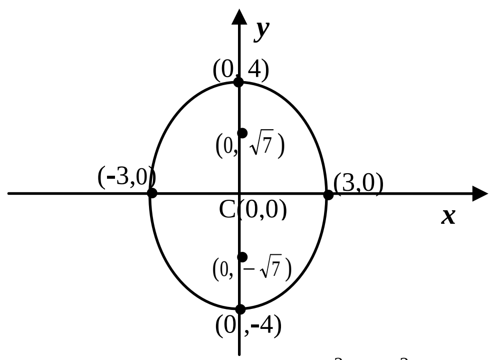


Figure 4.20: $16x^2 + 9y^2$

(II) Equation of shifted Ellipses:

When an ellipse is not at standard position but with center at a point C(h,k), then we can still obtain its equation by considering translation of the xy-axes in such a way that its origin translated to the point C(h,k). This result in a new $X'Y'$ coordinate system whose origin O′ is at C(h,k) so that the ellipse is at standard position relative to the $X'Y'$ system(see, Figure 4.21)

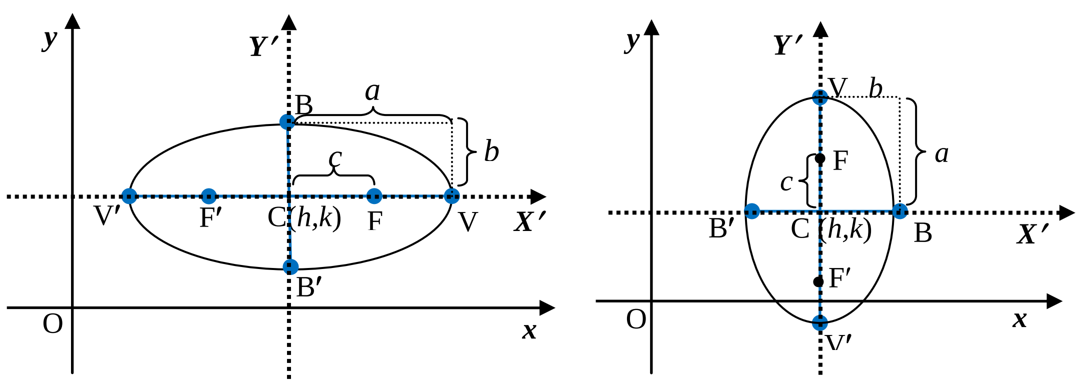


Fig. 4.21: (a) horizontal ellipse, center C(h,k) (b) vertical ellipse, center C(h,k)

Consequently, the equation of the horizontal and vertical ellipses relative to the new $X'Y'$ coordinate system with (x', y') coordinate points are

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1 \quad \text{and} \quad \frac{x'^2}{b^2}+\frac{y'^2}{a^2}=1, \qquad . \ . \ . \ . \ . \ . \ . \ . \ (I).$$

respectively. Since the origin of the new coordinate system is at the point (h,k) of the xy-coordinate system, the relationship between a point (x,y) of the $\boldsymbol{xy}$-coordinate system and (x',y') of the new coordinate system is given by $(x, y) = (x',y') + (h, k)$. That is,

$$x' = x - h, \quad \text{and} \quad y' = y - k.$$

Thus, in the original xy-coordinate system the equations of the horizontal and vertical ellipses with center C(h, k), lengths of major axis = $2a$ and minor axis = $2b$ are, respectively, given by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

(Standard equation of horizontal ellipse with center C(h,k))

and

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

(Standard equation of vertical ellipse with center C(h,k))

Example 4.19: The endpoints of the major axis of an ellipse are at (–3,4) and (7,4) and its eccentricity is 0.6. Find the equation of the ellipse and its foci.

Solution: The given vertices are at V′(–3,4) and V(7,4) implies that $2a = |V'V| = 10 \Rightarrow a = 5$; and the center C($h,k$) is the midpoint of V′V $\Rightarrow$ $(h,k) = (\frac{-3+7}{2}, \frac{4+4}{2}) = (2,4)$. Moreover, eccentricity = $c/a = 0.6 \Rightarrow c = 5\times0.6 = 3$. Hence, $b^2 = a^2 - c^2 = 25-9 = 16$. Note that the major axis V'V is horizontal. Therefore, using the standard equation of a horizontal ellipse, the equation of the ellipse is

$$\frac{(x-2)^2}{25} + \frac{(y-4)^2}{16} = 1.$$

Now, as the center (h,k) = (2,4), c=3 and V′V is horizontal, the foci are at $(h \pm c, k) = (2 \pm 3, 4)$.
That is, the foci are at F′(–1, 4) and F(5, 4).
Moreover, the endpoints of major axis are at $(h, k\pm b) = (2, 4\pm 4) \Rightarrow$ B′=(2,0) and B=(2,8).
The graph of the ellipse is sketched in Figure 4.22.

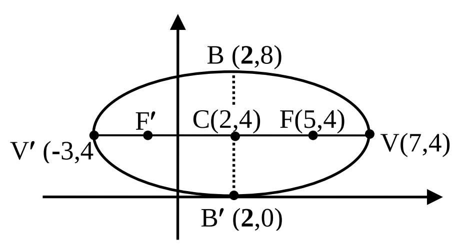


Figure 4.22

Example 4.20: Find the center, foci and vertices of $4x^2 + y^2 + 8x = 0$ and sketch its graph

Solution: Group the x-terms of the equation and complete the square:

$$4(x^2 + 2x) + y^2 = 0$$

$$\Rightarrow 4(x^2+2x+1) + y^2 = 4 \quad \text{(divide both sides by 4)}$$

$$\Rightarrow (x+1)^2 + \frac{y^2}{4} = 1$$

This is equation of a vertical ellipse (major axis parallel to the y-axis), center C=(h,k) = (–1,0),

$a=2,\ b=1. \Rightarrow c^2 = a^2 - b^2 = 4-1 \ \Rightarrow c=\sqrt{3}$

Thus, foci : F′(–1, –√3) and F(–1, √3) ,

Vertices: V = (–1, 2), V′= (–1, –2);

Endpoints of minor axis: B=(0,0), B′=(–2,0);

The graph of the ellipse is sketched in Figure 4.23.

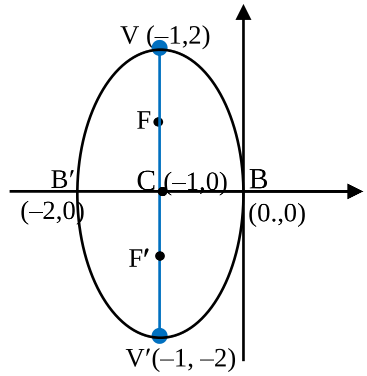


Figure 4.23: $4x^2 + y^2 + 8x = 0$

Remark: Consider the equation: $Ax^2 + Cy^2 + Dx + Ey + F = 0$,

when A and C have the same sign. So, without lose of generality, let A >0 and C >0.

By completing the squares you can show that this equation is equivalent to

$$A\left(x+\frac{D}{2A}\right)^2 + C\left(y+\frac{E}{2C}\right)^2 = \frac{D^2C+E^2A-4ACF}{4AC}.$$

From this you can conclude that the given equation represents:-

- an ellipse with center $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$ if $D^2C + E^2A - 4ACF > 0$.
- If $D^2C + E^2A - 4ACF = 0$, the equation is satisfied by the point $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$ only. In this case, the locus of the equation is called a **point-ellipse** (degenerate ellipse).
- If $D^2C + E^2A - 4ACF < 0$, then the equation has no locus.

Exercise 4.4.2

For questions 1 to 13, find an equation of the ellipse with the given properties and sketch its graph.

1. Foci at (± 2, 0) and a vertex at (5, 0)
2. A focus at (0, –3) and vertices at (0, ±5)
3. Foci at (2, 3) , (2,7) and a vertex at (2, 0)
4. Foci at (0, –1) , (8,–1) and a vertex at (9, –1)
5. Center at (6,1) , one focus at (3,1) and one vertex at (10, 1)
6. Foci at (2, ±1) and the length of the major axis is 4.
7. Foci at (2, 0), (2, 6) and the length of the minor axis is 5.

8. The distance between its foci is $2\sqrt{5}$ and the endpoints of its minor axis are (–1, –2) and (3,–2).
9. Vertices at (±5, 0) and the ellipse passes through (–3, 4).
10. Center at (1, 4), a vertex at (10, 4) , and one of the endpoints of the minor axis is (1, 2).
11. The ellipse passes through (–1, 1) and $\left(\frac{1}{2},-2\right)$ with center at origin.
12. The endpoints of the major axis are (3, –4) and (3, 4) , and the ellipse passes through the origin
13. The endpoints of the minor axis are (3, –2) and (3, 2) , and the ellipse passes through the origin

For questions 14 to 22 find the center, foci and vertices of the ellipse having the given equation and sketch its graph.

14. $\frac{x^2}{9}+\frac{y^2}{5}=1$
15. $5x^2+y^2=25$
16. $x^2+9y^2=9$
17. $\frac{(x-2)^2}{9}+\frac{(y+3)^2}{16}=1$
18. $(x+1)^2+2(y+2)^2=3$
19. $x^2+9y^2-2x+18y+1=0$
20. $9x^2+4y^2-18x=27$
21. $x^2+2y^2-6x+4y=-7$
22. $4x^2+y^2+2x-10y=6$
23. Consider the equation $2x^2+4y^2+8x-16y+F=0$. Find all values of F such that the graph of the equation
    (a) is an ellipse. (b) is a point. (c) consists of no points at all.

4.5 Hyperbolas

By the end of this section, you should

- know the geometric definition of a hyperbola.
- know the meaning of the center, vertices, foci and transverse axis of a hyperbola.
- be able to find equation of a hyperbola whose transverse axis is horizontal or vertical.
- be able to identify equations representing hyperbolas.
- be able to find the center, vertices, foci, and asymptotes of a hyperbola and sketch the hyperbola.

4.5.1 Definition of a hyperbola

Definition 4.5: Let F and F′ be two fixed points in the plane. A **hyperbola** is the set of all points in the plane such that the difference of the distance of each point from F and F′ is constant. We shall denote the constant by 2a, for some $a > 0$. That is, a point P is on the hyperbola if and only if $|PF'| - |PF| = 2a$ (or $|PF| - |PF'| = 2a$, whichever is positive) . The two fixed points F and F′ are called the foci of the hyperbola.

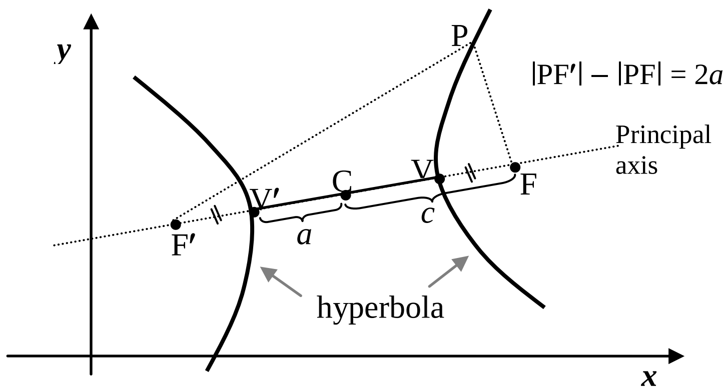


Figure 4.24: Hyperbola

Figure 4.24 illustrates the definition of hyperbola. Notice that the definition of hyperbola is similar to that of an ellipse, the only change is that the sum of distances has become the difference of distances. Here, for the difference of any two unequal values, we take the higher value minus the smaller so that $a > 0$ in the definition. The following terminologies, notations and relationships are also important with regard to a hyperbola. Refer to Figure 4.24 for the following discussion.

- The line through the two foci F′ and F is called the **principal axis** of the hyperbola. The point on the principal axis at halfway between the two foci, that is, the midpoint of F′F, is called the **center** of the hyperbola and represented by C. We denote the distance between the two foci by 2c. That is, $|F'F| = 2c$ or $|CF| = c = |CF'|$. Noting also that $|PF'| < |F'F| + |PF|$ in Δ PF′F and $|PF'| - |PF| = 2a$, you can show that $a < c$.
- The points V′ and V where the hyperbola crosses the principal axis are called **vertices** of the hyperbola. The line segment V′V is called the **transverse axis** of the hyperbola. So, as V′ and V are on the hyperbola, the definition requires that $|V'F| - |V'F'| = |VF'| - |VF|$. From this, you can obtain that $|V'F'| = |VF|$. Consequently,
  (i) C is the midpoint of also V′V; that is, $|CV'| = |CV|$ **.**
  (ii) $|V'V| = |V'F| - |VF| = |V'F| - |V'F'| = 2a$. (The length of the transverse axis is 2a)
  (iii) $|V'C| = a = |CV|$ (This follows from (i) and (ii).)

- The **eccentricity *e*** of a hyperbola is defined to be the ratio of the distance between its foci to the length of its transverse axis. That is, similar to the definition of eccentricity of an ellipse, the eccentricity of a hyperbola is

$$e = \frac{|F'F|}{|V'V|} = \frac{c}{a} \qquad \text{(But here, } e > 1 \text{ because } c > a\text{)}$$

Exercise 4.5.1

Use the definition of hyperbola and the given information to answer or solve each of the following problems.

1. Suppose C is the center, F′ and F are the foci, and V′ and V are the vertices of the hyperbola, as in Figure 4.24, with $|CV| = a$ and $|CF| = c$. Then, show that each of the followings hold.
   (a) If P is any point on the hyperbola, then $|PF| - |PF'| = \pm 2a$.
   (Note: Taking that $|PF| - |PF'| = k$, a constant, show that $k = \pm 2a$.)
   (b) $a > c$.
2. Consider a hyperbola whose foci are $(\pm 2, 0)$ and contains the point P(2, 3).
   (a) Where is the center of the hyperbola?
   (b) Determine the principal axis of the hyperbola.
   (c) Find the length of the transverse axis of the hyperbola.
   (d) Find the coordinates of the vertices of the hyperbola.
   (e) Determine whether each of the following points is on the hyperbola or not.
   (i) $(-2, 3)$ (ii) $(-2, -3)$ (iii) $(2, -3)$ (iv) $(3, 4)$ (v) $(\sqrt{13}, 6)$
3. Suppose the vertices of a hyperbola are at $(0, \pm 2)$ and its eccentricity is 1.5. Then,
   (a) Find the foci of the hyperbola.
   (b)) Determine whether each of the following points is on the hyperbola or not.
   (i) $(\sqrt{5}, 3)$ (ii) $(2, 3)$ (iii) $(\sqrt{5}, -3)$ (iv) $(3, \sqrt{5})$

4.5.2 Equation of a hyperbola

We are now ready to derive equation of a hyperbola. But, for simplicity, we consider first the equation of a standard hyperbola with center at origin. A standard hyperbola is the one whose principal axis (or transverse axis) is parallel to either of the coordinate axes.

I. Equation of a standard hyperbola with center at origin.

There are two possible situations, namely, when the transverse axis lies on *x*-axis (called horizontal hyperbola) and when the transverse axis lies on *y*-axis (called vertical hyperbola). We first consider a horizontal hyperbola with center C(0,0), vertices V′(–*a*, 0), V (*a*, 0) and foci F′(–*c*, 0), F (*c*, 0).

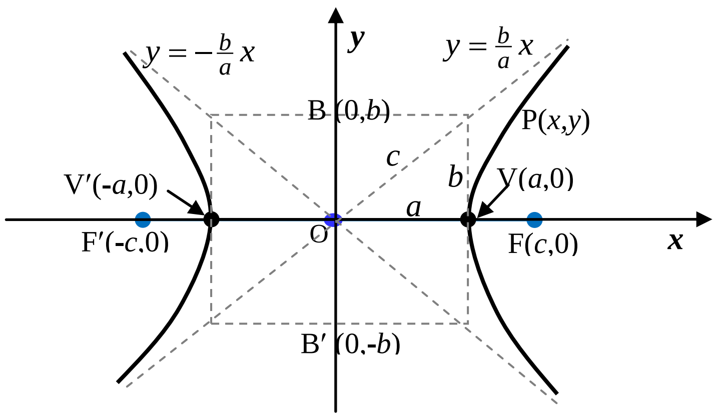


Figure 4.25: Horizontal hyperbola centered at origin

Notice that $c^2 - a^2 > 0$ as $c > a$. Hence, we can put $b^2 = c^2 - a^2$ for some positive *b*. That is, $a^2 + b^2 = c^2$ so that *a*, *b*, *c* are sides of a right triangle (see, Figure 4.25). The line segment BB′ perpendicular to the transverse axis at C and with endpoints B(0,*b*) and B′(0,–*b*) is called **conjugate axis** of the hyperbola. Observe that the midpoint of the conjugate axis is C and its length is |BB′| = 2*b*. (*b* will play important role in equation of the hyperbola and its graph).

Now, for any point P(*x*,*y*) on the hyperbola it holds that |PF′| – |PF| = 2*a* .

That is, $\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=2a$

or $\sqrt{(x+c)^2+y^2}=2a+\sqrt{(x-c)^2+y^2}$

Squaring both sides we get

$$x^2+2cx+c^2+y^2=4a^2+4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2$$

which simplifies to $a\sqrt{(x-c)^2+y^2}=cx-a^2$

Again squaring both sides and rearranging, we get $(c^2-a^2)x^2-y^2=a^2(c^2-a^2+y^2)$.

Recall that we set $b^2=c^2-a^2$. So, using this in the above equation and dividing both sides by a^2b^2, the equation becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(Equation of horizontal hyperbola with center C(0,0), vertices ($\pm a$, 0), foci ($\pm c$, 0), where $c^2 = a^2 + b^2$)

Note that this hyperbola has no y-intercept because if $x = 0$, then $-y^2 = b^2$ which is not possible. The hyperbola is symmetric with respect to both x-and y- axes.

Also, from this equation we get

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1 \quad \text{implies that} \quad x^2 \geq a^2 \text{ . So, } \quad |x| = \sqrt{x^2} \geq \sqrt{a^2} = a.$$

Therefore, we have $x \geq a$ or $x \leq -a$. This means that the hyperbola consists of two parts, called its *branches*. Moreover, if we solve for y from the equation we get $y = \pm\frac{b}{a}\sqrt{x^2 - a^2} \to \pm\frac{b}{a}x$ as $x \to \infty$.

This means the hyperbola will approach (but never reaches) the line $y = \pm\frac{b}{a}x$ as x gets larger and larger. That is, the lines $y = \pm\frac{b}{a}x$ are the **asymptotes** of the hyperbola.

In sketching a hyperbola, it is best to draw the rectangle formed by the line $y = \pm b$ and $x = \pm a$ and the to draw the asymptotes which are along the diagonals of the rectangle (as shown by the dashed lines in Figure 4.25). The hyperbola lies outside the rectangle and inside the asymptotes. It opens around the foci.

Example 4.21: Find an equation of the hyperbola whose foci are F′(–5, 0) and F(5, 0) and contains point P(5, 16/3).

Solution: It is horizontal hyperbola with center (0,0) and $c = 5$. In addition, as P(5, 16/3) is on the hyperbola we have that |PF′| – |PF| = 2a. That is,

$$\sqrt{(5+5)^2 + \left(\tfrac{16}{3}\right)^2} - \sqrt{(5-5)^2 + \left(\tfrac{16}{3}\right)^2} = 2a$$

$\Rightarrow$ a =3. (So, its vertices are (–3, 0) and (3, 0)).

Now, using the relationship $b^2 = c^2 - a^2$, we get $b^2 = 25 - 9 = 16$.

Therefore, the equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

You may find the asymptotes and sketch the hyperbola.

For a vertical hyperbola with center at origin (i.e., when transverse axis lies on *y*-axis), by reversing the role of *x* and *y* we obtain the following equation which is illustrated in Figure 4.26.

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

(Equation of vertical hyperbola with center C(0,0), foci $(0, \pm c)$, vertices $(0, \pm a)$, where $c^2 = a^2 + b^2$ and asymptotes $y = \pm(a/b)x$)

Note:
- For a vertical hyperbola, the coefficient of y^2 is positive and that of x^2 is negative .
- a^2 is always the denominator of the positive term.

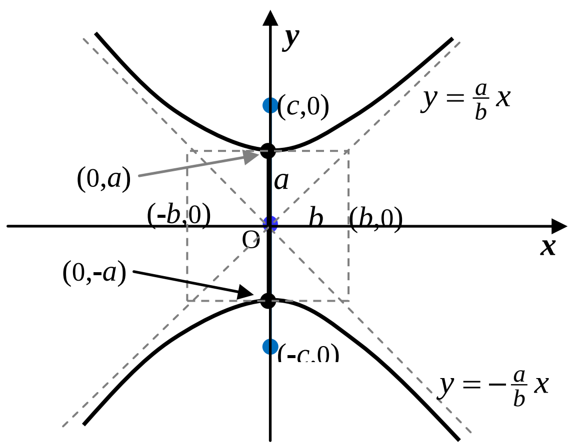


Figure 4.26: Vertical hyperbola centered at origin

Example 4.22: Find the foci and equation of the hyperbola with vertices V′(0,–1) and V(0, 1) and an asymptote $y = 2x$.

Solution: It is a vertical hyperbola with center C(0,0) and $a = |CV| = 1$. Since an asymptote of such vertical hyperbola is $y = (a/b)x$ and the slope of the given asymptote is 2, we have $a/b = 2$ $\Rightarrow 1/b = 2 \Rightarrow b = 1/2$. Thus, $c^2 = a^2 + b^2 = 1 + 1/4 = 5/4$.

So, the foci are $\left(0, \pm\sqrt{5}/2\right)$ and the equation of the hyperbola is $y^2 - 4x^2 = 1$.

(You may sketch the hyperbola)

(II) Equation of shifted hyperbolas:

The center of a horizontal or vertical hyperbola may be not at origin but at some other point C(h,k) as shown in Figure 4.27. In this case, we form the equation of the hyperbolas by using the translation of the xy-coordinate system that shifts its origin to the point C(h, k). As discussed in Section 4.4, the effect of this translation is just replacing x and y by $x–h$ and $y–k$, respectively, in the equation of the desired hyperbola.

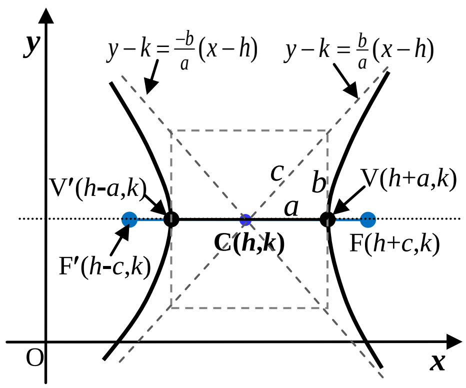
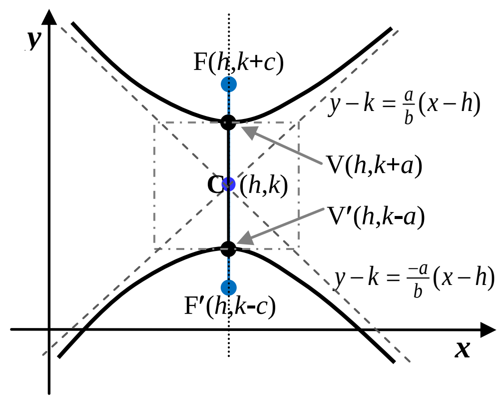


Figure 4.27: (a) Horizontal hyperbola
with center C(h,k)



(b) Vertical hyperbola
with center C(h,k)

Therefore, the standard equation of a horizontal hyperbola (transverse axis parallel to x-axis) with center C(h,k), length of transverse axis =2a, and length of conjugate axis =2b is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Center: C(h,k),
Vertices: V′ ($h–a$,k), V($h+a$, k),
Foci : F′($h–c$, k), F($h+c$, k), where $c^2 = a^2 + b^2$
Asymptotes: $y-k=\pm\frac{b}{a}(x-h)$

Similarly, the standard equation of a vertical hyperbola (transverse axis parallel to y-axis) with center C(h,k), length of transverse axis =2a, and length of conjugate axis =2b is

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

Center: C(h,k),
Vertices: V′ (h,$k–a$), V(h, $k+a$),
Foci : F′(h, $k–c$), F(h, $k+c$), where $c^2 = a^2 + b^2$
Asymptotes: $y-k=\pm\frac{a}{b}(x-h)$

Example 4.23: Find the foci, vertices and the asymptotes of the hyperbola whose equation is

$$4(x+1)^2 - (y-2)^2 = 4$$

and sketch the hyperbola.

Solution: Dividing both sides of the equation by 4 yields

$$(x+1)^2 - \frac{(y-2)^2}{4} = 1.$$

This is equation of a hyperbola with center C(–1, 2). Note that the 'x^2-term' is positive indicates that the hyperbola is horizontal (principal axis y=2), a=1, b=2, and $c^2 = a^2+b^2 \Rightarrow c = \sqrt{5}$. As a result the foci are at $\left(-1-\sqrt{5}, 2\right)$ and $\left(-1+\sqrt{5}, 2\right)$, vertices are at (–2,2) and (0,2) and the asymptotes are the lines y–2=±2(x+1), that is, y=2x+4 and y=–2x. Consequently, the hyperbola is sketched as in Figure 4.28.

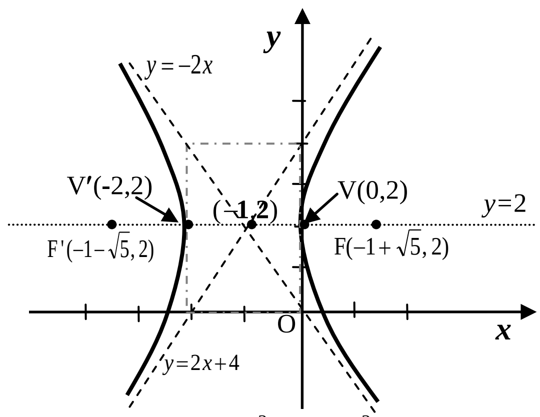


Figure 4.28: $4(x+1)^2 - (y-2)^2 = 4$

Example 4.24: Find the foci of the conic $9x^2 - 4y^2 - 72x + 8y + 176 = 0$ and sketch its graph.

Solution: Group the x-terms and y-terms of the equation and complete their squares:

$\Rightarrow\ 9x^2 - 72x - 4y^2 + 8y = -176$ (Multiply both sides by –1)

$\Rightarrow\ -9x^2 + 72x + 4y^2 - 8y = 176$

$\Rightarrow\ 4(y^2 - 2y) - 9(x^2 - 8x) = 176$

$\Rightarrow\ 4(y^2 - 2y + \mathbf{1^2}) - 9(x^2 - 8x + \mathbf{4^2}) = 176 + \mathbf{4 - 144}$

$\Rightarrow\ 4(y-1)^2 - 9(x-4)^2 = 36$ (Next, divide each by 36)

$\Rightarrow\ \frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} = 1$

This is standard equation of a hyperbola whose transverse axis is parallel to the y-axis (as its 'y^2 term' is positive) with center C (4,1), a^2=9 and b^2=4 . $\Rightarrow\ c^2 = a^2 + b^2 = 13 \Rightarrow c = \sqrt{13}$.

Thus, foci are $F'(4, 1-\sqrt{13})$ and $F(4, 1+\sqrt{13})$, and vertices (4, 1±3), i.e,, V′(4,–2) and V(4, 4). Moreover, the asymptotes are $y - k = \pm\frac{a}{b}(x-h)$. Hence, the asymptotes are

l_1: $y - 1 = \frac{3}{2}(x-4)$ and l_2: $y - 1 = -\frac{3}{2}(x-4)$. The hyperbola is sketched in Figure 4.29

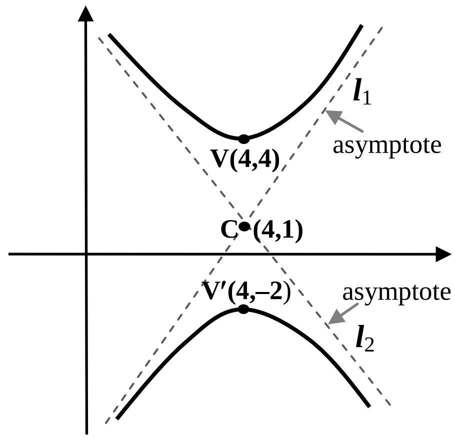


Figure 4.29: $\frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} = 1$

Example 4.25: Determine the locus or type of the conic section given by the equation

$$-x^2 + y^2 + 4x - 2y = 3.$$

Solution: Grouping the *x*-terms and *y*-terms of the equation and completing their squares yield

$$(y-1)^2 - (x-2)^2 = 0$$

$$\Rightarrow \quad (y-1)^2 = (x-2)^2$$

$$\Rightarrow \quad y-1 = \pm\sqrt{(x-2)^2} = \pm(x-2)$$

This represents pair of two lines intersecting at (2, 1), namely, $y = x-1$ and $y = -x+3$.

Remark: Consider the equation: $Ax^2 + Cy^2 + Dx + Ey + F = 0$ when $AC < 0$;

(i.e., A and C have opposite signs). Then, by completing the squares of x-terms and y-terms you can convert the equation to the following form:

$$A\left(x + \frac{D}{2A}\right)^2 + C\left(y + \frac{E}{2C}\right)^2 = \frac{D^2C + E^2A - 4ACF}{4AC}.$$

Now, letting $\Delta = D^2C + E^2A - 4ACF$, you can conclude the following:

- If $\Delta \neq 0$, the equation represents a hyperbola with center $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$.
- If $\Delta = 0$, the equation becomes $y + \frac{E}{2C} = \pm\sqrt{\left|\frac{A}{C}\right|}\left(x + \frac{D}{2A}\right)$ which are two lines intersecting at $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$. In this case, it is called **degenerate** hyperbola.

Exercise 4.5.2

For questions 1 to 9, find an equation of the hyperbola having the given properties and sketch its graph.

1. Center at the origin, a focus at (5, 0), and a vertex at (3, 0)
2. Center at the origin, a focus at (0,–5), and a vertex at (0, –3).
3. Center at the origin, ***x***-intercepts ±3, an asymptote $y = 2x$.
4. Center at the origin, a vertex at (2, 0) , and passing through $(4, \sqrt{3})$.
5. Center at (4, 2), a vertex at (7, 2) , and an asymptote $3y = 4x-10$.
6. Foci at (–2, –1) and $F_2(-2, 9)$, length of transverse axis 6.
7. Foci at (1, 3) and (7, 3), and vertices at (2,3) and (6, 3).
8. Vertices at (±3, 0), and asymptotes $y = \pm 2x$
9. Eccentricity e=1.5, endpoints of transversal axis at $(2, 2)$ and $(6, 2)$.

For questions 10 to 17 find the center, foci, vertices and asymptotes of the hyperbola having the given equation and sketch its graph.

10. $\frac{x^2}{64} - \frac{y^2}{36} = 1$
11. $y^2 - x^2 = 9$
12. $x^2 - y^2 = 9$
13. $(y+1)^2 - 4(y+2)^2 = 8$
14. $\frac{(x-2)^2}{9} - \frac{(y+3)^2}{16} = 1$
15. $4x^2 - y^2 + 2y - 5 = 0$
16. $2x^2 - 3y^2 - 4x + 12y + 8 = 0$
17. $-16x^2 + 9y^2 - 64x + 90y + 305 = 0$
18. Find an equation of hyperbola whose major axis is parallel to the x-axis, has a focus at $(2,1)$ and its vertices are at the endpoints of a diameter of the circle $x^2 + y^2 - 2y = 0$.
19. A satellite moves along a hyperbolic curve whose horizontal transverse axis is 24 km and an asymptote $y = \frac{5}{12}x + 2$. Then what is the eccentricity of the hyperbola?

20 Two regions A and B are separated by a sea. The shores are roughly in a shape of hyperbolic curves with asymptotes $y = \pm 3x$ and a focus at (30,0) taking a coordinate system with origin at the center of the hyperbola. What is the shortest distance between the regions in kms?

21. Determine the type of curve represented by the equation

$$\frac{x^2}{k} + \frac{y^2}{k-16} = 1$$

In each of the following cases: (a) $k < 0$, (b) $0 < k < 16$, (c) $k > 16$

4.6 The General Second Degree Equation

By the end of this section, you should

- know the general form of second degree equation representing conic sections whose lines of symmetry are not necessarily parallel to the coordinate axes.
- know the rotation formula for rotating the coordinate axes.
- be able to find equivalent equation of a conic section under rotation of the reference axes.
- be able to apply the rotation formula to find a suitable coordinate system in which a given general second degree equation is converted to a simpler standard form.
- be able to convert a given general second degree equation to an equivalent simpler standard form of equation of a conic section.
- be able to identify a conic section that a given general second degree equation represents and sketch the corresponding conic section.

In the previous sections we have seen that, except in degenerate cases, the graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

is a circle, parabola, ellipse or hyperbola. The construction of these equations was based on the assumption that the axis of symmetry of a conic section is parallel to one of the coordinate axes. The assumption seems to be quite restrictive because the axis of symmetry for a parabola, ellipse, or hyperbola can be any oblique line as indicated in their corresponding definitions (See Figures 4.12, 4.18 and 4.24).

However, the reason why we have assumed that is not only for simplicity but there is always a coordinate system whose one of the axes is parallel to a desired line of symmetry. In particular, we can rotate the axes of our *xy*-coordinate system, whenever needed, so as to form a new *x*'*y*'-coordinate system such that either the x'- axis or y'-axis is parallel to the desired line of symmetry. Toward this end, let us review the notion of rotation of axes.

4.6.1 Rotation of Coordinate Axes

A rotation of the x and y coordinate axes by an angle θ about the origin O(0,0) creates a new $x'y'$-coordinate system whose x'-axis is the line obtained by rotating the x-axis by angle θ about O and $\boldsymbol{y'}$-axis is the line obtained by rotating the $\boldsymbol{y}$-axis in the same way. This makes a point P to have two sets of coordinates denoted by (x,y) and (x',y') relative to the xy- and $x'y'$-coordinate axes, respectively.(See Figure 4.30).

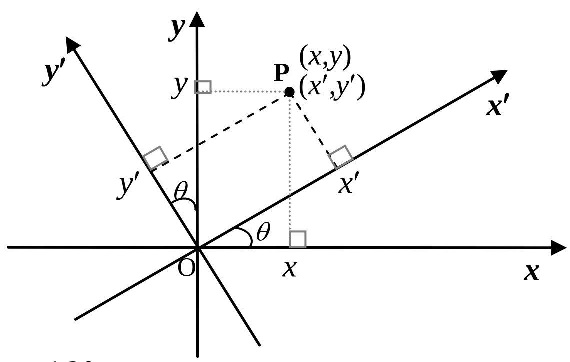


Figure: 4.30

The angle θ considered in the above discussion is called the angle of rotation. Our aim is to find the relationships between the coordinates (x,y) and the coordinates (x',y') of the same point P.

To find this relationships, let P(x,y) be any point in xy-plane, θ be an angle of rotation (i.e., θ is angle between x and x' axes) and ϕ be the angle between OP and $\boldsymbol{x'}$-axis (See Figure 4.31).

So, letting $|OP| = r$ observe that

$$x' = r\cos\phi, \qquad y' = r\sin\phi \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

and

$$x = r\cos(\theta+\phi), \qquad y = r\sin(\theta+\phi) \quad . \; . \; . \; . \quad (2)$$

Then, using the trigonometric identities

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

and (1), the equations in (2) become

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned} \quad . \; . \; . \quad (3)$$

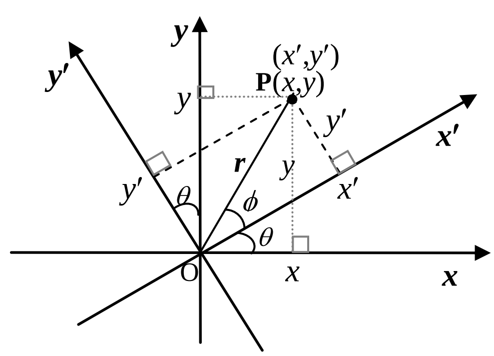


Figure: 4.31

Moreover, these equations can be solved for x' and y' in terms of x and y to obtain

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned} \quad . \; . \; . \quad (4)$$

The Equations (3) and (4) are called **rotation formulas**. It follows that if the angle of rotation θ is given, then Equation (3) can be used to determine the x and y coordinates of a point P if we know its x' and y' coordinates. Similarly, Equation (4) can be used to determine the x' and y' coordinates of P if we know its x and y coordinates.

Example 4.26: Suppose the ***x*** and ***y*** coordinate axes are rotated by π/4 about the origin.

(a) Find the coordinates of P(1, 2) relative to the new ***x***' and ***y***' axes.

(b) Find the equation of the curve $xy = 1$ relative to the new $x'y'$-coordinate system and sketch its graph.

Solution: The given information about P and the curve are relative to the xy-coordinate system and we need to express them in terms of x' and y' coordinates relative to the new $x'y'$-coordinate system obtained under the rotation of the original axes by $\theta = \pi/4$ rad about the origin. Thus, we use $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ in the relevant rotation formula to obtain the following.

(a) Since P(1,2) has the coordinates x=1 and y=2, its x' and y' coordinates are, using formula (4)

$$x' = \frac{\sqrt{2}}{2}(1) + \frac{\sqrt{2}}{2}(2) = \frac{3\sqrt{2}}{2}$$
$$y' = -\frac{\sqrt{2}}{2}(1) + \frac{\sqrt{2}}{2}(2) = \frac{\sqrt{2}}{2}$$

Therefore, the coordinates of P relative to the new ***x***' and ***y***' axes are $\left(\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

(b) We need to express x and y in the equation $xy = 1$ in terms of x' and y' using the rotation formula (3). So, again since $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, we obtain from formula (3):

$$x = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y' \quad \text{and} \quad y = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$$

Therefore, $xy = 1 \quad \Rightarrow \quad \left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) = 1$

$$\Rightarrow \quad \left(\frac{\sqrt{2}}{2}x'\right)^2 - \left(\frac{\sqrt{2}}{2}y'\right)^2 = 1$$

$$\Rightarrow \quad \frac{x'^2}{2} - \frac{y'^2}{2} = 1$$

Note that this is an equation of a hyperbola with center at origin vertices $(-\sqrt{2},0)$ and $(\sqrt{2},0)$ in the $x'y'$-coordinate system with principal axis on ***x***'-axis. Since the ***x*** and **y**- axes were rotated though an angle of π/4 to obtain ***x***' and ***y***'-axes, the hyperbola can be sketched as in Figure 4.32. (You may use Formula (3) to show that the vertices $(-\sqrt{2},0)$ and $(\sqrt{2},0)$ are $(-1,-1)$ and $(1,1)$, respectively, relative to the ***x*** and ***y***-axes).

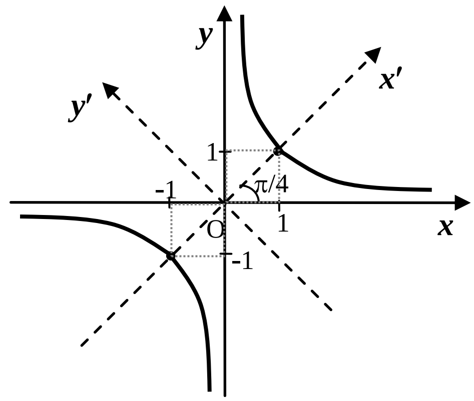


Figure 4.32: $xy = 1$

Example 4.27: Find an equation of the ellipse whose center is the origin, vertices are (–4,–3) and (4, 3), and length of minor axis is 6.

Solution: The position of the ellipse is as shown in Figure 4.33.

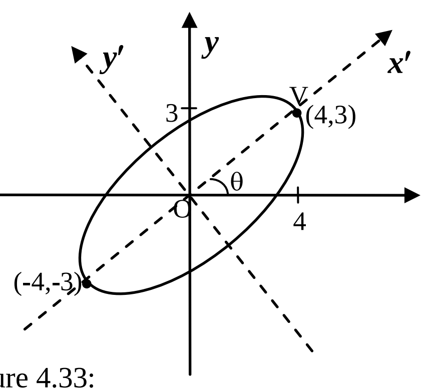


Figure 4.33:

To apply the standard equation of ellipse we use the $x'y'$-coordinate system such that the $\boldsymbol{x'}$-axis coincide with the major axis of the ellipse. Therefore, the equation of the ellipse relative to the $x'y'$ system is

$$\frac{x'^2}{a^2} + \frac{y'^2}{b'^2} = 1.$$

Moreover, from the given information, $a^2 = |OV|^2 = 3^2+4^2 = 25$; and

length of minor axis = $2b = 6 \Rightarrow b = 3$. So, $b^2 = 9$.

Hence, the equation of the ellipse relative to the $x'y'$ -coordinate system is

$$\frac{x'^2}{25} + \frac{y'^2}{9} = 1 \qquad \text{or} \qquad 9x'^2 + 25y'^2 = 225 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (1)$$

Now we use the rotation formula to express the equation relative to our xy-coordinate system. So, let θ the angle between $\boldsymbol{x}$-axis and $\boldsymbol{x'}$-axis. Then, observe that

$$\cos\theta = 4/5 \quad \text{and} \quad \sin\theta = 3/5.$$

Thus, using rotation formula (4) we get:

$$x' = \; x\cos\theta + y\sin\theta = \tfrac{4}{5}x + \tfrac{3}{5}y$$
$$y' = -x\sin\theta + y\cos\theta = \tfrac{-3}{5}x + \tfrac{4}{5}y$$

Now we substitute these for x' and y' in (1) to obtain

$$9\left(\tfrac{4}{5}x + \tfrac{3}{5}y\right)^2 + 25\left(\tfrac{-3}{5}x + \tfrac{4}{5}y\right)^2 = 225$$

And simplifying this we get

$$369x^2 - 384xy + 481y^2 - 5625 = 0$$

which is the equation of the ellipse in the xy- coordinate system.

Exercise 4.6.1

1. Suppose the ***xy***-coordinate axes are rotated 60° counterclockwise about the origin to obtain the new ***x'y'***-coordinate system.
   (a) If each of the following are coordinates of points relative to the xy-system, find the coordinates of the points relative to the ***x′y′***- system.

   (i) (5, 0) (ii) (1, 4) (iii) (0, 1) (iv) (−1/2, 5/2) (v) (−2, −1)

   (b) Find the equation of the following lines and conics relative to the new ***x′y′***- system.

   (i) $x = 5$ (iv) $(x-1)^2 + y^2 = 4$ (vii) $x^2 + 4y^2 - 4x = 0$

   (ii) $x - 2y = 1$ (v) $x^2 - 4y = 1$ (viii) $x^2 - 4y^2 = 1$

   (iii) $x^2 + y^2 = 1$ (vi) $4x^2 + (y-2)^2 = 4$ (ix) $-x^2 + y^2 - 2y = 0$

2 Suppose the ***xy***-coordinate axes are rotated 30° counterclockwise about the origin to obtain the new ***x'y'***-coordinate system. If the following points are with respect to the new ***x'y'***-system, what is the coordinates of each point with respect to the old ***xy***-system?

(a) (0, 2) (b) (−2, 4) (c) (1, −3) (d) $\left(\sqrt{3}, -\sqrt{3}\right)$

4.6.2 Analysis of the General Second Degree Equations

In the previous sections we have seen that the equation of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0, \quad . . . . \quad (1)$$

represents a conic section (a parabola, ellipse or hyperbola) whose axis of symmetry is parallel to one of the coordinate axes except in degenerate cases. In Subsection 4.6.1 we have also seen some examples of conic sections whose equations involve xy term when their lines of symmetry are not parallel to either of the axes. Now we would like to analyze the graph of any quadratic (second degree) equation in x and y of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad . . . . \quad (2)$$

where $B\neq0$. In order to analyze the graph of Equation (2), we usually need to convert it into an equation of type (1) in certain suitable reference system. To this end, we first prove the following Theorem.

Theorem 4.3: Consider a general second degree equation of the form (2), i..e.,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad \text{where } B\neq0, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

there is a rotation angle $\theta \in(0,\pi/2)$ through which the xy-coordinate system rotates to a new $x'y'$-coordinate system in which Equation (2) reduces to the form

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

Proof: Let the xy-coordinate system rotated by an angle θ about the origin to form a new $x'y'$-coordinate system. Then, from rotation formula (3), we have

$$x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta .$$

We can now substitute these for x and y in Equation (2) so that

$$A(x'\cos\theta - y'\sin\theta)^2 + B(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + C(x'\sin\theta + y'\cos\theta)^2$$
$$+ D(x'\cos\theta - y'\sin\theta) + E(x'\sin\theta + y'\cos\theta) + F = 0.$$

After some calculations, combining like terms (those involving x'^2, $x'y'$, y'^2, and so on), we get equation of the form

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

where $B' = 2(C–A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta)$.

Here the exact expressions for A', C', D', E' and F' are omitted as they are irrelevant. What we need is to get the angle of rotation θ for which Equation (4) has **no** $x'y'$ term, that is, $B' = 0$. This means that,

$$2(C–A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) = 0 .$$

Since $2\sin\theta\cos\theta = \sin2\theta$ and $\cos^2\theta - \sin^2\theta = \cos2\theta$, this equation is equivalent to

$$(C–A)\sin2\theta + B\cos2\theta = 0$$

or $\dfrac{\cos2\theta}{\sin2\theta} = \dfrac{A-C}{B}$, since $B\neq0$.

or $\cot2\theta = \dfrac{A-C}{B}$ . . . . (5).

That is, if we choose the angle of rotation θ satisfying (5), then $B' = 0$ in Equation (4) so that the resulting equation in $x'y'$-coordinate system is in the form of Equation (3). Moreover, we can always find an angle that satisfies $\cot(2\theta) = (A–C)/B$ for any $A, C, B \in \Re$, $B\neq0$ since the range of the cotangent function is the entire set of real numbers. Note also that since $2\theta \in(0, \pi)$, the angel of rotation θ can always be chosen so that $0 < \theta < \pi/2$. So, the Theorem is proved.

Remark: If A = C, then $\cot 2\theta = \frac{A-C}{B} = 0 \Rightarrow 2\theta = \pi/2 \Rightarrow \theta = \pi/4$.

Therefore, we can rewrite the result of the above Theorem as follows:

The rotation of the *xy*-coordinate system by angle θ creates an $x'y'$-coordinate system in which a general second degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, $B \neq 0$,

is converted to an equation $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$ (with no $x'y'$ term)

if we choose $\theta \in (0, \pi/2)$ such that

$$\tan 2\theta = \frac{B}{A - C}, \quad \text{if } A \neq C$$
$$\theta = \frac{\pi}{4}, \quad \text{if } A = C$$

Example 4.28: Use rotation of axes to eliminate the *xy* term in each of the following equations, describe the locus (type of conic section) and sketch the graph of the equation

a) $x^2 + 2xy + y^2 - 8\sqrt{2}x + 8\sqrt{2}y - 32 = 0$

(b) $73x^2 - 72xy + 52y^2 + 30x + 40y - 75 = 0$

Solution:

(a) Given: $x^2 + 2xy + y^2 - 8\sqrt{2}x + 8\sqrt{2}y - 32 = 0 \Rightarrow$ A= C=1. So, from the above Remark, the rotation angle is $\theta = \pi/4 \Rightarrow \cos\theta = \sin\theta = \frac{1}{\sqrt{2}}, \Rightarrow x = \frac{x'-y'}{\sqrt{2}}$ and $x = \frac{x'+y'}{\sqrt{2}}$.

Now we substitute these for *x* and *y* in the given equation:

$$\left(\frac{x'-y'}{\sqrt{2}}\right)^2 + 2\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) + \left(\frac{x'+y'}{\sqrt{2}}\right)^2 - 8\sqrt{2}\left(\frac{x'-y'}{\sqrt{2}}\right) + 8\sqrt{2}\left(\frac{x'+y'}{\sqrt{2}}\right) - 32 = 0$$

Expanding the squared expressions, combining like terms and simplifying, we obtain

$$x'^2 + 8y' - 16 = 0 \quad \text{or,} \quad x'^2 = -8(y' - 2)$$

This is an equation of a parabola. Its vertex is $(h', k') = (0,2)$ relative to the $x'y'$-system, principal axis is on y'-axis and open towards negative y' direction. (You can show that its vertex is (h, k) = $\left(-\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$ relative to the *xy*-system). The graph of the equation is sketched in Figure 4.34.

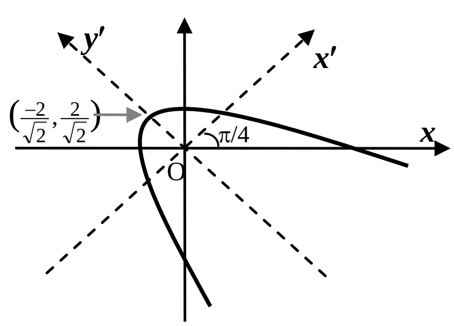


Figure 4.34: $x^2 + 2xy + y^2 - 8\sqrt{2}x + 8\sqrt{2}y - 32 = 0$

(b) Given: $73x^2 - 72xy + 52y^2 + 30x + 40y - 75 = 0 \quad \Rightarrow \quad A = 73,\ B = -72$ and $C = 52$.
Hence,

$\tan 2\theta = \frac{B}{A-C} = -\frac{72}{21} = -\frac{24}{7} \Rightarrow$ The terminal side of 2θ is through $(-7, 24)$ since $0 < 2\theta < \pi$.

$\Rightarrow \cos 2\theta = \frac{-7}{25}$. Now as $0 < \theta < \pi/2$, both $\cos\theta$ and $\sin\theta$ are positive. Hence,

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\frac{1-7/25}{2}} = \frac{3}{5} \quad \text{and} \quad \sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1+7/25}{2}} = \frac{4}{5}$$

This implies the x'-axis is through the coordinate point (3,4), that is the line $y = (4/3)x$.
Therefore, using the rotation formula (3), we get

$$x = \frac{3x' - 4y'}{5} \quad \text{and} \quad y = \frac{4x' + 3y'}{5}$$

Now we substitute these for x and y in the given equation to obtain

$$\tfrac{73}{25}(3x' - 4y)^2 - \tfrac{72}{25}(3x' - 4y')(4x' + 3y') + \tfrac{52}{25}(4x' + 3y')^2 + \tfrac{30}{5}(3x' - 4y') + \tfrac{40}{5}(4x' + 3y') - 75 = 0.$$

Expanding the squared expressions, combining like terms and simplifying, we obtain

$$25x'^2 + 100y'^2 + 50x' - 75 = 0$$

Completing the square for x' terms and divide by 100 to get

$$\frac{(x'+1)^2}{4} + y'^2 = 1$$

which is an ellipse with center at $(h', k') = (-1, 0)$ relative to the $x'y'$-system, major axis on x'-axis (which is the line $y=(4/3)x$), length of major axis =4 and length of minor axis =2. (You can show that the center is $(h, k) = \left(-\frac{3}{5}, -\frac{4}{5}\right)$ relative to the xy-system). The graph of the equation is sketched in Figure 4.35.

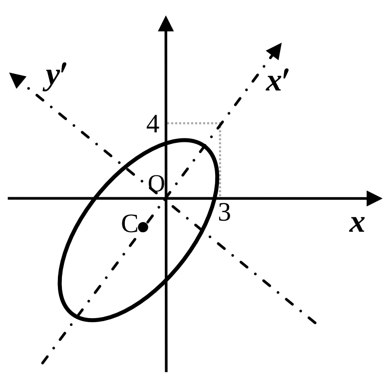


Figure 4.35: $73x^2 - 72xy + 52y^2 + 30x + 40y - 75 = 0$

Exercise 4.6.2

1. Find an equation of the conic section having the given properties and sketch its graph.
   (a) Ellipsi with center at origin, foci at (–2,2) and (2, 2), and length of major axis $2\sqrt{8}$.
   (b) Parabola whose vertex is at (3, 4) and focus (–5,–2)
   (c) Hyperbola whose foci are (–2, 2) and (2, –2), and length of transverse axis $2\sqrt{2}$.
2. Use rotation of axes to eliminate the *xy* term in each of the following equations, describe the locus (type of conic section) and sketch the graph of the equation.
   (a) $17x^2 - 12xy + 8y^2 - 36 = 0$
   (b) $8x^2 + 24xy + y^2 - 1 = 0$
   (c) $x^2 - 2xy + y^2 - 5y = 0$
   (d) $2x^2 + xy = 0$
   (e) $5x^2 + 6xy + 5y^2 - 4x + 4y - 4 = 0$
   (f) $x^2 + 4xy + 4y^2 + 2x - 2y + 1 = 0$
3. Show that if $B > 0$, then the graph of

$$x^2 + Bxy = F \ ,$$

is a hyperbola if $F \neq 0$, and two intersecting lines if $F = 0$.