TEXT BOOK OF TRANSFORMATION GEOMETRY

General Equations of Reflection

Let S_L be reflection on a line $L: ax + by + c = 0$.

Then, $S_L(x, y) = (x', y')$ where

$$\begin{cases} x' = x - \dfrac{2a(ax+by+c)}{a^2+b^2} \\ y' = y - \dfrac{2b(ax+by+c)}{a^2+b^2} \end{cases}$$

General Equations of Rotation

Let $R_{C,\theta}$ be a CCW rotation through an angle of θ

With center $C = (a, b)$ .Then, $R_{C,\theta}(x, y) = (x', y')$

Where $\begin{cases} x' = (x - a)cos\theta - (y - b)sin\theta + a \\ y' = (x - a)sin\theta + (y - b)cos\theta + b \end{cases}$

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TRANSFORMATIONS

1.1 Revision on Mappings

Definition: Let X and Y be nonempty sets. Then, a *mapping* f from X to Y is a rule which assigns to every element x in X exactly one (unique) value $f(x)$ in Y, here, $f(x)$ is called the *image* of x under f. The set X is said to be the *domain of* f and Y is the co-domain of f. The set of all images of f is called *range* of f. In this definition of mappings, the word unique (exactly one) refers to the idea of *well definedness*. A rule which assigns to every element in the domain (in X) some value in the co domain (in Y) is said to be a mapping if it is well defined.

To show well-defined ness, it suffices to show that $f(x) = y,\ f(x) = z \Rightarrow y = z$.

Notation: The mapping f from X to Y is denoted symbolically by $f : X \to Y$.

Examples

1. Let $g : R^2 \to R^2$ be given by $g(x, y) = (2x, 3y)$. Show that g is a mapping.

Solution: Clearly g is a rule which assigns to each value in R^2 a value in R^2. Now, let's show that g is *well-defined*. Suppose $g(x, y) = (a,b) \wedge g(x, y) = (c,d)$

$$\begin{aligned} g(x, y) = (a,b) \wedge g(x, y) = (c,d) &\Rightarrow (2x,3y) = (a,b) \wedge (2x,3y) = (c,d) \\ &\Rightarrow 2x = a, 3y = b \wedge 2x = c, 3y = d \\ &\Rightarrow a = c \wedge b = d \Rightarrow (a,b) = (c,d) \end{aligned}$$

This implies that the image of any point (x, y) in R^2 is unique and hence g is well defined and it is a mapping.

2*. Let Z be set of integers. Consider the set $S=\{x\in Z:|x-1|\le 2\}$. Define $h:Z\to S$ by $h(x)=x^2$. Is h a mapping or not?

Solution: Here, $S=\{x\in Z:|x-1|\le 2\}=\{x\in Z:-2\le x-1\le 2\}$

$$=\{x\in Z:-1\le x\le 3\}=\{-1,0,1,2,3\}$$

As we see, $x=-1$ is in S. But there is no integer in the domain such that $h(x)=-1$. This means $x=-1$ has no pre-image. Hence, h is not a mapping.

1.2 Types of Mappings

Definitions:

a) One-to-one (Injective) mapping: A mapping $f:X\to Y$ is said to be a *one-to-one (injective)* mapping if and only if f sends distinct elements of X in to distinct elements of Y. This means $x\ne y\Rightarrow f(x)\ne f(y)$. In other words, f is one to one if and only if $f(x)=f(y)\Rightarrow x=y$.

b) Onto (Surjective) mapping: A mapping $f:X\to Y$ is said to be *onto* mapping if and only if for every point y in Y, there exists an element x in X such that

$y=f(x)$. Or if the image of f is the whole of Y. That is every element of Y has at least one pre-image in X.

c) Bijective mapping: A mapping is said to be bijective if and only if it is both one to one and onto mapping.

Examples:

1. Verify that the following mappings are *one to one* but not *onto.*

$a)$ $f:N\to N$ given by $f(x)=2x$

$b)$ $g:Z\to ZxZ$ given by $g(n)=(n,0)$

$c)$ $h:NxN\to N$ given by $h(m,n)=2^m.3^n$

$d)$ $f:M_{2x2}\to M_{2x2}$ given by $f(A)=A^{-1}$

Solution:

a) i) **One to one:** Assume $f(x) = f(y)$ for any two numbers x, y in N. Then,

$f(x) = f(y) \Rightarrow 2x = 2y \Rightarrow x = y$. So, f is one-to-one.

ii) **Onto:** Let x be in the co-domain of f. Particularly, $y = 3$ (you can select any other odd natural number as well). Then, if $\exists x \in N$ in the domain of f, such that $f(x) = y$, then f is on to. But, $f(x) = 3 \Rightarrow 2x = 3 \Rightarrow x = \frac{3}{2}$. The number which maps to 3 under the given function is $\frac{3}{2}$. But $\frac{3}{2}$ is not member of the domain (natural number). So, the map is not onto.

b) i) **One to one:** Assume $g(m) = g(n)$ for any two integers m, n in Z. Then,

$g(m) = g(n) \Rightarrow (m,0) = (n,0) \Rightarrow m = n$ and thus $m = n \Rightarrow (m,0) = (n,0)$.

So, f is one-to-one.

ii) **Onto:** For each $(m,n), n \neq 0$ in the co-domain of g, we can not find a preimage in Z, such that $g(m) = (m,n)$ because the second coordinates of all image members is zero, or of the form $(m,0)$ .So, the map is not onto.

c) i) **One to one:** Here, $h(m,n) = h(a,b) \Rightarrow 2^m.3^n = 2^a.3^b \Rightarrow 2^{m-a} = 3^{b-n}$.

Since the basis are different, we have

$2^{m-a} = 3^{b-n} \Rightarrow m - a = 0, b - n = 0 \Rightarrow m = a, b = n \Rightarrow (m,n) = (a,b)$.

So, f is one-to-one.

ii) **Onto:** For each y in the co-domain of h, we can not find a pre-image in $N \times N$, such that $h(m,n) = y$. For instance if we take any natural number which is not a common multiple of 2 and 3, like 1,5,7 and so on we cannot find a pre-image. So, the map is not onto.

d) i) **One to one:** Using property of inverse,

$f(A) = f(B) \Rightarrow A^{-1} = B^{-1} \Rightarrow (A^{-1})^{-1} = (B^{-1})^{-1} \Rightarrow A = B$. So, f is one-to-one.

ii) **Onto:** For singular matrix, we cannot find a pre-image in the co-domain because only non-singular matrices have inverses. So, the map is not onto.

2*. Let Z be set of integers. Consider the set $T = \{x \in Z : |x+2| < 3\}$. Define $h: T \to Z$ by $h(x) = x^2$. Is h a mapping? If it is a mapping, is it one to one?

Solution: Here, $T = \{x \in Z : |x+2| < 3\} = \{x \in Z : -3 < x+2 < 3\}$
$= \{x \in Z : -5 < x < 1\} = \{-4, -3, -2, -1, 0\}$

As we see, $h(x) = x^2 \Rightarrow h(-4) = 16, h(-3) = 9, h(-2) = 4, h(-1) = 1, h(0) = 0$.

This means for every element in T, we can get an image in Z.

Besides, every element in T, maps into a unique element in Z. Therefore, h is a one to one mapping.

3. Verify that the following mappings are *onto* but not *one to one.*

a) $f: R \to R^+$ given by $f(x) = |x|$ b) $f: ZxZ \to Z$ given by $f(x, y) = x + y$

c) $f: M_{2x2} \to R$ given by $f(A) = \det(A)$ d) $h: R \to R^+$ given by $h(x) = e^{x^2}$

Solution:

a) Since $f(x) = f(-x)$, the map is not one to one.

For instance, $f(2) = f(-2) = 2$. But for every, positive real number x, $f(x) = |x| = x$, That means there exist at least itself such that $f(x) = x, \forall x \in R^+$.

Thus, the map is onto.

b) Since $f(2x, -x) = x, f(x, 0) = x,$ and so on the map is not one to one.
For instance, $f(4, -2) = f(2, 0) = 2$ but $(4, -2) \neq (2, 0)$. But for every, integer $x, \exists (x, 0), \ni f(x, 0) = x + 0 = x$. That means the map is onto.

c) For $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, B = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix}, f(A) = f(B) = \det A = \det B = 1$ but $A \neq B$.

So. The map is not one to one. But for every, real number $x, \exists\ A = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \ni f(A) = \det A = x$. That means the map is onto.

d) For $x = 3,\ y = -3, h(3) = h(-3) = e^9$ but $3 \neq -3$. So. The map is not one to one.
Now for every positive real number y if $\exists x$ such that $h(x) = y$ the map will be onto. However, $h(x) = y \Rightarrow e^{x^2} = y \Rightarrow x^2 = \ln y \Rightarrow x = \pm\sqrt{\ln y}$.

So, $\forall y \in R^+, \exists x = \pm\sqrt{\ln y}, \ni h(x) = e^{(\pm\sqrt{\ln y})^2} = e^{\ln y} = y$. That means the map is onto.

3. Verify that the following mappings are bijective.

$a)\ f: R^2 \to R^2$ given by $f(x,y)=(2x, y-1)$ $\quad b)\ f: R^2 \to R^2,\ f(x,y)=(\sqrt[3]{x-1}, y^3+3)$

$c)\ f: R^2 \to R^2$ given by $f(x,y)=(x+y, x-y)$ $\quad d)\ f: R \to R^+$ given by $f(x)=e^x$

Solution:

a) Assume $f(x,y)=f(z,w)$ for any two points (x,y) and (z,w) in R^2. Then, $(2x, y-1)=(2z, w-1)$. But from equality of order pairs, this equality is true if and only if $\begin{cases} 2x=2z \\ y-1=w-1 \end{cases} \Rightarrow x=z,\ y=w \Rightarrow (x,y)=(z,w)$. So, f is one-to-one.

ii) Let $(a,b)\in R^2$ be in the co-domain of f .Then, if $\exists (x,y)\in R^2$ in the domain of f , such that $f(x,y)=(a,b)$, then f is on to.

But, $f(x,y)=(2x,y-1)=(a,b)\Rightarrow 2x=a,\ y-1=b \Rightarrow x=\frac{a}{2},\ y=b+1$.Thus, we can find

$(x,y)=(\frac{a}{2}, b+1)\in R^2$ such that $f(x,y)=f(\frac{a}{2}, b+1)=(a,b),\ \forall (a,b)\in R^2$.

So f is on to. Therefore, the given map is bijective.

b) Assume $f(x,y)=f(z,w)$ for any two points (x,y) and (z,w) in R^2. Then,

$f(x,y)=f(z,w)\Rightarrow (\sqrt[3]{x-1}, y^3-1)=(\sqrt[3]{z-1}, w^3-1)$. But from equality of order pairs, this equality is true if and only if

$\begin{cases} \sqrt[3]{x-1}=\sqrt[3]{z-1} \\ y^3-1=w^3-1 \end{cases} \Rightarrow x-1=z-1,\ y^3=w^3 \Rightarrow x=z, y=w \Rightarrow (x,y)=(z,w)$. So, f is one-to-one.

ii) Let $(a,b)\in R^2$ be in the co-domain of f .Then, if $\exists (x,y)\in R^2$ in the domain of f , such that $f(x,y)=(a,b)$, then f is on to.

But, $f(x,y)=(\sqrt[3]{x-1}, y^3-1)=(a,b)\Rightarrow \sqrt[3]{x-1}=a,\ y^3-1=b \Rightarrow x=a^3+1,\ y=\sqrt[3]{b+1}$.

Since both the expressions $x=a^3+1,\ y=\sqrt[3]{b+1}$ are always defined, we can find $(x,y)=(a^3+1, \sqrt[3]{b+1})\in R^2$ such that

$f(x,y)=f(a^3+1, \sqrt[3]{b+1})=(a,b),\ \forall (a,b)\in R^2$. So f is on to. Therefore, the given map is bijective.

c) Assume $f(x,y)=f(z,w)$ for any two points (x,y) and (z,w) in R^2.

Then, $f(x,y)=f(z,w)\Rightarrow(x+y,x-y)=(z+w,z-w)$. But from equality of order pairs, this equality is true if and only if $\begin{cases}x+y=z+w\\x-y=z-w\end{cases}\Rightarrow 2x=2z\Rightarrow x=z$.

Again using $x=z$ in $x+y=z+w$, we get $y=w$.

Hence, $f(x,y)=f(z,w)\Rightarrow(x,y)=(z,w)$.

So, f is one-to-one.

ii) Let $(a,b)\in R^2$ be in the co-domain of f .Then, if $\exists(x,y)\in R^2$ in the domain of f , such that $f(x,y)=(a,b)$, then f is on to.

But, $f(x,y)=(x+y,x-y)=(a,b)\Rightarrow x+y=a,\ \ x-y=b\Rightarrow x=\frac{a+b}{2},\ y=\frac{a-b}{2}$.

This means we can find $(x,y)=(\frac{a+b}{2},\frac{a-b}{2})\in R^2$ such that

$f(x,y)=f(\frac{a+b}{2},\frac{a-b}{2})=(a,b),\ \forall(a,b)\in R^2$. So f is on to.

Therefore, the given map is bijective.

d) i) Assume $f(x)=f(y)$ for any twonumbers x and y in R.

Then, $f(x)=f(y)\Rightarrow e^x=e^y\Rightarrow x=y$. Hence, $f(x)=f(y)\Rightarrow x=y$.

So, f is one-to-one.

ii) Now for every positive real number y if $\exists x$ such that $h(x)=y$ the map will be onto. However, $f(x)=y\Rightarrow e^x=y\Rightarrow x=\ln y$.

So, $\forall y\in R^+,\exists x=\ln y,\ni f(x)=e^{\ln y}=y$. That means the map is onto.

Therefore, the given map is bijective.

Transformation Mappings :

Definition: *Transformation* is a one-to-one mapping from a set X onto itself. In other words, the map $f: X \to X$ is said to be a transformation if and only if it is one to one and onto. This means that for every point P in the domain there is a unique point Q such that $f(P) = Q$ and conversely, for every point R in the range there is a unique point S in the domain such that $f(S) = R$.

Examples

1. Let $f: R \to R$ be given by $f(x) = ax,\ a \neq 0,\ a \in R$. Show that f is a transformation.

Solution: To show that $f(x) = ax$ is a transformation, we need to show that it is one to one and onto.

i) Assume $f(x) = f(y)$. Then, $ax = ay \Rightarrow x = y$, since $a \neq 0$. So, f is one-to-one

ii) Let $y \in R$ be in the range of f. Then, if $\exists x \in R$ in the domain of f, such that $f(x) = y$, then f is on to. But for $a \neq 0$, we can find $x = y/a \in R$ such that $f(x) = f(y/a) = y$. So f is on to. Hence, f is a transformation.

2. Verify that the following mappings are transformations.

$a)$ $g: R^2 \to R^2$ given by $g(x, y) = (x + y + 1, x - y - 1)$

$b)$ $f: R^2 \to R^2$ given by $f(x, y) = (x, x - y)$

Solution:

a) i) Assume $g(x, y) = g(z, w)$ for any two points (x, y) and (z, w) in R^2. Then,

$(x + y + 1, x - y - 1) = (z + w + 1, z - w - 1)$.

But from equality of order pairs, this equality is true if and only if

$\begin{cases} x + y + 1 = z + w + 1 \\ x - y - 1 = z - w - 1 \end{cases} \Rightarrow 2x = 2z \Rightarrow x = z,\ y = w$. This gives $(x, y) = (z, w)$ So, g is one-to-one.

ii) Let $(a, b) \in R^2$ be in the co-domain of g. Then, if $\exists (x, y) \in R^2$ in the domain of g, such that $g(x, y) = (a, b)$, then g is on to.

But, $g(x,y)=(x+y+1,x-y-1)=(a,b)$

$$\Rightarrow \begin{cases} x+y+1=a \\ x-y-1=b \end{cases} \Rightarrow 2x=a+b \Rightarrow x=\frac{a+b}{2},\ y=\frac{a-b-2}{2}$$

Thus, we can find $(x,y)=(\frac{a+b}{2},\frac{a-b-2}{2})\in R^2$ such that $g(x,y)=(a,b)$. So g is on to.

Therefore, the given map g is a transformation.

b) Assume $f(x,y)=f(z,w)$ for any two points (x,y) and (z,w) in R^2.

Then, $f(x,y)=f(z,w)\Rightarrow(x,x-y)=(z,z-w)$.

But from equality of order pairs, this equality is true if and only if

$$\begin{cases} x=z \\ x-y=z-w \end{cases} \Rightarrow x=z,\, y=w \Rightarrow (x,y)=(z,w).$$

So, f is one-to-one.

ii) Let $(a,b)\in R^2$ be in the co-domain of f .Then, if $\exists(x,y)\in R^2$ in the domain of f , such that $f(x,y)=(a,b)$, then f is on to. But,

$f(x,y)=(x,x-y)=(a,b)\Rightarrow x=a,\ \ x-y=b\Rightarrow y=a-b$.

This means we can find $(x,y)=(a,a-b)\in R^2$ such that

$f(x,y)=f(a,a-b)=(a,b),\ \forall(a,b)\in R^2$. So f is on to.

Therefore, the given map is a transformation.

Equality of Transformations: Two transformations f and g on the same set from X to X are said to be equal if and only if they have the same value for each x in X. That is, $f=g \Leftrightarrow f(x)=g(x),\ \forall x\in X$.

Examples:

1. Let f and g be transformations on R^2 given by $f(x,y)=(2ax^5-3,4y)$ and $g(x,y)=(6x^5+2b,4y)$. If $f=g$, find the constants a and b .

Solution: By definition of equality,

$$\begin{aligned} f=g &\Leftrightarrow f(x,y)=g(x,y),\forall(x,y)\in R^2 \\ &\Leftrightarrow (2ax^5-3,4y)=(6x^5+2b,4y) \\ &\Leftrightarrow 2ax^5-3=6x^5+2b \\ &\Leftrightarrow 2a=6,-3=2b \Leftrightarrow a=3,b=-3/2 \end{aligned}$$

2. Let f and g be transformations on the whole R^2 such that $f(1,3) = (3,-10)$ and $g(x,y) = (a+x, 2-by)$. If $f = g$, find the constants a and b.

Solution: By definition of equality, $f = g \Leftrightarrow f(x,y) = g(x,y), \forall (x,y) \in R^2$.

$$\text{Partticularly}, g(1,3) = f(1,3) \Rightarrow (a+1, 2-3b) = (3,-10)$$
$$\Rightarrow a+1 = 3, 2-3b = -10 \Rightarrow a = 2, b = -4$$

1.3 Composition of Transformations and Their Properties

Let $f : X \to Y$ and $g : Y \to Z$ be mappings. Then for each $x \in X$, $f(x) \in Y$. Thus, there exists $y \in Y$ such that $f(x) = y$. Besides, as g is a mapping from Y to Z for each $y \in Y$, there exists $z \in Z$ such that $g(y) = z$. Thus, $g(y) = g(f(x)) = z$ which makes sense to write $g(f(x))$ as a mapping from X to Z. This mapping is evaluated by applying f first on the elements of X followed by g. This is defined as $g \circ f(x) = g(f(x))$ for each $x \in X$. So, the mapping $gof\,(\text{"}f \text{ follwed by g"})$ is called the composition mapping. In such cases, one has to remember that the range of the first mapping is a subset of the domain of the second mapping.

In particular, composition of transformation is defined as $(g \circ f)(x) = g(f(x))$ where f and g are transformations on the same set X.

Proposition 1.1: Composition of mappings is associative

If $h : X \to Y, g : Y \to Z$ and $f : Z \to W$ are mappings, then the compositions $(f \circ g) \circ h$ and $f \circ (g \circ h)$ represent the same mapping from X in to W. That is $(f \circ g) \circ h = f \circ (g \circ h)$. Particularly, $(f \circ g) \circ h = f \circ (g \circ h)$ holds if f, g and h are transformations on the same set X.

Proof: Since the domain of h is X, by definition of composition of mappings we can see that the domain of $(f \circ g) \circ h$ is also X. But, the domain of $f \circ (g \circ h)$ is the same as the domain of $g \circ h$ and the domain of $g \circ h$ is X. Hence, the domain of $(f \circ g) \circ h$ is the same as that of $f \circ (g \circ h)$, that is, X.

So, $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are mappings with the same domain.

Furthermore, for any $x \in X$,

$$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x))), \forall x \in X$$
$$\text{Also}, (f \circ (g \circ h))(x) = f((g \circ h)(x)) = f(g(h(x))), \forall x \in X$$

Hence, by definition of equality of mappings $(f \circ g) \circ h = f \circ (g \circ h)$.

Note: Composition of two transformations need not be commutative. That is even though both $f \circ g$ and $g \circ f$ exists and have the same domain and co-domain, $f \circ g$ and $g \circ f$ may not be equal.

Examples:

1. Let $f : R \to R, g : R \to R$ be defined by $f(x) = x - 3$ and $g(x) = 2x + 5$.

Solution: Here, $(f \circ g)(x) = f(g(x)) = f(2x+5) = 2x+2$ and

$$(g \circ f)(x) = g(f(x)) = g(x-3) = 2x - 1.$$

Thus, $2x + 2 \neq 2x - 1 \Rightarrow f \circ g \neq g \circ f$.

Hence, composition of mappings is not *commutative*.

2. Let $f : R \to R, g : R \to R$ be defined by $f(x) = 3x + 8, g(x) = 2x + k$. Find the value of the constant k such that $g \circ f = f \circ g$.

Solution: Here, by definition of equality of transformations,

$$g \circ f = f \circ g \Rightarrow (g \circ f)(x) = (f \circ g)(x), \forall x \in R$$
$$\Rightarrow 6x + 16 + k = 6x + 3k + 8$$
$$\Rightarrow 2k = 8 \Rightarrow k = 4$$

3. Let $f(x, y) = (ax + 8, 3y - 5)$ and $g(x, y) = (7x, 4y + b)$ be the transformations. If $(g \circ f)(x, y) = (14x + 8, 12y + 23 - b)$, then, find the constants a and b .

Solution:

Here, $(f \circ g)(x, y) = (14x + 8, 12y + 23 - b)$

$$\Rightarrow f(g(x, y)) = (14x + 8, 12y + 23 - b)$$
$$\Rightarrow f(7x, 4y + b) = (14x + 8, 12y + 23 - b)$$
$$\Rightarrow (7ax + 8, 12y + 3b - 5) = (14x + 8, 12y + 23 - b)$$
$$\Rightarrow 7ax + 8 = 14x + 8,\ 12y + 3b - 5 = 12y + 23 - b$$
$$\Rightarrow 7a = 14,\ 3b - 5 = 23 - b$$
$$\Rightarrow a = 2,\ b = 7$$

Proposition 1.2: The composition of transformations on the same set are again transformations.

Proof: Let $f : X \to X, g : X \to X$ be any two transformations on set X. We need to show that $f \circ g$ is also a transformation. That means we need to verify that $f \circ g$ is one to one and onto. By definition of composition $f \circ g$ is a mapping from X into X. To show $f \circ g$ is one to one, let x and y be arbitrary elements of X such that $f \circ g(x) = f \circ g(y)$.

Then, $f \circ g(x) = f \circ g(y) \Rightarrow f(g(x)) = f(g(y)) \Rightarrow g(x) = g(y) \Rightarrow x = y$ (because both f and g are one to one). Thus, $f \circ g$ is one to one.

To show $f \circ g$ is onto, let t be any element in X (considering X as co-domain of f), since f is onto there exists an element y in X (in the domain) such that $f(y) = t$.

Again, g is onto corresponding to the element y in X there is an element x in X suchthat $g(x) = y$. As a result, $(f \circ g)(x) = f(g(x)) = f(y) = t$.

Thus, $f \circ g$ is onto. Hence, we have got that $f \circ g$ is one to one and onto on the set X. Therefore, $f \circ g$ is a transformation whenever f and g are transformations on X.

1.4 Identity and Inverse Transformations

Definition: A transformation from a set X into X denoted by i is said to be *identity transformation* if and only if $i(x) = x, \forall x \in X$.

Any two transformations f and g from X to X are said to be *inverse of each other* if both $g \circ f$ and $f \circ g$ are identity transformations.

That is $(g \circ f)(x) = (f \circ g)(x) = i(x) = x, \ \forall x \in X$, then f is called the inverse of g and g is called the inverse of f. We denote the inverse of a transformation f by f^{-1}(Read as "the inverse of f" or $f-$inverse).

Example: Verify that $f(x) = 5x - 20$ and $g(x) = \frac{1}{5}x + 4$ are inverse of each other.

Solution: Using the definition, we have

$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{5}x + 4\right) = 5\left(\frac{1}{5}x + 4\right) - 20 = x + 20 - 20 = x$ and

$(g \circ f)(x) = g(f(x)) = g(5x - 20) = \frac{1}{5}(5x - 20) + 4 = x - 4 + 4 = x$

Thus, $(g \circ f)(x) = (f \circ g)(x) = i(x) = x, \ \forall x \in X$. Hence, $f^{-1} = g$ and $g^{-1} = f$.

Remark: Any mappings which have inverse are called *invertible* mappings. Only these mappings which are bijective have inverses. That means a mapping is *invertible* if and only if it is both one to one and onto. Thus, any transformation has an inverse because transformations are one to one and onto mappings on the same set.

Finding Inverse of a transformation: Now let's see how can we find the inverse of transformations. Since every transformation f is bijective, its inverse denoted by f^{-1} always exists. But there is no hard and fast rule on how to find f^{-1} from the formula of f. Any way, one can use the following hints on how to find f^{-1} whenever the formula of f is given. Let $f: S \to S$ be a transformation such that $Y = f(X)$. Then, to find f^{-1} :

Step-1: Interchange X and Y in the formula of f

Step-2: Solve for Y (for coordinates of Y) in terms of X (coordinates of X).

Step-3: Equate $f^{-1}(X) = Y$ from $Y = f(X)$. That will be the formula of f^{-1}.

Examples:

1. Find the inverses of the following transformations

$a)\ f: R \to R$ given by $f(x) = \sqrt[3]{x} - 1$ $\qquad b)\ f: R \to R$ given by $f(x) = \dfrac{3e^x - 2}{e^x + 4}$

$c)\ g: R^2 \to R^2,\ g(x, y) = (2x - 1, y + 5)$ $\qquad d)\ f: R \to R,\ f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$

Solution:

a) $y = f(x) = \sqrt[3]{x} - 1$

Step-1: Interchange x and y .That is $x = \sqrt[3]{y} - 1$

Step-2: Solve for y in terms of x .That is

$\sqrt[3]{y} - 1 = x \Rightarrow \sqrt[3]{y} = x + 1 \Rightarrow y = (x+1)^3 = x^3 + 3x^2 + 3x + 1$

Step 3: Equate the value of y obtained in step 2 with $f^{-1}(x)$.

That is $f^{-1}(x) = x^3 + 3x^2 + 3x + 1$.

b) **Step-1:** Interchange x and y .

That is $y = f(x) = \dfrac{3e^x - 2}{e^x + 4} \Rightarrow x = \dfrac{3e^y - 2}{e^y + 4}$

Step-2: Solve for y in terms of x .That is

$$x = \frac{3e^y - 2}{e^y + 4} \Rightarrow 3e^y - 2 = x(e^y + 4) \Rightarrow e^y(3 - x) = 4x + 2 \Rightarrow e^y = \frac{4x+2}{3-x} \Rightarrow y = \ln\left(\frac{4x+2}{3-x}\right)$$

Step-3: Equate the value of y in step-2 with $f^{-1}(x)$.

That is $f^{-1}(x) = \ln\left(\dfrac{4x+2}{3-x}\right)$

c) For $g(x,y)=(2x-1,y+5)$,let $Y=g(X)$ where $X=(x,y),\ Y=(z,w)$.

Step-1: Interchange coordinates of X and Y:

$g(Y)=X \Rightarrow g(z,w)=(x,y) \Rightarrow (2z-1,w+5)=(x,y)$

Step-2: Solve for coordinates of Y in terms of coordinates of X .That is

$(2z-1,w+5)=(x,y) \Rightarrow 2z-1=x,\ w+5=y \Rightarrow z=\frac{x}{2}+\frac{1}{2},\quad w=y-5$

Step-3: Equate the coordinates of Y obtained in step 2 with $f^{-1}(x,y)$.

Hence, $g^{-1}(X)=Y \Rightarrow g^{-1}(x,y)=(z,w) \Rightarrow g^{-1}(x,y)=(\frac{x}{2}+\frac{1}{2},y-5)$

Thus, $g(x,y)=(2x-1,y+5) \Leftrightarrow g^{-1}(x,y)=(\frac{x}{2}+\frac{1}{2},y-5)$

2. Let $f(x)=\dfrac{x^3}{x^2+1}$. Find x such that $f^{-1}(x)=2$.

Solution: From the definition of inverse, we have that $f^{-1}(y)=x \Leftrightarrow y=f(x)$

Thus, $f^{-1}(x)=2 \Leftrightarrow x=f(2) \Leftrightarrow x=8/5$.

Preposition 1.3: The inverse of a transformation is unique. Besides, $(f^{-1})^{-1}=f$.

Proof: Let f be a transformation whose inverses are g and h. That is $f^{-1}=g$ and $f^{-1}=h$. We need to show $g=h$. Here, $f^{-1}=g \Rightarrow f\circ g=g\circ f=i$ and $f^{-1}=h \Rightarrow f\circ h=h\circ f=i$. But, $g=i\circ g=(h\circ f)\circ g=h\circ(f\circ g)=h\circ i=h$ (Because composition is associative as well as g and h are inverses of f).

From this we can conclude that the inverse of a transformation is unique.

Examples:

1. Let $f(x)=\frac{1}{3}x+2$. Find the value of a if $h(x)=\frac{a}{4}x-6$ is the inverse of f .

Solution: Here, $f(x)=\frac{1}{3}x+2 \Rightarrow f^{-1}(x)=3x-6$. Then, from the uniqueness of inverse, we have $\frac{a}{4}x-6=3x-6 \Rightarrow \frac{a}{4}=3 \Rightarrow a=12$.

2. Let $(g \circ f)^{-1}(x, y) = h(x, y)$ where $g(x, y) = (1-x, -y), h(x, y) = (x+1, y+3)$.

Then, find the formula for $f(x, y)$.

Solution: Let's apply the property $(f^{-1})^{-1} = f$.

Here, $(g \circ f)^{-1}(x, y) = h(x, y)$

$$\Rightarrow [(g \circ f)^{-1}]^{-1}(x, y) = h^{-1}(x, y)$$
$$\Rightarrow (g \circ f)(x, y) = h^{-1}(x, y) \qquad [\text{Using the property}[(g \circ f)^{-1}]^{-1} = g \circ f\,]$$
$$\Rightarrow g^{-1} \circ (g \circ f)(x, y) = g^{-1} \circ h^{-1}(x, y), [\text{Operate both sides by } g^{-1}]$$
$$\Rightarrow f(x, y) = (g^{-1} \circ h^{-1})(x, y) \qquad [\text{Using the property } g^{-1} \circ g = i\,]$$

But $h(x, y) = (x+1, y+3) \Rightarrow h^{-1}(x, y) = (x-1, y-3)$ and

$g(x, y) = (1-x, -y) \Rightarrow g^{-1}(x, y) = (1-x, -y)$

Therefore, $f(x, y) = (g^{-1} \circ h^{-1})(x, y) = g^{-1}(h^{-1}(x, y)) = g^{-1}(x-1, y-3) = (2-x, 3-y$

Preposition 1.4: The inverse of a transformation is again a transformation.

Proof: Let $f : X \to X$ be any transformation on set X. Then, f^{-1} also exists as f is bijective. Now, we need to show f^{-1} is also a transformation.

(i) **One-to-ones:** Let a and b be arbitrary elements in X. Since f is bijective, there exists unique $x, y \in X, \ni f(x) = a, f(y) = b$.

But, $f(x) = a, f(y) = b \Rightarrow x = f^{-1}(a), y = f^{-1}(b)$.

Now, assume that $f^{-1}(a) = f^{-1}(b)$.

But, $f^{-1}(a) = f^{-1}(b) \Rightarrow x = y \Rightarrow f(x) = f(y) \Rightarrow a = b$

Thus, $f^{-1}(a) = f^{-1}(b) \Rightarrow a = b$. Hence, f^{-1} is one to one.

(ii) **Onto ness:** Let x be arbitrary element in X. Since f is onto, $f(x) \in X$, so, for every $x \in X, \exists f(x) \in X, \ni f^{-1}(f(x)) = x$. Hence, f^{-1} is onto. Therefore, from (i) and (ii), whenever f is a transformation on set X and so is f^{-1}.

Proposition 1.5: ***(Reverse Law of Inverse)***

For any two transformations f and g, $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

1.5 Fixed Points of Mappings and Involution

Definition: Let $f : S \to T$ be a mapping. Then, a point $x_0 \in S$ (in the domain of f) is said to be a fixed point of f if and only if $f(x_0) = x_0$. Generally, the set of fixed points of a mapping f is the set given by $S = \{x : f(x) = x\}$.

Examples: Determine the fixed points of the following mappings.

$a)$ $f : R \to R$ given by $f(x) = x^3 - 3x$

$b)$ $g : R^2 \to R^2$ given by $g(x, y) = (x^3, y^3)$

$c)$ $f : R^2 \to R^2$, $f(x, y) = (x, x + y - 3)$

$d)$ $h : R^2 \to R^2$ given by $h(x, y) = (|x|, \sqrt[3]{y})$

$e)$ $t : R^2 \to R^2$, $t(x, y) = (x + 1, y - 1)$

$f)$ $g : R^2 \to R^2$, $g(x, y) = (y - x, 6 - x)$

Solution:

$$a)\ S = \{x : f(x) = x\} = \{x : x^3 - 3x = x\} = \{x : x^3 - 4x = 0\}$$
$$= \{x : x(x^2 - 4) = 0\} = \{x : x(x - 2)(x + 2) = 0\}$$
$$= \{x : x = 0,\ 2,\ -2\} = \{0, 2, -2\}$$

$$b)\ S = \{(x, y) : g(x, y) = (x, y)\} = \{(x, y) : (x^3, y^3) = (x, y)\}$$
$$= \{(x, y) : x^3 = x,\ y^3 = y\} = \{(x, y) : x^3 - x = 0,\ y^3 - y = 0\}$$
$$= \{(x, y) : x = 0, 1, -1,\ y = 0, 1, -1\}$$
$$= \{(0,0)(0,1), (0,-1), (1,0), (1,1), (1,-1), (-1,0), (-1,1), (-1,-1)$$

$$c)\ S = \{(x, y) : f(x, y) = (x, y)\}$$
$$= \{(x, y) : (x, x + y - 3) = (x, y)\}$$
$$= \{(x, y) : x = x, x + y - 3 = y\}$$
$$= \{(x, y) : x = 3,\ y \in R\} = \{(3, y) : y \in R\}$$

$$d)\ S = \{(x, y) : h(x, y) = (x, y)\} = \{(x, y) : (|x|, \sqrt[3]{y}) = (x, y)\}$$
$$= \{(x, y) : |x| = x,\ \sqrt[3]{y} = y\}$$
$$= \{(x, y) : x \geq 0,\ y - y^3 = 0\}$$
$$= \{(x, y) : x \geq 0,\ y = 0, 1, -1\}$$
$$= \{(x, 0), (x, 1), (x, -1),\ \forall x \geq 0\}$$

Here, for any $x \geq 0$ and $y = 0,\ 1,\ -1,\ (x, y)$ is the fixed point of $h(x, y) = (|x|, \sqrt[3]{y})$.

e) Here, $t(x, y) = (x+1, y-1) = (x, y)$

$$\Rightarrow x+1 = x,\ y-1 = y \Rightarrow 1 = 0,\ -1 = 0$$

But this is impossible. This means t has no fixed point.

$f)\ S = \{(x, y) : g(x, y) = (x, y)\}$

$$= \{(x, y) : (y-x, 6-x) = (x, y)\}$$
$$= \{(x, y) : y-x = x, 6-x = y\} = \{(x, y) : y = 2x, 6-x = y\}$$
$$= \{(x, y) : 2x = 6-x\} = \{(x, y) : x = 2, y = 4\} = \{(2,4)\}$$

Note: From the above examples, we can conclude that a given mapping can have exactly one fixed point, finitely many fixed points or infinitely many fixed points. On the other hand, part (f), shows that there are mappings that have no fixed points.

Involution: A non-identity transformation α is said to be an *involution* if and only if $\alpha^2 = \alpha \circ \alpha = i$. That means $\alpha^2(x) = (\alpha \circ \alpha)(x) = \alpha(\alpha(x)) = i(x) = x$ for all x in the domain of α.

Examples: Verify whether the following transformations are involution or not.

$a)\ \beta : R \to R$ given by $\beta(x) = 1-x$

$b)\ h : R^2 \to R^2,\ h(x, y) = (-x+7, -y-2)$

$c)\ \alpha : R \to R$ given by $\alpha(x) = x+3$

$d)\ g : R^2 \to R^2,\ g(x, y) = (x-3, y+5)$

Solution:

$a)\ \beta^2(x) = \beta \circ \beta(x) = \beta(\beta(x)) = \beta(1-x) = x = i(x) \Rightarrow \beta^2 = i$.

So, β is an involution.

$b)\ h^2(x, y) = h(h(x, y)) = h(-x+7, -y-2) = (x, y) = i(x, y) \Rightarrow h^2 = i$.

So, h is an involution.

$c)\ \alpha^2(x) = \alpha(\alpha(x)) = \alpha(x+3) = x+6 \neq x = i(x) \Rightarrow \alpha^2 \neq i \Rightarrow \alpha$ is not an involution .

$d)\ g^2(x, y) = g(g(x, y)) = g(x-3, y+5) = (x-6, y+10) \neq (x, y) = i(x, y) \Rightarrow g^2 \neq i$.

Hence, g is not an involution.

1.6 Collineations and Dilatations

Definition: A transformation f is said to be a *collineation* if and only if the image of any line l under f is again a line. In other words, for any point $P \in l$ the image $f(P) \in f(l)$. Further more; f is said to be a *dilatation* if and only if the image of any line l under f is a line parallel to l. That is $f(l) // l$ whenever f is a collineation then f is said to be a dilatation.

Examples:

1. Let $\alpha : R^2 \to R^2$ be given by $\alpha(x, y) = (x+1, y-2)$. Show that α is a dilatation.

Solution: First let's show that α is a transformation. But, to show that α is a transformation, we need to show that it is one to one and onto.

One- to- one: Assume $\alpha(x, y) = \alpha(z, w)$ for any two points (x, y) and (z, w) in R^2. Then, $(x+1, y-2) = (z+1, w-2)$. But from equality of order pairs, this equality is true if and only if $x+1 = z+1$ and $y-2 = w-2$. This gives $x = z$ and $y = w$ which implies $(x, y) = (z, w)$. So, α is one-to-one.

On to ness: Let $(a, b) \in R^2$ be in the co-domain of α. Then, if $\exists (x, y) \in R^2$ in the domain of α, such that $\alpha(x, y) = (a, b)$, then α is on to. But, $\alpha(x, y) = (x+1, y-2) = (a, b) \Rightarrow x+1 = a, \ \ y-2 = b \Rightarrow x = a-1, \ y = b+2.$ Thus, we can find $(x, y) = (a-1, b+2) \in R^2$ such that $\alpha(x, y) = (a, b)$. So α is onto. Therefore, the given map α is a transformation. To show that α is a collineation we need to show the image of an arbitrary line $l : ax + by + c = 0$ is again a line.

Let (x, y) be any point on l. Then, the image $(x', y') = \alpha(x, y) = (x+1, y-2)$.

Solving this for x and y we get $x = x'-1, \ y = y'+2$. So,the image line will be

$l' : a(x'-1) + b(y'+2) + c = 0 \Rightarrow ax' + by' + 2b + c - a = 0$ and this is equation of a line.

Hence we can say that α is a collineation. Besides, l' has the same slope to that of l which means l' is parallel to l. In other words, $l // \alpha(l)$. Therefore, α is a dilatation.

2. Let $\alpha : R^2 \to R^2$ be given by $\alpha(x, y) = (-y, x+1)$. Then, find

a) The image of the line $l : x + 3y = 6$ under α.

b) The pre- image of the line $l' : 2x + y + 3 = 0$ under α.

Solution: *Images and pre-images:* Let α be a transformation. Then, for any two objects P and Q, if $\alpha(P) = Q$, the object Q is called the image of the object P under α and the object P is called the pre-image of Q under α. Besides, to find the image of any line l under any coollination α:

First: Select any two points P and Q from the given line (the selection is arbitrary you can select in any way you want !) and find their images $P' = \alpha(P)$ and $Q' = \alpha(Q)$.

Second: Form the equation of a line using the image points P' and Q'. This is the required image line under α.

a) Using the above steps, select $P = (0,2)$ and $Q = (6,0)$ on $l : x + 3y = 6$ (you can select any other points). Now find their images.

That is $P' = \alpha(P) = \alpha(0,2) = (-2,1), Q' = \alpha(Q) = \alpha(6,0) = (0,7)$

Hence, the image line is the line through $P' = (-2,1)$ and $Q' = (0,7)$.

That is the slope is $m = \dfrac{\Delta y}{\Delta x} = \dfrac{6}{2} = 3$.

Then, using slope intercept form $l' : y = mx + b = 3x + b \Rightarrow y = 3x + 7$.

b) In this case, we are asked the pre-image, that means what is the line whose image is given. To, do so, select two points P' and Q' on the image line $l' : 2x + y + 3 = 0$. Say $P' = (0,-3)$ and $Q' = (-2,1)$. Then, find two points $P = (a,b)$ and $Q = (c,d)$ such that $\alpha(P) = P'$ and $\alpha(Q) = Q'$.

Thus, $\alpha(P) = P' \Rightarrow \alpha(a,b) = (-b, a+1) = (0,-3) \Rightarrow a = -4, b = 0 \Rightarrow P = (-4,0)$ and $\alpha(Q) = Q' \Rightarrow \alpha(c,d) = (-d, c+1) = (-2,1) \Rightarrow d = 2, c = 0 \Rightarrow Q = (0,2)$

Hence, the pre-image of the line $l' : 2x + y + 3 = 0$ is a line through $P = (-4,0)$ and $Q = (0,2)$. That is the slope is $m = \dfrac{\Delta y}{\Delta x} = \dfrac{2}{4} = \dfrac{1}{2}$. Then, using slope intercept form $l : y = mx + b = \dfrac{1}{2}x + b \Rightarrow y = \dfrac{1}{2}x + 2 \Rightarrow x - 2y + 4 = 0$.

Remarks:

1. The main difference between collineation and dilatation is that any collineation maps a pair of parallel lines to a pair of parallel line but a dilatation maps every line to a line parallel to the given line. This means a transformation α is a collineation if and only if for any two lines m and n, $m // n \Rightarrow \alpha(m) // \alpha(n)$. A transformation α is a dilatation if and only if for any line m, $m // \alpha(m)$.

2. If $\alpha(x, y) = (x', y')$ where $x' = ax + by + h,\ y' = cx + dy + k$, then the necessary and sufficient conditions on the coefficients of x and y such that α to be a transformation is that $ad - bc \neq 0$.(This is known as transformation test).

Examples:

1. Define $\alpha : R^2 \to R^2$ by $\alpha(x, y)) = (-y, x)$. Show that α is a collineation but not a dilatation.

Solution: Clearly, α is a transformation. Besides, for any two arbitrary parallel lines $m : ax + by + c = 0$ and $n : ax + by + k = 0$(Parallel lines differ by a constant), $m' = \alpha(m) : ay - bx + c = 0,\ n' = \alpha(n) = ay - bx + k = 0$. Still, m' and n' have the same slope which means they are parallel. $i.e\ m // n \Rightarrow \alpha(m) // \alpha(n)$. Thus, α is a collineation. But, if we consider only the single line $m : ax + by + c = 0$ separately, $m' = \alpha(m) : ay - bx + c = 0$. In this case, slope of line m is $-\frac{a}{b}$ while that of m' is $\frac{b}{a}$ which gives the product of their slope is -1. This means $i.e\ m \perp \alpha(m) = m'$. In other words, m and $\alpha(m) = m'$ are not parallel lines. Particularly, take the line $m : 6x - 2y + 5 = 0$. Then, its image under α is $\alpha(m) = m' : 2x + 6y + 5 = 0$. Consequently, α is a collineation but not a dilatation.

2. Find the value of the constant t for which $\alpha(x, y) = (4x - ty + 7, 8x + 6y - 2)$ will not be a transformation.

Solution: By the second part of the above ramark, α will bot be a transformation if and only if $4.(6) - 8.(-t) = 0 \Rightarrow 24 + 8t = 0 \Rightarrow 8t = -24 \Rightarrow t = -3$.

Problem Set 1.1

1*. Let Z, N and W be the set of integers, natural numbers and whole numbers.

a) Define $f : Z \to N$ by $f(x) = x^2$. Is f a mapping? If so, is it one to one?

b) Define $g : N \to Z$ by $g(x) = x^2$. Is g a mapping? If so, is it one to one?

c) Consider the set $T = \{x \in Z : |x-1| \leq 2\}$. Define $h : T \to Z$ by $h(x) = x^3$. Is h a mapping? If so, is it one to one? is it onto?

d) Let $S = \{x \in Z : |x+2| \leq 3\}, \mathrm{T} = \{0, 1, 4, 9, 16, 25\}$.

Define $r : S \to T$ by $r(x) = x^2$. Is this relation a mapping? If not, **justify!** If it is a mapping, is it bijective?

Answer : $a)$ It is not a mapping because for $x = 0 \in Z$, $f(0) = 0^2 = 0 \notin N$.
This mean there is no image in the set of natural numbers N for $x = 0$.
$b)$ It is a mapping but neither one to one nor onto.
$c)$ It is a mapping, it is one to one but not onto.
$d)$ It is a mapping, it is onto but not one to one.

2. Determine whether the following mappings are transformations or not on R^2

$a)$ $f(x, y) = (e^x, y^3)$ $b)$ $g(x, y) = (-2x, y+3)$ $c)$ $h(x, y) = (3y, x+2)$
$d)$ $\alpha(x, y) = (3y-2, 2x+1)$ $e)$ $\varphi(x, y) = (x^3 - x, y)$ $f)$ $\beta(x, y) = (\sqrt[3]{x-1}, y+2)$
$g)$ $\delta(x, y) = (2y - x, x-2)$ $h)$ $\partial(x, y) = (x^3, y^3)$ $i)$ $h(x, y) = (\sqrt{y-1}, x^3)$
$j)$ $\psi(x, y) = (x - y + 3, 2x + 3y - 8)$ $k)$ $\eta(x, y) = (x + y + 1, x - y - 1)$

Answer : Only b, c, d, f, g, h, j, k are Transformations

3. Find the fixed points (if any) of each of the following mappings

$a)$ $\alpha : R^2 \to R^2$, given by $\alpha(x, y) = (y^3, \sqrt[5]{x})$ $b)$ $f : R^2 \to R^2$, $f(x, y) = (|x|, y - x)$
$c)$ $\eta : R^2 \to R^2, \eta(x, y) = (x + y + 1, x - y - 1)$ $d)$ $\varphi : R^2 \to R^2$, $\varphi(x, y) = (x^3 - x, y)$
$e)$ $k : R^2 \to R^2$, $k(x, y) = (|x|, |y|)$ $f)$ $f : R^2 \to R^2$, $f(x, y) = (x + y, x - y)$
$g)$ $f : R^2 \to R^2$, $f(x, y) = (x + y - 2, y + 3)$ $h)$ $k : R^2 \to R^2$, $k(x, y) = (x, y - x)$

Answer : $a)$ $S = \{(0,0), (1,1), (-1,-1)\}$ $b)$ $S = \{(0, y) : y \in R\}$ $c)$ $S = \{(-1,-1)\}$
$d)$ $S = \{(0, y), (\sqrt{2}, y), (-\sqrt{2}, y) : y \in R\}$ $e)$ $S = \{(x, y) : x, y \geq 0\}$
$f)$ $S = \{(0,0)\}$ $g)$ It has no fixed points $h)$ $S = \{(0, y) : y \in R\}$

4. Let $f_n : Z \to Z$ be a map given by $f_n(x) = nx, \forall x \in Z$. For which values of n, f_n is one to one? Onto? **Answer** : One to one $n \neq 0$ and onto $n = \pm 1$

5. Let $f : R \to R, g : R \to R$ be defined by $f(x) = kx + 8, g(x) = 2x + 4$. Find the value of the constant k such that $g \circ f = f \circ g$ **Answer** : $k = 3$

6. Consider the transformations $f(x, y) = (ax, 3y + 2)$ and $g(x, y) = (5x, 4y + 2b)$. If $(g \circ f)(x, y) = (10x, 12y - 40)$, then, find a and b. **Answer** : $a = 2, b = -7$

7. If S is a finite set, show that $f : S \to S$ is one-to-one if and only if it is onto.

8. Find the formula for the inverse of the following functions.

$a)$ $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$ b) $h(x, y) = (2 - x, 6 - y)$ $c)$ $\beta(x, y) = (\sqrt[3]{x-1}, y + 2)$

$d)$ $\delta(x, y) = (x - y, x + y)$ $e)$ $\psi(x, y) = (x, y - x)$

Answer : $a)$ $f^{-1}(x) = \dfrac{e^x - 1}{e^x + 1}$ b) $f^{-1}(x, y) = (2 - x, 6 - y)$

9. Let $\alpha : R^2 \to R^2$ be given by $\alpha\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 - x \\ y + 2t - 6 \end{pmatrix}$. What value of t makes α an *involution*? **Answer** : $t = 3$

10. *Consider the line $L : y = x + 3$ such that $f : L \to R$ is given by $f(x, y) = 2x, \forall (x, y) \in L$. Show that f is bijective. If $f(P) = 3$, find the coordinates of P. **Answer** : $P = (3/2, 9/2)$

11. Let h be any transformation from R to R defined by $h(x) = 3x - 6$. If $g(x) = \dfrac{a}{2}x + 2$ is the *inverse* of $h(x)$, then what must be a? **Answer** : $a = 2/3$

12. Let α and β be any two transformations with $\beta(P) = 2\alpha(P) - P, \forall P \in R^2$. If $\beta(7,3) = (9,9)$, then calculate $\alpha(7,3)$. **Answer** : (8,6)

13. If $\alpha(x, y) = (x', y')$ where $x' = ax + by + h, y' = cx + dy + k$, then find the necessary and sufficient conditions on the coefficients of x and y such that α to be a transformation. **Answer** : $ad - bc \neq 0$

14. Find the constant c for which $\alpha(x, y) = (5x - 3y + 2, cx + 6y - 5)$ will not be a transformation. **Answer** : $c = -10$

15. Let $\psi : R^2 \to R^2$ be given by $\psi(x, y) = (x - y + 7, 2x - 3)$. Show that ψ is a transformation and find ψ^{-1}. **Answer** : $\psi^{-1}(x, y) = (\frac{y+3}{2}, \frac{y-2x+17}{2})$

16. Define $g : R^2 \to R^2$ by $g(x, y)) = (ax + c, by + d)$ where a and b are non-zero. Show that $a)$ g is a transformation $b)$ g is a dilatation if and only if $a = b$

17. Show that $\alpha : R^2 \to R^2$ given by $\alpha(x, y) = (3y, x - y)$ is a collineation and find the pre-image of the line $l : y = 3x + 2$ under α. **Answer** : $x - 10y - 2 = 0$

18. Define $\psi : R^2 \to R^2$ by $\psi(x, y) = (5 - y, x + 1)$. Show that ψ is a collineation but not a dilatation.

19. Show that $\beta : R^2 \to R^2$ given by $\beta(x, y) = (x + 2, y - 3)$ is a dilatation.

20. Given $\alpha(x, y) = (3mx - 2x + 1, my + 6y - 1)$. What should be the value of m so that α is a dilatation? **Answer** : $m = 4$

21. *Prove:* If α is an involution, then for any transformation β, $\beta \circ \alpha \circ \beta^{-1}$ is also an involution.

22. If f an involution such that $f(1,5) = (2,1)$, find $f^{-1}(1,5)$ and $f^{-1}(2,1)$.

23*. Show that any collineation sends a pair of parallel lines to a pair of parallel lines.

24. Find the condition on a and b such that $\psi(x, y) = (ay, \frac{x}{b})$ is an involution.

Answer : $a = b$

25. For what value (s) of k is $\psi(x, y) = \left((3k^2 - 1)y, \frac{3}{22}x \right)$ an involution?

Answer : $k = \pm 5/3$

26*. Let $\beta : S \to R$ be a mapping given by $\beta(t) = \frac{1 - 2t}{|2t - 1| - 1}$ where the set S is given by $S = \{t : 0 < t < 1\}$.

Show that β is a bijective mapping but not a transformation.

27. Consider the napping $f(x)=\sin x$. Define an equivalence relation $xRy \Leftrightarrow \sin x = \sin y$. Find the equivalence class determined by $y=\dfrac{\pi}{2}$

Answer: $R_{\pi/2}=\{\pi/2+n\pi;\ n\in Z\}$

28. Let $\beta: M_{2x2}\to M_{2x2}$ be a mapping given by $\beta(A)=A^{-1},\ \forall A\in M_{2x2}$ where M_{2x2} is the set of all invertible $2x2$ matrices. Show that β is an involution and calculate all the fixed points of β.

28. Show that the map $\alpha: C\to C$ given by $\alpha(x+yi)=x-yi$ is a transformation with infinite number of fixed points.

29. For each order pair (a,b) of integers define $\lambda_{a,b}: Z\to Z$ by $\lambda_{a,b}(n)=an+b$. For which pair (a,b) is $\lambda_{a,b}$ injective (one to one)? For which pair (a,b) is $\lambda_{a,b}$ surjective (onto)?

Answer: $\lambda_{a,b}$ is injective for $a\neq 0, \forall b\in Z$, $\lambda_{a,b}$ is surjective only for $a=\pm 1$

30. Let $M=(a,b)$ and $\psi: R^2\to R^2$ be a mapping with the property that for any point P in R^2 the mid point of P and $\psi(P)$ is M. Prove that ψ is a transformation with fixed point M itself.

31*. Let L be the set of all lines in the plane and let α be a collineation in R^2. Then, α is a transformation from L to L.

1.7 Definitions and examples of Transformation Groups

Definition: Let G be the set of all transformations on a non empty set S. Then the system $(G,\circ)$ is said to be a transformation group if and only if the following conditions are satisfied:

i) For all f, g in G, $g \circ f$ is in G.

ii) For all f in G, $\exists g \in G \ni f \circ g = g \circ f = i$ and denoted by $f^{-1} = g$.

iii) For all f in G, $\exists i \in G \ni f \circ i = i \circ f = f$.

Examples

1. Let $f_{ab} : R \to R$ be defined by $f_{ab}(x) = ax + b, a \neq 0.$

Let $G = \{f_{ab} / a, b \in R, a \neq 0\}$. Show that $(G,\circ)$ forms a transformation group.

Solution: To check whether the elements of G are transformation or not, for any f_{ab} in G. $f_{ab}(x) = f_{ab}(y) \Rightarrow ax + b = ay + b \Rightarrow x = y$ and for each $r \in R$, there exists $x = \dfrac{r-b}{a}, a \neq 0$ in R such that

$f_{ab}(x) = f_{ab}(\dfrac{r-b}{a}) = a(\dfrac{r-b}{a}) + b = r - b + b = r$. Thus, any element f_{ab} in G is a transformation.

i) Taking $a = 1, b = 0,$ we get $f_{10}(x) = x, \forall x \in R$. So, $i = f_{10}$ which is the identity transformation contained in G.

ii) For any f_{ab} and f_{cd} in G,

$f_{ab} \circ f_{cd}(x) = f_{ab}(f_{cd}(x)) = f_{ab}(cx + d) = acx + ad + b = f_{ac,ad+b}(x)$

Since f_{ab} and f_{cd} in G, $a \neq 0, c \neq 0 \Rightarrow ac \neq 0$. Thus, $f_{ab} \circ f_{cd} = f_{ac,ad+b} \in G$. This shows closure property holds true.

iii) To have inverse for any f_{ab} in G, we have to find f_{cd} in G such that $f_{ab} \circ f_{cd} = f_{cd} \circ f_{ab} = i = f_{10}$.

But $f_{ab} \circ f_{cd}(x) = f_{ac,ad+b}(x) = f_{10}(x) = x \Rightarrow acx + ad + b = x$.

So, $acx + ad + b = x \Rightarrow ac = 1, ad + b = 0 \Rightarrow c = \dfrac{1}{a} = a^{-1}, d = -\dfrac{b}{a} = -a^{-1}b$.

Since $a \neq 0$, c and d are defined and hence $(f_{ab})^{-1} = f_{a^{-1},-a^{-1}b}$ because

$f_{ab} \circ f_{a^{-1}-a^{-1}b}(x) = f_{ab}(a^{-1}x + -a^{-1}b) = a(a^{-1}x + -a^{-1}b) + b = x - b + b = x = f_{10}(x)$.

Hence, $(f_{ab})^{-1} = f_{a^{-1},-a^{-1}b}$ is the required inverse of f_{ab} in G. Therefore, by definition $(G,\circ)$ forms a transformation group.

2. Consider the transformations $f(x)=x,\ g(x)=-x,\ h(x)=\frac{1}{x},\ k(x)=-\frac{1}{x}$ defined on the domain $R/\{0\}$. Show that $(G,\circ)$ forms abelian group of transformation where $G=\{f,\ g,\ h,\ k\}$.

Solution: To show whether a finite set together with a binary operation forms a group or not is simple by making a table called Cayley table. Each cell is filled using the calculation as follow:

$$f\circ f(x)=f(f(x))=f(x)=x\Rightarrow f\circ f=f,$$
$$f\circ g(x)=f(g(x))=f(-x)=-x=g(x)\Rightarrow f\circ g=g$$
$$g\circ h(x)=g(h(x))=g(\frac{1}{x})=-\frac{1}{x}=k(x)\text{ and so on.}$$

$\circ$	f	g	h	k
f	f	g	h	k
g	g	f	k	h
h	h	k	f	g
k	k	h	g	f

From this table, we can infer the following results.

i) Existence of Identity: In the table, $f\circ g=g\circ f=g$ and so on for all elements of G. Thus f is the identity transformation in G.

ii) Existence of inverse: As we see from the table, $f\circ f=f, g\circ g=f,\ h\circ h=f, k\circ k=f$ where f is identity. So every element is its own inverse.

iii) Closure property: If all the cells in the body of Cayley table is filled with elements from the set, then closurity holds which is true in our case. Hence, $(G,\circ)$ forms transformation group.

3. Let β_k be a mapping for $k\neq 0$ given by $\beta_k(x,y)=(x',y')$ where $\begin{cases}x'=x\\y'=ky\end{cases}$.

Show that the set $G=\{\beta_k : k\neq 0\}$ forms abelian group of transformations with composition. Particularly, give the identity element and the inverse of $\beta_{-3}(x,y)=(x,-3y)$ in this group.

Proof: Apply the definition.

4. Show that the set of all transformations on the plane forms a transformation group.

Proof: Apply the definition.

1.8 Criteria for Transformation Groups

So far we discussed about transformation groups and we saw how to show a given set of transformations forms a transformation group using definitions and or using Cayley table. But, some times definitions alone are not easy or efficient to use in any cases. That is why mathematicians are looking for a short cut method to use.

In transformation geometry, they developed the following theorem as a test for a transformation groups.

Theorem 1.1 (Test for a transformation groups):

Let G be a nonempty set of transformations on a set S. Then, G with composition is a transformation group if and only if the following conditions are satisfied:

$a)\ f \in G \Rightarrow f^{-1} \in G,\ \forall f \in G$

$b)\ f, g \in G \Rightarrow f \circ g \in G,\ \forall f, g \in G$

Proof: Suppose $(G,\circ)$ is a transformation group. We need to show conditions (a) and (b) hold true. Since $(G,\circ)$ is a transformation group, from the definition $\forall f, g \in G,\ f^{-1} \in G$ and $f \circ g \in G$ whish implies that conditions (a) and (b) are true. Conversely, suppose conditions (a) and (b) hold true. We need to show $(G,\circ)$ is a transformation group.

i) Existence of Inverse: Since $G \neq \Phi$, $\exists f \in G$ but $\forall f \in G,\ f^{-1} \in G$ from condition (a). So, G contains inverse transformation.

ii) Closure property: From condition (b), $f, g \in G \Rightarrow f \circ g \in G,\ \forall f, g \in G.$

iii) Existence of Identity: $\forall f \in G$, $f^{-1} \in G$ from condition (a) and from condition (b), $f^{-1} \circ f \in G$ and $f \circ f^{-1} \in G$.

But, $f^{-1} \circ f = i \in G$ and $f \circ f^{-1} = i \in G$.

Thus, G contains identity transformation. Therefore, by definition, $(G,\circ)$ forms a transformation group.

Examples:

1. Let $g_a : R \to R$ be defined by $g_a(x) = ax,\ a \neq 0.$ Let $G = \{g_a \,/\, a \in R,\ a \neq 0\}$. Show that $(G,\circ)$ forms a transformation group.

Solution: So far, we have seen that g_a is a transformation. Using the above test for transformation group, we will verify the problem as follow:

i) For any $g_a \in G,\ \ g_a(x) = ax \Rightarrow g_a^{-1}(x) = \dfrac{1}{a}x = a^{-1}x \Rightarrow g_a^{-1} \in G.$

ii) For any $g_a,\ g_b \in G,\ g_a \circ g_b(x) = g_a(g_b(x)) = abx = g_{ab}(x) \Rightarrow g_a \circ g_b = g_{ab}$

Since $g_{ab} \in G$,and $g_a \circ g_b = g_{ab}$, we get $g_a \circ g_b \in G$. Hence, by test of transformation group, $(G,\circ)$ forms a transformation group.

2. Let $f_a : R^2 \to R^2;\ f_a(x,y) = (2a - x, y),\ \forall a \in R$ such that $G = \{f_a : \forall a \in R\}$.

Using criteria of transformation group, determine whether $(G,\circ)$ is transformation group or not.

Solution: For any $f_a \in G,\ \ f_a(x,y) = (2a - x, y)$. First of all, we need to show that every element of G is transformation.

Now suppose, $f_a(x,y) = f_a(z,w)$.

$$\begin{aligned} f_a(x,y) = f_a(z,w) &\Rightarrow (2a - x, y) = (2a - z, w) \\ &\Rightarrow 2a - x = 2a - z,\ y = w \\ &\Rightarrow x = z,\ y = w \Rightarrow (x,y) = (z,w) \end{aligned}$$

This shows that each f_a is one to one.

Besides, to each $(z,w) \in R^2$ (in the co-domain), $\exists (x,y) = (2a - z, w) \in R^2$ such that $f_a(x,y) = f_a(2a - z, w) = (z,w)$.

So, each f_a is also onto.

Therefore, from these explanations every element of G is a transformation.

For any $f_a \in G$,

$$f_a(x,y) = (2a - x, y) \Rightarrow f_a^{-1}(x,y) = (2a - x, y) \Rightarrow f_a^{-1} = f_a \Rightarrow f_a^{-1} \in G.$$

But for any two elements $f_a,\ f_b \in G$,

$$f_a \circ f_b(x,y) = f_a(2b - x, y) = (x + 2a - 2b, y) \neq f_r(x,y),\ \forall r \in R \Rightarrow f_a \circ f_b \notin G.$$

This means the second condition of the above theorem (test for a transformation group) fails.

As a result, $(G,\circ)$ does not form transformation group.

Theorem 1.2 (Cancellation Laws on Transformation Groups):

Let G be a transformation group. Then, for α, β, σ in G

a) $\alpha \circ \beta = \alpha \circ \sigma \Rightarrow \beta = \sigma$ (This is called Left Cancellation Law)

b) $\alpha \circ \beta = \sigma \circ \beta \Rightarrow \alpha = \sigma$ (This is called Right Cancellation Law)

Proof: Let G be a transformation group. Then, for α, β, σ in G

a) $\alpha \circ \beta = \alpha \circ \sigma \Rightarrow \alpha^{-1} \circ (\alpha \circ \beta) = \alpha^{-1} \circ (\alpha \circ \sigma) \Rightarrow \beta = \sigma$

b) $\alpha \circ \beta = \sigma \circ \beta \Rightarrow (\alpha \circ \beta) \circ \beta^{-1} = (\sigma \circ \beta) \circ \beta^{-1} \Rightarrow \alpha = \sigma$

Theorem 1.3: In any transformation group G, for any α, β in G, the equation $\alpha \circ \sigma = \beta$ has a unique solution for σ in G which is given by $\sigma = \alpha^{-1} \circ \beta$.

Proof: For α, β in G, $\alpha \circ \sigma = \beta \Rightarrow \alpha^{-1} \circ (\alpha \circ \sigma) = \alpha^{-1} \circ \beta \Rightarrow \sigma = \alpha^{-1} \circ \beta \in G$

Hence, the equation has a solution in G. For the uniqueness of this solution, assume there are two different solutions say σ, θ, then

$\alpha \circ \sigma = \beta,\ \alpha \circ \theta = \beta \Rightarrow \alpha \circ \sigma = \alpha \circ \theta \Rightarrow \sigma = \theta$ (by Left cancellation)

Example: Let $(G, \circ)$ be a transformation group such that α, β, σ in G. If $\alpha(x, y) = (3 - x, 5 - y), \beta(x, y) = (3 - 2x, 5 - 3y)$. Find σ such that $\alpha \circ \sigma = \beta$

Solution: By the above theorem, $\alpha \circ \sigma = \beta$ has a unique solution for σ in G which is given by $\sigma = \alpha^{-1} \circ \beta$.

But after some ups and downs we get $\alpha^{-1}(x, y) = (3 - x, 5 - y)$.

Therefore, $\sigma(x, y) = \alpha^{-1} \circ \beta(x, y) = \alpha^{-1}(\beta(x, y))$

$$= \alpha^{-1}(3 - 2x, 5 - 3y) = (2x, 3y)$$

Problem Set 1.2

1. Consider the following transformations on $R/\{1,0\}$.

$$f(x)=x,\ g(x)=\frac{1}{x},\ h(x)=1-x,\ k(x)=\frac{1}{1-x},\ m(x)=\frac{x-1}{x},\ n(x)=\frac{x}{x-1}.$$

If $G=\{f,\ g,\ h,\ k,\ m,\ n\}$, show that $(G,\circ)$ forms a transformation group.

2. Consider the set $G=\{f,\ g,\ h,\ k\}$ where $f,\ g,\ h,\ k$ are transformations from R^2 to R^2 given by:

$$f(x,y)=(x,y),\qquad g(x,y)=(2-y,x)$$
$$h(x,y)=(y,2-x),\quad k(x)=(2-x,2-y)$$

Show that $(G,\circ)$ forms an abelian transformation group.

3. Prove that if a is a fixed element of a group G, and $\alpha:G\to G$ is defined by $\alpha(x)=ax,\ \forall x\in G$, then α is a transformation.

4. Let $f:R\to R$ be defined by $f_{ab}(x)=ax+b,\ a\neq 0.$ Let $G=\{f_{ab}/a,b\in R,\ a\neq 0\}$, $H=\{f_{1b}:b\in R\}$. Show that

a) $(H,\circ)$ is a subgroup of the transformation group $(G,\circ)$

b) $R=\{(a,b):\int_a^b f(x)dx=\int_a^b g(x)dx,\ \forall f,g\in G\}$ is an equivalence relation

5. Let α_k be a mapping for $k\neq 0$ given by $\alpha_k(x,y)=(x',y')$ where $\begin{cases}x'=kx\\ y'=y\end{cases}$.

Show that $G=\{\alpha_k:k\neq 0\}$ forms abelian group of transformations with composition. Give the identity element and the inverse of $\alpha_5(x,y)=(5x,y)$ in this group. List the involution elements (if any).

Answer : Identity element, $i=\alpha_1$, Inverse of α_5 is $\alpha_{\frac{1}{5}}(x,y)=(\frac{1}{5}x,y)$,

The set of all involution elements : $\{\alpha_1,\alpha_{-1}\}$

6. *Let α_a, β_b be maps for $a, b \neq 0$ given by $\alpha_a(x, y) = (ax, y),\ \beta_b(x, y) = (x, by)$. Consider the set $G = \{\alpha_a \circ \beta_b : a, b \neq 0\}$. Show that $(G, \circ)$ forms abelian group of transformations. Give the identity element and the inverse of $\alpha_{\frac{1}{3}} \circ \beta_2$ in this group. List all the involution elements .

Answer : Identity element, $i = \alpha_1 \circ \beta_1$, Inverse of $\alpha_{\frac{1}{3}} \circ \beta_2$ is $\alpha_3 \circ \beta_{\frac{1}{2}}(x, y) = (3x, \frac{1}{2}y)$,

The set of all involution elements : $\{\alpha_1 \circ \beta_1, \alpha_1 \circ \beta_{-1}, \alpha_{-1} \circ \beta_1, \alpha_{-1} \circ \beta_{-1}\}$

7. Let G be a nonempty set of transformations on a set S. Then, $(G, \circ)$ forms a transformation group if and only if $f, g \in G \Rightarrow f \circ g^{-1} \in G,\ \forall f, g \in G.$

8. Let G be any group. Define $f : G \to G$ by $f(a) = a^{-1}, \forall a \in G$. Show that f is a transformation.

9. Suppose α, β, σ are transformation such that $\alpha \circ \sigma(X) = \beta(X)$ where $\alpha(x, y) = (-y, x)$ and $\beta(x, y) = (1 - x, -y - 10)$. Find equation of σ.

Answer : $\sigma(x, y) = (-y - 10, x - 1)$

10. In a transformation group G, if $f = f^{-1}, \forall f \in G$, then prove that G is abelian.

11. Let G be a finite transformation group. Show that in a Cayley table of G, each element appears exactly once in each raw and in each column.

12. Let G be a transformation group. If $f \circ g = g$, then show that $f = i$.

13. If G is the set of transformations given by $G = \{f, g, h\}$, then complete the following Cayley table of G. Which element of G is the identity? Give the inverse of each element. Which element (s) of G are involutions?

$\circ$	f	g	h
f		g	
g			
h			

14. In a transformation group G, if $(\alpha \circ \beta)^2 = \alpha^2 \circ \beta^2$ for each α, β in G, prove that G is abelian group of transformation.

15. Let $(G, \circ)$ be a transformation group with the property that $\forall f \in G \Rightarrow f \circ f = f^2 = i.$ Prove that; every element of G is its own inverse.

16. In a transformation group G, if all the non-identity elements are involution, then show that G is abelian.

17. Let α, β, σ be elements of a transformation group G such that $\alpha \circ \sigma(x, y) = \beta(x, y), \forall (x, y) \in R^2$ where $\alpha(x, y) = (-3y, 2x+1)$ and $\beta(x, y) = (9y, 4x+1)$. Find the equation of σ.

18. Let $(G, *)$ be arbitrary group. Prove that there is a group T of transformation on G and a one to one mapping $g: G \to T$ which assigns to each $a \in G$, a transformation $f_a \in T$ such that $f_{b*a} = f_b \circ f_a, \forall a, b \in G$.

CHAPTER-2

AFFINE GEOMETRY

2.1 Introduction to Affine Spaces

Affine geometry is a type of geometry that can be used to make analysis in a plane and space whose major entries or elements are vectors. The properties and conditions of affine geometry work in any dimensional space (finite or infinite). As a result, affine geometry is treated as dimension free. The space that we will work using this geometry is called affine space (which is formally defined below). In this space, there is no specified origin.

As vectors are the main elements in this geometry, most of the theorems and problems that we know in Linear Algebra courses are treated in this unit only by using vectors and vector analysis rather than using coordinates and Synthetic analysis. Now, we will prove and solve different theorems and problems using vector method in order to understand the notion of affine geometry and affine spaces. Finally, you may appreciate to what extent vectors are useful in analysis of theorems and problems that we already know in different branches of Mathematics.

Definition: A non-empty set W is said to be an *affine space* associated with the vector space V if and only if the following three conditions are satisfied:

i. To every two points A, B in W, there exists a unique vector $u = \overrightarrow{AB} = B - A$ in V, and u is the zero vector if and only if $A = B$.

ii. To any point A in W and any vector u in V there exists a unique point B such that $u = \overrightarrow{AB}$.

iii. For any three points A, B, C in W, we have $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

In any affine space W, its elements are called vectors. Most of the concepts like norm, projection, cross product and other concepts of vectors in Euclidean Geometry are similarly defined in affine geometry.

Examples:

1. Let $V = R^3$ be the vector space with the usual operations. Determine whether each of the following sets are affine spaces associated with V or not

$a)\ W = \{(x, y, z) : x = y = z\}$
$b)\ W = \{(x, y, z) : x + y + z = 0\}$
$c)\ W = \{(x, 0, 0) : x \in R\}$
$d)\ W = \{(x, y, z) : x - 2y + z = 7\}$

Solution: Follows directly from the definition.

2. Let $V = M_{2\times 2}$ be the vector space of 2×2 matrices with the usual operations. Verify that $W = \{A \in M_{2\times 2} : A = A^t\}$ forms an affine spaces associated with V.

Perpendicular and Parallel vectors:

Definition: Two non-zero vectors $\vec{a}$ and $\vec{b}$ are said to be orthogonal or perpendicular if their dot product is zero.

Denoted by $\vec{a} \perp \vec{b}$. Thus, $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a}.\vec{b} = 0$.

Note that if $\vec{a}$ is perpendicular to $\vec{b}$ and k is any scalar, then $\vec{a}$ is also perpendicular to $k\vec{b}$ because $\vec{a} \cdot k\vec{b} = k(\vec{a}.\vec{b}) = k.0 = 0 \Rightarrow \vec{a} \perp k\vec{b}$.

Example: Show that the vectors $\vec{a} = (3, -1, 5)$ and $\vec{b} = (1, 3, 2)$ are orthogonal.

Solution: Here, $\vec{a}.\vec{b} = 3 - 13 + 10 = 0 \Rightarrow \vec{a} \perp \vec{b}$.

Definition: Any two non-zero vectors are said to be parallel if and only if one is the scalar multiple of the other.

Let $\vec{a}$ and $\vec{b}$ be any two non-zero vectors. Then, $\vec{a} // \vec{b} \Leftrightarrow \exists t \neq 0 \in R, \ni \vec{a} = t\vec{b}$ and note that if $\vec{a} // \vec{b}, \ni \vec{a} = t\vec{b}$ we always have $\vec{b} // \vec{a}, \ni \vec{b} = \frac{1}{t}\vec{a}$.

The scalar t is said to be norm (length) ratio and given by $|t| = \frac{\|\vec{a}\|}{\|\vec{b}\|}$.

Examples

1. Verify that the vectors $\vec{a}=(2,-1,3)$ and $\vec{b}=(-6,3,-9)$ are parallel.

Solution: Here, $\vec{b}=(-6,3,-9)=-3(2,-1,3)=-3\vec{a}\Rightarrow \exists t=-3, \ni \vec{b}=-3\vec{a}$.

Therefore, the two vectors are parallel. That is $\vec{a}//\vec{b}$.

2. Let $\vec{a}=(m,3,-4)$ and $\vec{b}=(2,-n,8)$. Find the values of m and n so that the vectors $\vec{a}$ and $\vec{b}$ are parallel.

Solution: Here,

$$\vec{a}//\vec{b}\Leftrightarrow \exists t\neq 0\in R, \ni \vec{b}=t\vec{a}\Leftrightarrow (2,-n,8)=t(m,3,-4)$$
$$\Leftrightarrow 2=m.t,\ -n=3t,\ 8=-4t\Leftrightarrow t=-2\Rightarrow m=-1,\ n=6$$

Proposition 2.1:

i) Parallelism relation denoted by $//$ on vectors is an equivalence relation.

ii) If $\vec{a}$ and $\vec{b}$ are not parallel, then $\forall\ r,t\in R$, the equation $r\vec{a}=t\vec{b}$ has a unique solution $r=t=0$

Proof:

i) For any vector $\vec{a}=1.\vec{a}\Rightarrow \vec{a}//\vec{a}$ and thus $//$ is *reflexive*. Suppose $\vec{a}//\vec{b}$. Then, $\exists t\in R, \ni \vec{a}=t\vec{b}\Rightarrow \vec{b}=\frac{1}{t}\vec{a}\Rightarrow \vec{b}//\vec{a}$. Thus, $//$ is *symmetric*. If $\vec{a}//\vec{b}$ and $\vec{b}//\vec{c}$, then $\vec{a}=r\vec{b}$ and $\vec{b}=t\vec{c}$. Therefore $\vec{a}=\overrightarrow{rtc}\Rightarrow \vec{a}//\vec{c}$ so $//$ is *transitive*. From the three conditions, $//$ is an equivalent relation on vectors.

ii) Suppose $r\neq 0$ such that $r\vec{a}=t\vec{b}$, then $\vec{a}=\frac{t}{r}\vec{b}\Rightarrow \vec{a}//\vec{b}$ which contradicts with the assumption that $\vec{a}$ and $\vec{b}$ are not parallel. Therefore $r=0$ and similarly by assuming $t\neq 0$ it can be shown $t=0$.

2.2 Geometry in Affine Space

Since the elements of affine space are vectors, the geometric theorems that we know are analyzed using vector analysis. Now, let's analyze some theorems and problems using vectors in order to understand affine geometry and affine space.

Proposition 2.2:

1. The line segment joining the mid-points of two sides of a triangle is parallel to the third side and its length is half of that side.

Proof: Consider ΔABC below (figure 2.1a) where M and N are the mid-points of $\overline{AC}$ and $\overline{BC}$.

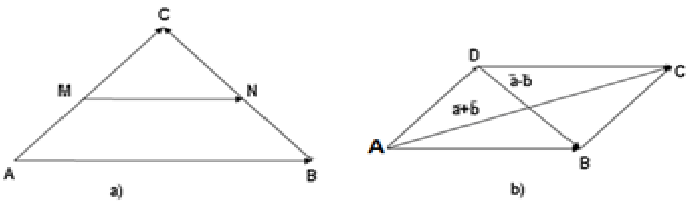


Figure 2.1

From the third condition in the definition of affine space, we have

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} \Rightarrow 2\overrightarrow{MC} = \overrightarrow{AB} + 2\overrightarrow{NC}$$
$$\Rightarrow 2\overrightarrow{MC} - 2\overrightarrow{NC} = \overrightarrow{AB}$$
$$\Rightarrow 2(C-M) - 2(C-N) = \overrightarrow{AB}$$
$$\Rightarrow 2\overrightarrow{MN} = \overrightarrow{AB} \Rightarrow \overrightarrow{MN} = \tfrac{1}{2}\overrightarrow{AB}$$

Besides, since $\overrightarrow{MN} = \tfrac{1}{2}\overrightarrow{AB}$, $\overrightarrow{MN}$ is a scalar multiple of $\overrightarrow{AB}$ with scalar $t = \tfrac{1}{2}$. Hence, $\overrightarrow{MN} \,//\, \overrightarrow{AB}$.

2. The diagonals of a rhombus are perpendicular.

Proof: Let $ABCD$ be a rhombus with diagonals $\overrightarrow{AC}$ and $\overrightarrow{BD}$ as shown in figure 2.1b above. To show that the diagonals $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are perpendicular, it suffices to show that their dot product is zero. Let $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$ such that $\overrightarrow{AC} = \vec{a} + \vec{b}$, $\overrightarrow{DB} = \vec{a} - \vec{b}$.

Besides, since the four sides of a rhombus are congruent $\overline{AB} = \|\vec{a}\| = \|\vec{b}\|$.

Thus, $\overrightarrow{AC}.\overrightarrow{DB} = (\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} - \vec{b}\cdot\vec{b} = \|\vec{a}\|^2 - \|\vec{b}\|^2 = 0$.

Hence the diagonals are perpendicular to each other.

3. **The cosine laws:** In ΔABC, suppose the lengths of the sides of the triangle opposite to the angles $\angle A, \angle B, \angle C$ respectively are a, b, c. Then,

$$a^2 = b^2 + c^2 - 2bc\cos\angle A$$
$$b^2 = a^2 + c^2 - 2ac\cos\angle B$$
$$c^2 = a^2 + b^2 - 2ab\cos\angle C$$

Proof: Consider figure 2.2b. In this diagram, the arrow indicates in which direction we considered the vectors along the sides so that to determine what angle between the vectors to be taken. Because angle between two vectors is the angle formed by the vectors when they share (made to share) the same initial points or tails. For instance, the angle between the vectors $\overrightarrow{AC}$ and $\overrightarrow{CB}$ is θ given by $\theta = \pi - \angle C$

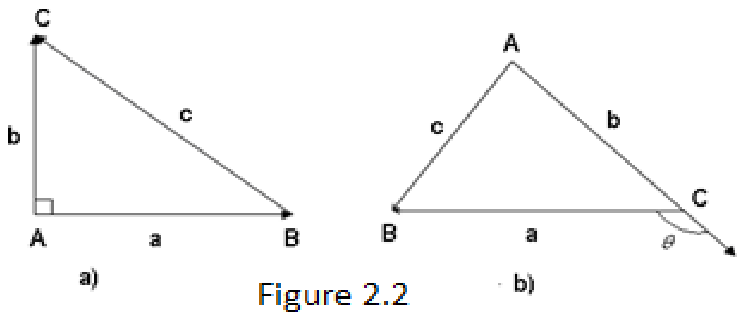


Figure 2.2

Here, we have

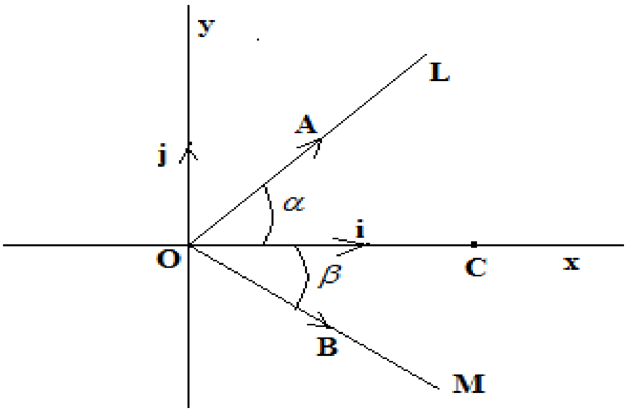
$$\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB} \Rightarrow (\overrightarrow{AC} + \overrightarrow{CB}).(\overrightarrow{AC} + \overrightarrow{CB}) = \overrightarrow{AB}.\overrightarrow{AB}$$
$$\Rightarrow \overrightarrow{AC}^2 + \overrightarrow{CB}^2 + 2\overrightarrow{AC}.\overrightarrow{CB} = \overrightarrow{AB}^2$$
$$\Rightarrow b^2 + a^2 + 2\|\overrightarrow{AC}\|\|\overrightarrow{CB}\|\cos\theta = c^2$$
$$\Rightarrow b^2 + a^2 + 2ab\cos(\pi - \angle C) = c^2$$
$$\Rightarrow c^2 = a^2 + b^2 - 2ab\cos\angle C, \quad \cos(\pi - \angle C) = -\cos\angle C$$

Similarly, the other identities can also be derived.

4. **Sum and Difference Formula of Trigonometric:** Let α and β be any two angles. Then,

$$i)\ \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$ii)\ \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$iii)\ \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$iv)\ \cos(\alpha-\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

Proof: Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be two unit vectors along the lines $\overleftrightarrow{OL}$ and $\overleftrightarrow{OM}$ making angles of α and β with the x-axis respectively as shown in the diagram below.



Here;

$a)\ \overrightarrow{OA}=\cos\alpha.\mathbf{i}+\sin\alpha.\mathbf{j},\quad \overrightarrow{OB}=\cos\beta.\mathbf{i}-\sin\beta.\mathbf{j}$

$b)\ \angle MOL=\angle MOC+\angle COL=\alpha+\beta$

Now, to prove (i), let's proceed as follow.

$$\begin{aligned}\overrightarrow{OA}\times\overrightarrow{OB}&=(\cos\alpha.\mathbf{i}+\sin\alpha.\mathbf{j})\times(\cos\beta.\mathbf{i}-\sin\beta.\mathbf{j})\\&=\cos\alpha\cos\beta(\mathbf{i}\times\mathbf{i})-\cos\alpha\sin\beta(\mathbf{i}\times\mathbf{j})+\sin\alpha\cos\beta(\mathbf{j}\times\mathbf{i})-\sin\alpha\sin\beta(\mathbf{j}\times\mathbf{j})\\&=-\cos\alpha\sin\beta(\mathbf{i}\times\mathbf{j})+\sin\alpha\cos\beta(\mathbf{j}\times\mathbf{i}),\ \ [\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=0]\\&=\cos\alpha\sin\beta(\mathbf{j}\times\mathbf{i})+\sin\alpha\cos\beta(\mathbf{j}\times\mathbf{i}),\ \ [\mathbf{i}\times\mathbf{j}=-\mathbf{j}\times\mathbf{i}]\\&=[\cos\alpha\sin\beta+\sin\alpha\cos\beta](\mathbf{j}\times\mathbf{i})\\&=[\cos\alpha\sin\beta+\sin\alpha\cos\beta]\mathbf{k},\qquad[\ \mathbf{j}\times\mathbf{i}=-\mathbf{k}]\ldots\ldots\ldots\ldots(i)\end{aligned}$$

On the other hand, using $\left\|\overrightarrow{OA}\right\|=\left\|\overrightarrow{OB}\right\|=1, \angle MOL=\alpha+\beta$, we have

$$\overrightarrow{OA}\times\overrightarrow{OB}=\left\|\overrightarrow{OA}\right\|.\left\|\overrightarrow{OB}\right\|\sin\angle MOL.\mathbf{k}=\sin(\alpha+\beta)\mathbf{k}\ldots\ldots\ldots\ldots(ii)$$

Thus, from (i) and (ii), we have

$$\overrightarrow{OA}\times\overrightarrow{OB}=\left\|\overrightarrow{OA}\right\|.\left\|\overrightarrow{OB}\right\|\sin\angle MOL.\mathbf{k}=\sin(\alpha+\beta)\mathbf{k}=[\cos\alpha\sin\beta+\sin\alpha\cos\beta]\mathbf{k}$$

$$\Rightarrow\sin(\alpha+\beta)=\cos\alpha\sin\beta+\sin\alpha\cos\beta=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

To prove part (ii), let's appy dot product.

That is $\overrightarrow{OA}.\overrightarrow{OB} = (\cos\alpha.\mathbf{i} + \sin\alpha.\mathbf{j}).(\cos\beta.\mathbf{i} - \sin\beta.\mathbf{j}) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

On the other hand, $\overrightarrow{OA}.\overrightarrow{OB} = \left\|\overrightarrow{OA}\right\|.\left\|\overrightarrow{OB}\right\|\cos\angle MOL = \cos(\alpha+\beta)$

Combining the two equations, gives us $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

5. **The sine Law:** Suppose the sides of ΔABC are represented by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ where $\left\|\vec{a}\right\| = a, \left\|\vec{b}\right\| = b, \left\|\vec{c}\right\| = c$. Then, $\dfrac{a}{\sin\angle A} = \dfrac{b}{\sin\angle B} = \dfrac{c}{\sin\angle C}$.

Proof: Consider ΔABC with $\angle A = \alpha,\ \angle B = \beta,\ \angle C = \gamma$ as shown in the diagram.

Now, observe that:

i) The angle between the vectors $\vec{a}$ and $\vec{b}$ is $\pi - \alpha$. (Do you see why?)

ii) The angle between the vectors $\vec{a}$ and $\vec{c}$ is $\pi - \beta$. (Do you see why β?)

iii) The angle between the vectors $\vec{b}$ and $\vec{c}$ is $\pi - \gamma$. (Do you see why?)

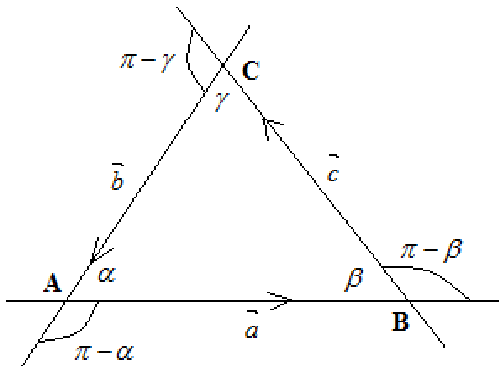
iv) For any angle θ, $\sin(\pi - \theta) = \sin\theta$ (Difference formula of angles)

Besides, by parallelogram law $\vec{a} + \vec{b} + \vec{c} = \vec{\mathbf{0}}$ (zero vector).

But $\vec{a} + \vec{b} + \vec{c} = \vec{\mathbf{0}} \Rightarrow \vec{a} + \vec{b} = -\vec{c}$. Here, if we cross by $\vec{a}$ both sides, we get

$\vec{a} \times (\vec{a} + \vec{b}) = \vec{a} \times -\vec{c} \Rightarrow \vec{a} \times \vec{a} + \vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$

$$\Rightarrow \vec{a} \times \vec{b} = \vec{c} \times \vec{a},\ \ [\text{because } \vec{a} \times \vec{a} = 0, -\vec{a} \times \vec{c} = \vec{c} \times \vec{a}]$$



Similarly, if we cross by $\vec{b}$ both sides, we get

$\vec{b} \times (-\vec{c}) = \vec{b} \times (\vec{a} + \vec{b}) \Rightarrow -\vec{b} \times \vec{c} = \vec{b} \times \vec{a} + \vec{b} \times \vec{b} \Rightarrow -\vec{b} \times \vec{c} = -\vec{a} \times \vec{b} \Rightarrow \vec{b} \times \vec{c} = \vec{a} \times \vec{b}$

From the two results, we get that $\vec{b}\times\vec{c}=\vec{c}\times\vec{a}=\vec{a}\times\vec{b}$.

But if two vectors are equal, then their norms are equal and if the angle between any two non-zero vectors $\vec{a}$ and $\vec{b}$ is θ, then

$\left\|\vec{a}\times\vec{b}\right\|=\left\|\vec{a}\right\|\left\|\vec{b}\right\|\sin\theta, 0\le\theta\le\pi$.

Hence,

$$\begin{aligned}
\vec{b}\times\vec{c}=\vec{c}\times\vec{a}=\vec{a}\times\vec{b} &\Rightarrow \left\|\vec{b}\times\vec{c}\right\|=\left\|\vec{c}\times\vec{a}\right\|=\left\|\vec{a}\times\vec{b}\right\| \\
&\Rightarrow \left\|\vec{b}\right\|.\left\|\vec{c}\right\|\sin(\pi-\gamma)=\left\|\vec{c}\right\|.\left\|\vec{a}\right\|\sin(\pi-\beta)=\left\|\vec{a}\right\|.\left\|\vec{b}\right\|\sin(\pi-\alpha) \\
&\Rightarrow b.c.\sin\angle A=c.a\sin\angle B=a.b.\sin\angle C \\
&\Rightarrow \frac{1}{a.b.c}(b.c.\sin\angle A=c.a\sin\angle B=a.b.\sin\angle C) \\
&\Rightarrow \frac{\sin\angle A}{a}=\frac{\sin\angle B}{b}=\frac{\sin\angle C}{c}\Rightarrow\frac{a}{\sin\angle A}=\frac{b}{\sin\angle B}=\frac{c}{\sin\angle C}
\end{aligned}$$

In any ΔABC, the result $\dfrac{a}{\sin\angle A}=\dfrac{b}{\sin\angle B}=\dfrac{c}{\sin\angle C}$ is known as *The Sine Law*.

6. Using vectors, prove Pythagoras Theorem.

Proof: Consider ΔABC with right angle at vertex A. We need to show $c^2=a^2+b^2$.

From figure 2.2a, $\left\|\overrightarrow{AB}\right\|=a$, $\left\|\overrightarrow{AC}\right\|=b$, $\left\|\overrightarrow{BC}\right\|=c$, $\overrightarrow{AC}.\overrightarrow{AB}=0$ and

$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}\Rightarrow\overrightarrow{BC}=\overrightarrow{AC}-\overrightarrow{AB}$. Thus,

$$\begin{aligned}
c^2=\left\|\overrightarrow{BC}\right\|^2 &=\overrightarrow{BC}.\overrightarrow{BC} \\
&=(\overrightarrow{AC}-\overrightarrow{AB}).(\overrightarrow{AC}-\overrightarrow{AB}) \\
&=\overrightarrow{AC}^2-2\overrightarrow{AB}.\overrightarrow{AC}+\overrightarrow{AB}^2 \\
&=\overrightarrow{AC}^2+\overrightarrow{AB}^2, \qquad \overrightarrow{AC}.\overrightarrow{AB}=0 \\
&=a^2+b^2
\end{aligned}$$

2.3 Lines and Planes in Affine space

From Euclidean geometry, we know that any two distinct points determine a unique line whose equation can be determined using the concepts in coordinate geometry. But, in affine space lines and planes are defined using the concept of vectors. So our next discussion focuses on analysis of lines and planes in affine space using vectors.

2.3.1 Lines in Affine Geometry

Definition: Let W be an affine space. Then, a line in W through two different points A and B is defined as the set $< A, B >= \{X \in W : \overrightarrow{AX} = t\overrightarrow{AB}, t \in R\}$

or $l : X = A + t\overrightarrow{AB}, t \in R$. Here, the vector $\overrightarrow{AB}$ is called direction *vector* of the line and the scalar t is called parameter. Now, let l be any line through the points A and B in affine space. Then, $l :< A, B >= \{X \in W : \overrightarrow{AX} = t\overrightarrow{AB}, t \in R\}$.

Letting $X = (x, y, z),\ A = (x_0, y_0, z_0),\ \vec{d} = \overrightarrow{AB} = (a, b, c)$, we get that

$l : (x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$ (This is called vector equation of the line).

Parametric and Symmetric Equations of a line:

Equating corresponding components from this equation, we have $\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$

(This is called *Parametric Equation*).

Again, from the parametric equation (whenever a, b, c are all non zero), solving for the parameter t gives us, $\begin{cases} x = x_0 + ta \Rightarrow t = \dfrac{x - x_0}{a} \\ y = y_0 + tb \Rightarrow t = \dfrac{y - y_0}{b} \\ z = z_0 + tc \Rightarrow t = \dfrac{z - z_0}{c} \end{cases}$

Equating these values of t implies $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$ (This is known as the *Symmetric Equation*).

Remarks: The case when one of a, b, c is zero gives the following equations.

i) If $a = 0, x = x_0, \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$

ii) If $b = 0, y = y_0, \dfrac{x - x_0}{a} = \dfrac{z - z_0}{c}$

iii) If $c = 0, z = z_0, \dfrac{x - x_0}{a} = \dfrac{y - y_0}{b}$

Examples: Give the vector, parametric and symmetric equations of the line ℓ that passes:

a) Through the points $A(2,-1,1)$ *and* $B(1,3,-2)$

b) Through the point $A(1,3,0)$ and parallel to the vector $\vec{u} = 2i - 5j + 7k$

c) Through the point $A(2,3,-5)$ and parallel to the vector $\vec{u} = 4j + 8k$

Solution:

$a)$ VE : $l : X = A + t\overrightarrow{AB} = (2,-1,1) + t(-1,4,-3)$,

$$\text{Parametric Equations}: \begin{cases} x = 2 - t \\ y = -1 + 4t, \\ z = 1 - 3t \end{cases} \text{ and Symmetric Equation}: \frac{x-2}{-1} = \frac{y+1}{4} = \frac{z-1}{-3}$$

$b)$ Vector equation (VE) : $l : X = (1,3,0) + t(2,-5,7)$

$$\text{Parametric Equations (PE)}: \begin{cases} x = 1 + 2t \\ y = 3 - 5t \\ z = 7t \end{cases} \text{ and Symmetric equation (SE)}: \frac{x-1}{2} = \frac{y-3}{-5} = \frac{z}{7}$$

$$c)\ \text{VE}: l : X = (2,3,-5) + t(0,4,8), \text{PE}: \begin{cases} x = 2 \\ y = 3 + 4t, \\ z = -5 + 8t \end{cases} \text{ SE}: x = 2; \frac{y-3}{4} = \frac{z+5}{8}$$

Note that the vector and or the parametric equations of a line are not unique. That means we can find many parametric equations for the same line.

Proposition 2.3:

a) There is a unique line through any two points in affine space.

b) Any two different direction vectors of a line are parallel.

Proof: (Follows from the definition)

Parallel and Perpendicular Lines in Affine Space:

Definition: Any two lines $<A,B>$ and $<C,D>$ are said to be *parallel* if their direction vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel and they are said to be perpendicular if their direction vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are perpendicular. Besides, the angle between these lines is the same as the angle between their direction vectors.

Examples:

1. Determine whether the following pairs of lines are parallel, perpendicular or neither and for these which are neither find the angle between them.

a) $l: X = (1,2,3)+t(1,2,-1)$ and $m: X = (1,0,1)+t(-3,-6,3)$

b) The lines through $A=(1,3,5), B=(4,7,5)$ and $C=(5,-2,2), D=(1,1,7)$

c) The lines through $A=(2,-1,4), B=(2,-2,5)$ and $C=(3,4,3), D=(3,5,3)$

Solution:

a) Here, $\vec{u}=(1,2,-1),\ \vec{v}=(-3,-6,3) \Rightarrow \vec{v}=-3\vec{u} \Rightarrow \vec{u}\,//\,\vec{v}$.

Hence, from the definition, the lines are also *parallel.*

b) In this case, $\vec{u}=B-A=(3,4,0),\ \vec{v}=D-C=(-4,3,5) \Rightarrow \vec{v}.\vec{u}=0 \Rightarrow \vec{u}\perp\vec{v}$.

Hence, the lines are also *perpendicular*.

c) $\vec{u}=B-A=(0,-1,1),\ \vec{v}=D-C=(0,1,0)$. But those vectors are neither parallel nor perpendicular and so are the lines through these points. Let θ be the angle between the lines. Then, $\cos\theta=\dfrac{\vec{u}.\vec{v}}{\|\vec{u}\|\|\vec{v}\|}=\dfrac{-1}{\sqrt{2}} \Rightarrow \theta=\cos^{-1}(\dfrac{-1}{\sqrt{2}})=\dfrac{3\pi}{4}$.

Or it could be $\theta=\pi-\dfrac{3\pi}{4}=\dfrac{\pi}{4}$.

2. For what value of k are the line $l: x=2t, y=1-3t, z=-2-7t$ and the line $m: X=(2,3,-5)+r(3,k,-3)$ perpendicular in Affine space?

Solution: The direction vectors of the lines are $\vec{u}=2i-3j-7k, \vec{v}=3i+kj-3k$. Then, the two lines will be perpendicular if their direction vectors are perpendicular. So, $\vec{u}\perp\vec{v} \Rightarrow \vec{u}.\vec{v}=0 \Rightarrow 6-3k+21=0 \Rightarrow k=9$.

3. Find the value of a if the cosine value of the angle between the lines $l: x = y = z$ and $m: x = 1+t,\ y = at+5,\ \ z = 4+2t$ is $\frac{1}{\sqrt{3}}$.

Solution: The angle between the two lines is the same as the angle between their direction vectors $\vec{u} = i + j + k,\ \vec{v} = i + aj + 2k$

$$\text{So, } \cos\theta = \frac{1}{\sqrt{3}} \Rightarrow \frac{\vec{u}.\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{1}{\sqrt{3}} \Rightarrow \frac{a+3}{\sqrt{3}\sqrt{a^2+5}} = \frac{1}{\sqrt{3}} \Rightarrow a+3 = \sqrt{a^2+5}$$

$$\Rightarrow (a+3)^2 = a^2+5 \Rightarrow a^2+6a+9 = a^2+5 \Rightarrow a = -2/3$$

4. Let $l :< A, B >$ and $m :< C, D >$ be two non-perpendicular intersecting lines in affine space. Then, verify that $\tan\theta = \frac{\|\overrightarrow{AB} x \overrightarrow{CD}\|}{\overrightarrow{AB} \cdot \overrightarrow{CD}}$ where θ is the angle between the lines.

Proof: From the above definition, the angle between two lines in Affine Space is the same as the angle between their direction vectors. Thus, as θ is the angle between the lines, it is also θ is the angle between their direction vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Besides, from vector analysis we know that $\|\overrightarrow{AB} \times \overrightarrow{CD}\| = \|\overrightarrow{AB}\|\|\overrightarrow{CD}\|\sin\theta$ and $\overrightarrow{AB} \cdot \overrightarrow{CD} = \|\overrightarrow{AB}\|\|\overrightarrow{CD}\|\cos\theta$. Thus dividing the first relation by the second yields, $\frac{\|\overrightarrow{AB} \times \overrightarrow{CD}\|}{\overrightarrow{AB} \cdot \overrightarrow{CD}} = \frac{\|\overrightarrow{AB}\|\|\overrightarrow{CD}\|\sin\theta}{\|\overrightarrow{AB}\|\|\overrightarrow{CD}\|\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$.

Proposition 2.4: Two Lines $l: X = A + ru$ and $m: X = A + tv$ passing through the same point P are equal (identical) if and only if u and v are parallel.

Proof: Suppose $l = m$ and let $B \in l = m,\ B \neq A$. Then,

$$B = A + ru,\ B = A + tv \Rightarrow A + ru = A + tv$$
$$\Rightarrow ru = tv \Rightarrow u = \tfrac{t}{r}v, [\therefore\ B \neq A \Rightarrow r \neq 0]$$
$$\Rightarrow u // v$$

Conversely, let u and v be parallel. Then, there exists a scalar $k \neq 0$ such that $u = kv$. Let X be arbitrary point on line l.

Thus, $X \in l \Leftrightarrow X = A + ru \Leftrightarrow X = A + r(kv)$

$$\Leftrightarrow X = A + tv,\ t = rk \Leftrightarrow X \in m \Leftrightarrow l = m$$

Proposition 2.5: Given a line l and a point Q not on l. Then, there is exactly one line m through Q parallel to l.

Proof: Let $l: X = P + tu,$ then $m: X = Q + tu$ for any point Q is parallel to l (because by definition any two lines with the same direction vectors are parallel). This proves the existence of parallel line to l.

Now suppose there is another line n through Q parallel to l. In this case, it is given by $n: X = Q + rv,\ m // n \Rightarrow u // v$.

Therefore, from $m: X = Q + tu$ and $n: X = Q + rv$ with u parallel to v we get $m = n$ by the previous proposition.

Theorem 2.1(Length Ratio Theorem): Let l be a line passing through two different points A and B given by $l: P = A + r\overrightarrow{AB},\ r \in R$. Then,

$$a)\ |r| = \frac{AP}{AB} \qquad b)\ \left|\frac{r}{1-r}\right| = \frac{AP}{PB},\ if\ \ P \neq B$$

$c)$ P is between A and B if $0 < r < 1$

$d)$ B is between A and P if $r > 1$

Proof: a) From vector equation of the line $l: P = A + r\overrightarrow{AB},\ r \in R$, we have

$$P - A = r\overrightarrow{AB},\ r \in R \Rightarrow \overrightarrow{AP} = r\overrightarrow{AB} \Rightarrow \left\|\overrightarrow{AP}\right\| = \left\|r\overrightarrow{AB}\right\| = |r|\left\|\overrightarrow{AB}\right\| \Rightarrow |r| = \frac{\left\|\overrightarrow{AP}\right\|}{\left\|\overrightarrow{AB}\right\|} = \frac{AP}{AB}$$

b) For any point P on a line determined by two distinct points A and B, $\overrightarrow{AB} = \overrightarrow{AP} + \overrightarrow{PB}$. Besides, from equation of the given line we have $\overrightarrow{AP} = r\overrightarrow{AB}$ which gives us $\overrightarrow{AB} = \frac{1}{r}\overrightarrow{AP}$. From these relations, we get

$$\frac{1}{r}\overrightarrow{AP} = \overrightarrow{AP} + \overrightarrow{PB} \Rightarrow (1-r)\overrightarrow{AP} = r\overrightarrow{PB}$$

$$\Rightarrow \left\|(1-r)\overrightarrow{AP}\right\| = \left\|r\overrightarrow{PB}\right\|$$

$$\Rightarrow |1-r|AP = |r|PB \Rightarrow \frac{AP}{PB} = \left|\frac{r}{1-r}\right|$$

c) Here, we use the fact that for any vector $\vec{a}$ and a positive constatnt k, $\vec{a}$ and $k\vec{a}$ have the same direction and $\left\|k\vec{a}\right\| = k\left\|\vec{a}\right\|$.

For $0 < r < 1,\ \overrightarrow{AP} = r\overrightarrow{AB} \Rightarrow \left\|\overrightarrow{AP}\right\| = r\left\|\overrightarrow{AB}\right\| < \left\|\overrightarrow{AB}\right\| \Rightarrow AP = rAB < AB$.

This shows P is between A and B. The rest parts of the theorem can be derived from what we did here.

Examples:

1. If $A=(5,0,7)$ and $B=(2,-3,6)$, then find a point P on the line $<A,B>$ which satisfies $\dfrac{AP}{PB}=3.$

Solution: From part (b) of Length Ratio Theorem, on the line $l: P=A+r\overrightarrow{AB}$ if $P\neq B$, we have $\left|\dfrac{r}{1-r}\right|=\dfrac{AP}{PB}$.

But we need to find a point P which satisfies $\dfrac{AP}{PB}=3.$

Thus, $\left|\dfrac{r}{1-r}\right|=\dfrac{AP}{PB}=3 \Rightarrow \dfrac{r}{1-r}=\pm 3 \Rightarrow r=\dfrac{3}{4}\ or\ r=\dfrac{3}{2}$. Besides, any point P on the line determined by A and B is given by $l: P=A+r\overrightarrow{AB},\ r\in R.$

Using $r=\dfrac{3}{4}$, we get $P=A+\dfrac{3}{4}\overrightarrow{AB}=(5,0,7)+\dfrac{3}{4}(-3,-3,-1)=(\dfrac{11}{4},\dfrac{-9}{4},\dfrac{25}{4}).$

Using $r=\dfrac{3}{2}$, we get $P=A+\dfrac{3}{2}\overrightarrow{AB}=(5,0,7)+\dfrac{3}{2}(-3,-3,-1)=(\dfrac{1}{2},\dfrac{-9}{2},\dfrac{11}{2})$.

2. **Application of Line Equation in Affine Space:** Suppose an Engineer wants to paint a rectangular portion $ABCD$ of a building using white and yellow colors where two of the vertices are $A=(2,3,-1)$ and $B=(3,7,4)$. If he divides the region into two sub-rectangles by a line through P (on the segment $\overline{AB}$) and Q (on the segment $\overline{CD}$) so that to paint *white color* the region $APQD$ and *yellow color* the region $PBCQ$ where the ratio of the length painted white to the length painted yellow is known to be $\dfrac{AP}{PB}=\dfrac{3}{2}$. Find the coordinates of P where he should draw the dividing line to Q.

Solution: From Length Ratio Theorem, we have $\left|\dfrac{r}{1-r}\right|=\dfrac{AP}{PB}$. But we need to find a point P which satisfies $\dfrac{AP}{PB}=\dfrac{3}{2}.$

Thus, $\left|\frac{r}{1-r}\right| = \frac{AP}{PB} = \frac{3}{2} \Rightarrow \frac{r}{1-r} = \pm\frac{3}{2} \Rightarrow 2r = \pm 3(1-r)$

$$\Rightarrow 2r = 3-3r \text{ or } 2r = 3r-3 \Rightarrow r = 3/5 \text{ or } r = 3$$

Since we need a point P between A and B, only $r = 3/5$ is valid by part (c) of

Length Ratio Theorem. Besides, any point P on the line determined by A and B is given by $l: P = A + r\overrightarrow{AB},\ r \in R.$

Hence, using $r = \frac{3}{5}$, we get $P = A + \frac{3}{5}\overrightarrow{AB} = (2,3,-1) + \frac{3}{5}(1,4,5) = (\frac{13}{5}, \frac{27}{5}, 2).$

Here, if we use $r = 3$, we get $P = A + 3\overrightarrow{AB} = (2,3,-1) + 3(1,4,5) = (5,15,14).$ But we didn't do so. (Verify why this value of P is not used!).

Theorem 2.2 (Distance in affine space)

Let l be a line through A and B. If C is any point not on the given line l, then there is exactly one point P on the line l such that $\overrightarrow{CP}$ is perpendicular to $\overrightarrow{AB}$ given by $P = A + t\overrightarrow{AB},$ where $t = \frac{\overrightarrow{AC}.\overrightarrow{AB}}{\|AB\|}$.

Moreover, if Q is any other point on the line, then $CQ \geq CP$ and hence P is the point on l closest to C. Besides, the shortest distance from the line to the point C is given by $d = CP = \frac{\sqrt{AC^2 AB^2 - (\overrightarrow{AC}.\overrightarrow{AB})^2}}{AB} = \frac{\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\|}{AB}$

Proof: Let $P = A + t\overrightarrow{AB}$ and assume $\overrightarrow{CP}$ is perpendicular to $\overrightarrow{AB}$ as shown in figure 2.3 below.

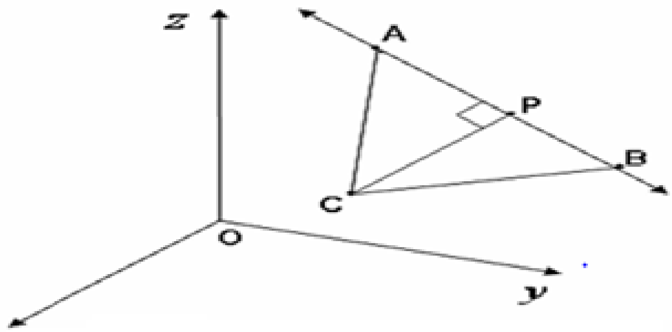


Figure 2.3 Distance in affine space

Then, $\overrightarrow{CP}.\overrightarrow{AB} = 0 \Rightarrow (P - C).\overrightarrow{AB} = 0$

$$\Rightarrow (A + t\overrightarrow{AB} - C).\overrightarrow{AB} = 0$$
$$\Rightarrow (\overrightarrow{CA} + t\overrightarrow{AB}).\overrightarrow{AB} = 0$$
$$\Rightarrow \overrightarrow{CA}.\overrightarrow{AB} + t(\overrightarrow{AB}.\overrightarrow{AB}) = 0$$
$$\Rightarrow -\overrightarrow{AC}.\overrightarrow{AB} + t(\overrightarrow{AB}.\overrightarrow{AB}) = 0$$
$$\Rightarrow t = \frac{\overrightarrow{AC}.\overrightarrow{AB}}{AB^2}, \qquad (AB^2 = \|AB\|^2)$$

The uniqueness of point P follows from the unique value of the parameter t.

Now for any other point Q, the inequality $CQ \geq CP$ follows from Pythagoras theorem. Because if Q is any other point on the line (on the either side of P), then $CQ^2 = CP^2 + PQ^2 \Rightarrow CQ^2 \geq CP^2 \Rightarrow CQ \geq CP.$ Equality holds when $P = Q$.

Finally, from the diagram as the vector $\overrightarrow{CP}$ is perpendicular to $\overrightarrow{PA}$, using Pythagoras theorem we get,

$$CP^2 = AC^2 - AP^2 = AC^2 - \left\| t\overrightarrow{AB} \right\|^2, \; P = A + t\overrightarrow{AB} \Rightarrow \overrightarrow{AP} = t\overrightarrow{AB}$$

$$= AC^2 - \left(\frac{\overrightarrow{AC}.\overrightarrow{AB}}{AB^2} \right)^2 AB^2$$

$$= \frac{AC^2.AB^2 - (\overrightarrow{AC}.\overrightarrow{AB})^2}{AB^2}$$

$$= \frac{\|AC\|^2 \|AB\|^2 - (\|AC\|\|AB\|\cos\theta)^2}{AB^2}$$

$$= \frac{\|AC\|^2 \|AB\|^2 [1 - \cos^2\theta]}{AB^2}$$

$$= \frac{\|AC^2\|\|AB\|^2 \sin^2\theta}{AB^2}, \; 1 - \cos^2\theta = \sin^2\theta$$

$$= \frac{(\|AC\|\|AB\|\sin\theta)^2}{AB^2}$$

$$= \frac{\left\| \overrightarrow{AC} \times \overrightarrow{AB} \right\|^2}{AB^2}$$

Taking square root both sides gives the required result.

Examples:

1. Let l be a line through the points $A = (-2,1,3)$ and $B = (1,2,4)$. Find a point P on this line which is closest to the origin and calculate the shortest distance from the line to the origin.

Solution: From the above theorem, the closest point on the line to a given point C (not on the line) is given by $P = A + t\overrightarrow{AB}$, where $t = \dfrac{\overrightarrow{AC}.\overrightarrow{AB}}{AB^2}$.

In our case, $A = (-2,1,3)$, $B = (1,2,4)$ and $C = (0,0,0)$. Thus,

$$\overrightarrow{AC} = <2,-1,-3>, \; \overrightarrow{AB} = <3,1,1>, \; \|AB\| = \sqrt{11} \Rightarrow t = \frac{<2,-1,-3>.<3,1,1>}{11} = \frac{2}{11}$$

$$\Rightarrow P = (-2,1,3) + \frac{2}{11} <3,1,1> = (-\frac{16}{11}, \frac{13}{11}, \frac{35}{11})$$

$$d = CP = \frac{\sqrt{AC^2 AB^2 - (\overrightarrow{AC}.\overrightarrow{AB})^2}}{AB} \Rightarrow d = \frac{\sqrt{154 - 4}}{\sqrt{11}} = \sqrt{\frac{150}{11}}$$

2. Find a point on the line $l: X = (1,2,3) + r(2,3,4)$ closest to $Q = (5,9,3)$.

Solution: From the above theorem, the closest point on the line to a given point Q is given by $P = A + t\overrightarrow{AB}$, where $t = \dfrac{\overrightarrow{AQ}.\overrightarrow{AB}}{AB^2}$ and $\overrightarrow{AB}$ is any direction vector of the line. In our case, $A = (1,2,3)$ and $\overrightarrow{AB} = (2,3,4)$.

Thus, $t = \dfrac{<4,7,0>.<2,3,4>}{29} = \dfrac{29}{29} = 1.$

Therefore, $P = A + t\overrightarrow{AB} \Rightarrow P = (1,2,3) + (2,3,4) = (3,5,7).$

2.3.2 Planes in Affine Space

Definition: Let W be an affine space.

Then, a plane π passing through three non-collinear points A, B, C is the set given by $<A, B, C> = \{X \in \pi : X = A + r\overrightarrow{AB} + t\overrightarrow{AC},\ r, t \in R\}$

From this equation, if we let $A = (x_0, y_0, z_0), \overrightarrow{AB} = (a,b,c), \overrightarrow{AC} = (d,e,f)$, then any arbitrary point $X = (x, y, z)$ on this plane is given by

$X = (x, y, z) = (x_0, y_0, z_0) + r(a,b,c) + t(d,e,f).$

This is known as vector equation of the plane. Now by equating components from the vector equation, we get $\begin{cases} x = x_0 + ra + td \\ y = y_0 + rb + te \\ z = z_0 + rc + tf \end{cases}$. This is called parametric equation of the plane where r, t are parameters.

Example: Find vector and parametric equations of a plane passing through the points $A = (1,2,3), B = (4,-1,-2), C = (1,0,-1)$.

Solution: First let's determine the vectors using the given points.

$\overrightarrow{AB} = (3,-3,-5), \overrightarrow{AC} = (0,-2,-4)$. Thus, the vector equation becomes,

$\pi :< A, B, C >= \{X \in \pi : X = (x, y, z) = (1,2,3) + r(3,-3,-5) + t(0,-2,-4)\}$

The parameter equation with parameters r, t becomes $\begin{cases} x = 1 + 3r \\ y = 2 - 3r - 2t \\ z = 3 - 5r - 4t \end{cases}$

Problem Set 2.1

1. Let $A=(3,10,9),\ B=(7,5,-1),\ D=(8,16,8)$. Show that $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are perpendicular. Find a fourth point D such that $ABCD$ forms a rectangle.
2. Show that R is an affine space over R.
3. Using vector method, prove that:

   a) An angle inscribed in a semicircle is right angle.

   b) The altitudes of a triangle are concurrent.

   c) The sum of the squares of the diagonals of a parallelogram is equal to twice the sum of the squares of its sides.
4. Using vector method, show that:

   a) $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

   b) $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
5. If $A=(0,4),\ B=(8,-2)$, find a point P on the line $\overleftrightarrow{AB}$ such that $\dfrac{AP}{PB}=\dfrac{1}{3}$.

   Answer : $P=(2,5/2)$
6. Show that a point P on the line $<A,B>$ is uniquely determined by the ratio $r=\dfrac{AP}{PB}$. If $A=(2,3,-1),\ B=(3,7,4)$, then find the coordinates of point P on $\overleftrightarrow{AB}$ such that $\dfrac{AP}{PB}=\dfrac{2}{5}$.
7. Let $A,\ B,\ C$ be points in affine space. Show that the set $\langle A,B,C\rangle$ is either a plane, a line or a single point.

   Particularly, if $A=B=C$, what is the geometric figure represented by the set $\langle A,B,C\rangle$?
8. If P is any point inside ΔABC, prove that $P=rA+sB+tC$ where $r+s+t=1$.

9. If P is any point inside ΔABC such that $P = 2rA - \frac{3}{4}B + \frac{1}{2}C$, then what must be the value of r?
10. Give the centroid of ΔDEF whose vertices are $D = (-3,5,6)$, $E = (-2,7,9)$ and $F = (2,1,7)$.
11. Let l be a line through $A = (7,1,9)$, $B = (-1,5,8)$. Show that the length of the projection of the segment CD is $L = \dfrac{17}{3}$ where $C = (1,2,3), D = (5,-2,6)$.
12. Find a point on the line $l : X = (1,2,3) + r(2,3,4)$ which is closest to $Q = (5,9,3)$. **Answer** : $(3,5,7)$
13. Let $l :< A, B >$ and $m :< C, D >$ be any two intersecting lines in affine space. Then, verify that $\cot\theta = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{\left\| \overrightarrow{AB} x \overrightarrow{CD} \right\|}$ where θ is the angle between the lines.

2.3.3 Collinearity in Affine Space

Definition: (Co-linearity in affine space): Any three points A, B, C in affine space are said to be collinear if and only if $A = aB + bC$ where $a + b = 1$.

Proposition 2.6: Suppose A, B, C, D are distinct points such that any three of them are not collinear. Then, $\overrightarrow{AB} = \overrightarrow{CD}$ if and only if $\overrightarrow{AB} // \overrightarrow{CD}$ and $\overrightarrow{AC} // \overrightarrow{BD}$.

Proof: If $\overrightarrow{AB} = \overrightarrow{CD}$, then $B - A = D - C \Rightarrow C - A = D - B \Rightarrow \overrightarrow{AC} = \overrightarrow{BD}$. But any two equal vectors are parallel and thus $\overrightarrow{AB} // \overrightarrow{CD}$ and $\overrightarrow{AC} // \overrightarrow{BD}$.

Conversely, suppose that $\overrightarrow{AB} // \overrightarrow{CD}$ and $\overrightarrow{AC} // \overrightarrow{BD}$. Then, for some scalars r and t, $\overrightarrow{AB} = r\overrightarrow{CD}$ and $\overrightarrow{AC} = t\overrightarrow{BD}$. This in turn implies, $B - A = r(D - C)$ and $C - A = t(D - B)$. Here, if we can show that either $r = 1$ or $t = 1$ we are done.

Now, subtracting the second relation from the first gives

$$B - C = r(D - C) - t(D - B) \Rightarrow (1 - t)B = (1 - r)C + (r - t)D$$

If we assume $t \neq 1$, we have $(1 - t)B = (1 - r)C + (r - t)D \Rightarrow B = \frac{1 - r}{1 - t}C + \frac{r - t}{1 - t}D$

This shows that $B = aC + bD$ with $a + b = 1$ where $a = \frac{1 - r}{1 - t}$, $b = \frac{r - t}{1 - t}$.

But this mean that B is on the line $\overleftrightarrow{CD}$ which implies that B, C, D are collinear but this is a contradiction with the hypothesis that any three of the points are not collinear. Hence, $t = 1$. Similarly one can get $r = 1$.

Therefore, $\overrightarrow{AC} = t\overrightarrow{BD} \Rightarrow C - A = D - B \Rightarrow B - A = D - C \Rightarrow \overrightarrow{AB} = \overrightarrow{CD}$.

Theorem 2.3 (The intercept Theorem):

Let A, B, C, D be non-collinear points such that $\overrightarrow{AB} // \overrightarrow{CD},\ \overrightarrow{AB} \neq \overrightarrow{CD}$. Then, the lines $< A, C >$ and $< B, D >$ intersect in a points O for which $\frac{\overrightarrow{OA}}{\overrightarrow{OC}} = \frac{\overrightarrow{OB}}{\overrightarrow{OD}} = \frac{\overrightarrow{AB}}{\overrightarrow{CD}}$ and $\frac{\overrightarrow{AC}}{\overrightarrow{OA}} = \frac{\overrightarrow{BD}}{\overrightarrow{OB}}$. Conversely, if the lines $< A, C >$ and $< B, D >$ intersect in a point O such that $\frac{\overrightarrow{OC}}{\overrightarrow{OA}} = \frac{\overrightarrow{OD}}{\overrightarrow{OB}}$, then $\overrightarrow{AB} // \overrightarrow{CD},\ \overrightarrow{AB} \neq \overrightarrow{CD}$.

Proof: Consider figure 2.4.

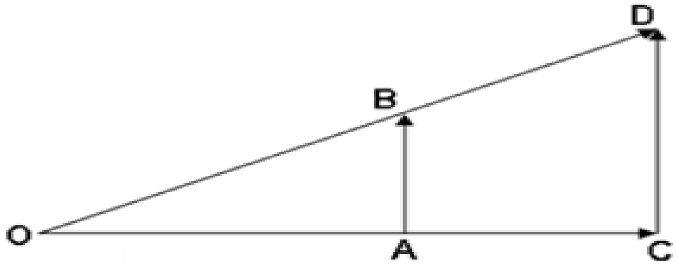


Figure 2.4

Since $\overrightarrow{AB} // \overrightarrow{CD},\ \overrightarrow{AB} \neq \overrightarrow{CD}$ the lines $< A, C >$ and $< B, D >$ intersect in a point. Otherwise, if they don't intersect, $ABDC$ becomes a parallelogram such that $\overrightarrow{AB} = \overrightarrow{CD}$ which is a contradiction. Now suppose they intersect at some point O (refer the figure above)and then consider ΔOAB and ΔOCD.In these triangles, $\angle OAB \cong \angle OCD,\ \angle OBA \cong \angle ODC$ (corresponding angles) as $\overrightarrow{AB} // \overrightarrow{CD}$.

Thus, $\Delta OAB \sim \Delta OCD$ by Angle-Angle similarity theorem.

As a result, $\frac{\overrightarrow{OA}}{\overrightarrow{OC}} = \frac{\overrightarrow{OB}}{\overrightarrow{OD}} = \frac{\overrightarrow{AB}}{\overrightarrow{CD}}$.

Besides, from this result,

$$\frac{\overrightarrow{OA}}{\overrightarrow{OC}} = \frac{\overrightarrow{OB}}{\overrightarrow{OD}} \Rightarrow \frac{\overrightarrow{OC}}{\overrightarrow{OA}} = \frac{\overrightarrow{OD}}{\overrightarrow{OB}}$$

$$\Rightarrow \frac{\overrightarrow{OC}}{\overrightarrow{OA}} - 1 = \frac{\overrightarrow{OD}}{\overrightarrow{OB}} - 1$$

$$\Rightarrow \frac{\overrightarrow{OC} - \overrightarrow{OA}}{\overrightarrow{OA}} = \frac{\overrightarrow{OD} - \overrightarrow{OB}}{\overrightarrow{OB}}$$

$$\Rightarrow \frac{\overrightarrow{AC}}{\overrightarrow{OA}} = \frac{\overrightarrow{BD}}{\overrightarrow{OB}}$$

Conversely, suppose the lines $< A, C >$ and $< B, D >$ intersect at some point O such that $\frac{\overrightarrow{OC}}{\overrightarrow{OA}} = \frac{\overrightarrow{OD}}{\overrightarrow{OB}}$. Let $\frac{\overrightarrow{OC}}{\overrightarrow{OA}} = \frac{\overrightarrow{OD}}{\overrightarrow{OB}} = k$. Then, $\overrightarrow{OC} = k\overrightarrow{OA}, \overrightarrow{OD} = k\overrightarrow{OB}$.

Thus, $\overrightarrow{OC} + \overrightarrow{CD} = \overrightarrow{OD} \Rightarrow \overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = k\overrightarrow{OB} - k\overrightarrow{OA} = k(\overrightarrow{OB} - \overrightarrow{OA})$.

On the other hand, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \Rightarrow \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

So, from these two results, $\overrightarrow{CD} = k(\overrightarrow{OB} - \overrightarrow{OA})$ and $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, we get $\overrightarrow{CD} = k\overrightarrow{AB}$. Therefore, $\overrightarrow{AB} // \overrightarrow{CD}$ besides $\overrightarrow{AB} \neq \overrightarrow{CD}$.

Otherwise, if we assume that $\overrightarrow{AB} = \overrightarrow{CD}$, then by Proposition 2.6, we get $\overrightarrow{AC} // \overrightarrow{BD}$ which in turn implies that the lines $< A, C >$ and $< B, D >$ are parallel. But this contradicts with the hypothesis.

Then, $\overrightarrow{AB} = \overrightarrow{CD}$ if and only if $\overrightarrow{AB} // \overrightarrow{CD}$ and $\overrightarrow{AC} // \overrightarrow{BD}$.

2.4 The Classical Theorems

Definition: Three points A, B, C are said to be collinear if and only if $\overrightarrow{AB} // \overrightarrow{BC}$. That means if all the lines lie on the same line it means. Two or more lines are said to be concurrent if and only if they pass through the same point. The common point at which the lines meet is called concurrency point or point of concurrency.

Now let's see the most useful theorems usually called ***Classical Theorems***.

Menelao's Theorem 2.4: Let D, E and F be points on the lines $< B,C >, < A,C >$ and $< A,B >$ of triangle ABC respectively. Then, the points D, E and F are collinear if and only if $\frac{\overrightarrow{BD}}{\overrightarrow{DC}} . \frac{\overrightarrow{CE}}{\overrightarrow{EA}} . \frac{\overrightarrow{AF}}{\overrightarrow{FB}} = -1$

Proof: Suppose D, E and F are points on the lines $< B,C >, < A,C >$ and $< A,B >$ respectively such that D, E and F are collinear.

We need to show that $\frac{\overrightarrow{BD}}{\overrightarrow{DC}} . \frac{\overrightarrow{CE}}{\overrightarrow{EA}} . \frac{\overrightarrow{AF}}{\overrightarrow{FB}} = -1$

Draw a line through B parallel to $\overrightarrow{AE}$ that intersects $\overline{DF}$ at a point P as shown in the figure 2.5 below.

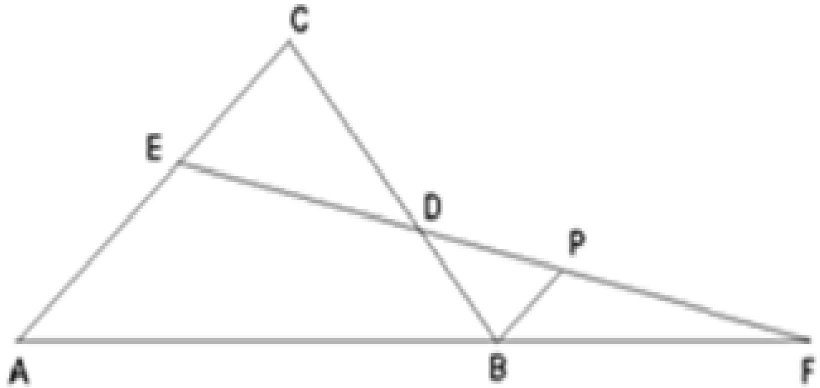


Figure 2.5

Since $\overrightarrow{BP} // \overrightarrow{AE}$, $\Delta DPB \sim \Delta DEC$, $\Delta FAE \sim \Delta FBP$ by Angle- Angle similarity theorem.

Using this result together with the intercept Theorem,

we get $\frac{\overrightarrow{DB}}{\overrightarrow{DC}} = \frac{\overrightarrow{PB}}{\overrightarrow{EC}}$ and $\frac{\overrightarrow{FA}}{\overrightarrow{FB}} = \frac{\overrightarrow{EA}}{\overrightarrow{PB}}$.

Multiplying these two equations, we get

$$\frac{\overrightarrow{DB}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{FA}}{\overrightarrow{FB}} = \frac{\overrightarrow{PB}}{\overrightarrow{EC}} \cdot \frac{\overrightarrow{EA}}{\overrightarrow{PB}} \Rightarrow \frac{\overrightarrow{DB}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{FA}}{\overrightarrow{FB}} \cdot \frac{\overrightarrow{EC}}{\overrightarrow{EA}} = 1$$

$$\Rightarrow \frac{\overrightarrow{DB}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{AF}}{\overrightarrow{FB}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}} = 1, \quad \overrightarrow{FA} = -\overrightarrow{AF},\ \overrightarrow{EC} = -\overrightarrow{CE}$$

$$\Rightarrow \frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{AF}}{\overrightarrow{FB}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}} = -1, \quad \overrightarrow{BD} = -\overrightarrow{DB}$$

Hence, co linearity of D, E and F implies that $\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}} \cdot \frac{\overrightarrow{AF}}{\overrightarrow{FB}} = -1$.

Suppose $\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}} \cdot \frac{\overrightarrow{AF}}{\overrightarrow{FB}} = -1$..$(i)$

We need to prove that the points D, E and F are collinear. Consider line $< D, E >$ and suppose that F is not on this line.

Assume that F' is another point on $< D, E >$ such that D, E and F' are collinear. Then, by the forward part

$$\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}} \cdot \frac{\overrightarrow{AF'}}{\overrightarrow{F'B}} = -1 \text{...}(ii)$$

So, from (i) and (ii) we get,

$$\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}} \cdot \frac{\overrightarrow{AF'}}{\overrightarrow{F'B}} = \frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{AF}}{\overrightarrow{FB}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}$$

$$\Rightarrow \frac{\overrightarrow{AF'}}{\overrightarrow{F'B}} = \frac{\overrightarrow{AF}}{\overrightarrow{FB}}$$

$$\Rightarrow \frac{F'-A}{B-F'} + 1 = \frac{F-A}{B-F} + 1$$

$$\Rightarrow \frac{B-A}{B-F'} = \frac{B-A}{B-F} \Rightarrow B - F' = B - F \Rightarrow F' = F$$

Therefore, if $\frac{\overrightarrow{BD}}{DC} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}} \cdot \frac{\overrightarrow{AF}}{\overrightarrow{FB}} = -1$, then the points D, E and F are collinear.

Ceva's Theorem 2.5: Let D, E and F be points on the lines $< B,C >$, $< A,C >$ and $< A,B >$ of triangle ABC respectively. Then, the lines $< A, D >$, $< B, E >$ and $< C,F >$ are concurrent if and only if $\frac{\overrightarrow{BD}}{\overrightarrow{DC}}.\frac{\overrightarrow{CE}}{\overrightarrow{EA}}.\frac{\overrightarrow{AF}}{\overrightarrow{FB}}=1$.

Proof: Suppose the lines $< A, D >$, $< B, E >$ and $< C,F >$ are concurrent at a point P. Draw lines through B and C which are parallel to the line $< A, D >$. Extend the line $< B, E >$ to a point Q and

$< C,F >$ to a point R (Refer figure 2.6 below).

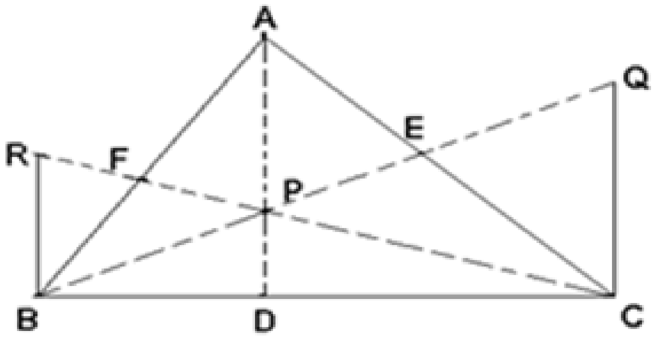


Figure 2.6

Since $\overrightarrow{AP} // \overrightarrow{QC}$, $\overrightarrow{AP} // \overrightarrow{RB}$ and $\overrightarrow{BR} // \overrightarrow{CQ}$, we have $\frac{\overrightarrow{CE}}{\overrightarrow{EA}}=\frac{\overrightarrow{CQ}}{\overrightarrow{PA}}$, $\frac{\overrightarrow{AF}}{\overrightarrow{FB}}=\frac{\overrightarrow{PA}}{\overrightarrow{BR}}$ and

$\frac{\overrightarrow{BD}}{\overrightarrow{DC}}=\frac{\overrightarrow{RP}}{\overrightarrow{PC}}=\frac{\overrightarrow{BR}}{\overrightarrow{CQ}}$.

Now, multiplying the three equations yields, $\frac{\overrightarrow{BD}}{\overrightarrow{DC}}.\frac{\overrightarrow{CE}}{\overrightarrow{EA}}.\frac{\overrightarrow{AF}}{\overrightarrow{FB}}=\frac{\overrightarrow{BR}}{\overrightarrow{CQ}}.\frac{\overrightarrow{CQ}}{\overrightarrow{PA}}.\frac{\overrightarrow{PA}}{\overrightarrow{BR}}=1$.

Conversely, suppose $\frac{\overrightarrow{BD}}{\overrightarrow{DC}}.\frac{\overrightarrow{CE}}{\overrightarrow{EA}}.\frac{\overrightarrow{AF}}{\overrightarrow{FB}}=1$.......................................(i)

Let the point P be the intersection of the lines $< A, D >$ and $< B, E >$. Assume the line $< C,F >$ does not pass through this point P. Then, we have another point F' on line $< A, B >$ such that the line $< C,F' >$ passes through the point P.

Then, by the forward part $\frac{\overrightarrow{BD}}{DC}.\frac{\overrightarrow{CE}}{\overrightarrow{EA}}.\frac{\overrightarrow{AF'}}{\overrightarrow{F'B}}=1$...............$(ii)$.

So, from (i) and (ii) we get,

$$\frac{\overrightarrow{BD}}{\overrightarrow{DC}}.\frac{\overrightarrow{CE}}{\overrightarrow{EA}}.\frac{\overrightarrow{AF'}}{\overrightarrow{F'B}}=\frac{\overrightarrow{BD}}{\overrightarrow{DC}}.\frac{\overrightarrow{AF}}{\overrightarrow{FB}}.\frac{\overrightarrow{CE}}{\overrightarrow{EA}}\Rightarrow\frac{\overrightarrow{AF'}}{\overrightarrow{F'B}}=\frac{\overrightarrow{AF}}{\overrightarrow{FB}}\Rightarrow\frac{\overrightarrow{AF'}}{\overrightarrow{F'B}}+1=\frac{\overrightarrow{AF}}{\overrightarrow{FB}}+1$$

$$\Rightarrow\frac{\overrightarrow{AF'}+\overrightarrow{F'B}}{\overrightarrow{F'B}}=\frac{\overrightarrow{AF}+\overrightarrow{FB}}{\overrightarrow{FB}}\Rightarrow\frac{\overrightarrow{AB}}{\overrightarrow{F'B}}=\frac{\overrightarrow{AB}}{\overrightarrow{FB}},\quad \overrightarrow{AF'}+\overrightarrow{F'B}=\overrightarrow{AB}$$

$$\Rightarrow\overrightarrow{F'B}=\overrightarrow{FB}\Rightarrow B-F'=B-F\Rightarrow F'=F$$

Therefore, the lines $<A, D>, <B, E>$ and $<C, F>$ are concurrent whenever

$$\frac{\overrightarrow{BD}}{\overrightarrow{DC}}.\frac{\overrightarrow{CE}}{\overrightarrow{EA}}.\frac{\overrightarrow{AF}}{\overrightarrow{FB}}=1.$$

Corollary 2.1: The three medians of any triangle are concurrent.

Proof: Consider ΔABC below (figure 2.7a) where AD, BE, CF are the medians.

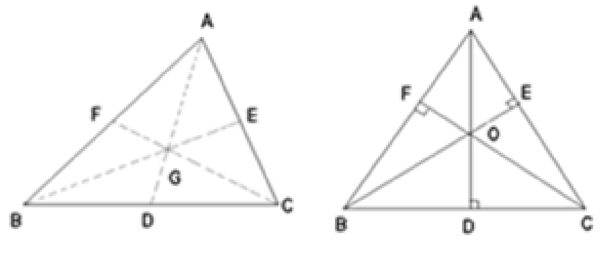


Figure 2.7

From the definition of medians, we know that $AF=FB,\ BD=DC,\ CE=EA$

Thus, $(AF)(BD)(CE)=(FB)(DC)(EA)\Rightarrow\frac{BD}{DC}.\frac{CE}{EA}.\frac{AF}{FB}=1.1.1=1$

Therefore, by Ceva's theorem the medians are concurrent.

Corollary 2.2: In any triangle, the three altitudes are concurrent

Proof: The proof is given for the case when triangle ABC is acute angled triangle. The case for obtuse follows the same reasoning and is left as exercise. Consider ΔAFC and ΔAEB (refer figure 2.7 b above). In these triangles $\angle BAE\equiv\angle BAE$ (it is common angle) and $\angle BEA\equiv\angle CFA$ (Both are right angles). Hence, $\Delta AFC\sim\Delta AEB$ by AA similarity theorem.

But $\Delta AFC\sim\Delta AEB\Rightarrow\frac{AF}{EA}=\frac{AC}{AB}$

Similarly in ΔBDA and ΔBFC, $\angle ABD \equiv \angle ABD$ (it is common angle) and $\angle BDA \equiv \angle BFC$ (Both are right angles). Hence, $\Delta BDA \sim \Delta BFC$ by AA similarity theorem. But $\Delta BDA \sim \Delta BFC \Rightarrow \frac{BD}{BF} = \frac{AB}{BC} \Rightarrow \frac{BD}{FB} = \frac{AB}{BC}$. Finally, by considering ΔCEB and ΔCDA, we get $\Delta CEB \sim \Delta CDA$. $\Delta CEB \sim \Delta CDA \Rightarrow \frac{CE}{DC} = \frac{BC}{AC}$

Multiply theses three ratios above will lead to $\frac{BD}{DC}.\frac{CE}{EA}.\frac{AF}{FB} = 1$.

Therefore, by Ceva's theorem the three altitudes are concurrent.

Corollary 2.3: The angle bisectors of a triangle intersect at a common point.

Proof: Use the same reasoning as above.

Desargue's Theorem 2.6: Let ΔABC and ΔDEF be given where their respective (corresponding) sides are parallel. Then, the lines $< A, D >, < B, E >$ and $< C, F >$ are concurrent and $\frac{\overrightarrow{AB}}{\overrightarrow{DE}} = \frac{\overrightarrow{AC}}{\overrightarrow{DF}} = \frac{\overrightarrow{BC}}{\overrightarrow{EF}}$.

Proof: Given ΔABC and ΔDEF with $\overrightarrow{AB} // \overrightarrow{DE}$, $\overrightarrow{AC} // \overrightarrow{DF}$ and $\overrightarrow{BC} // \overrightarrow{EF}$ (refer figure 2.8 below). Since $\overrightarrow{AB} // \overrightarrow{DE}$, $\overrightarrow{AC} // \overrightarrow{DF}$ and $\overrightarrow{BC} // \overrightarrow{EF}$, then there exist scalars k, r, t such that $\frac{\overrightarrow{AB}}{\overrightarrow{DE}} = k$, $\frac{\overrightarrow{AC}}{\overrightarrow{DF}} = r$, $\frac{\overrightarrow{BC}}{\overrightarrow{EF}} = t$. Now we need to show that these scalars are equal in order to prove that the sides are proportional.

Using vector addition,

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \Rightarrow k\,\overrightarrow{DE} + t\,\overrightarrow{EF} = r\,\overrightarrow{DF} = r(\overrightarrow{DE} + \overrightarrow{EF})$$

$$\Rightarrow (k - r)\overrightarrow{DE} = (r - t)\overrightarrow{EF}$$

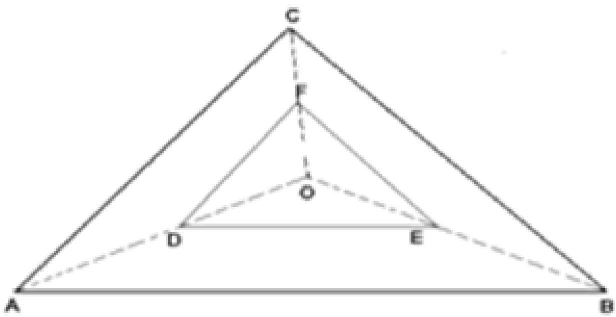


Figure 2.8

Since $\overrightarrow{DE}$ is not parallel to $\overrightarrow{EF}$, then by proposition 2.1, the equation $(k-r)\overrightarrow{DE}=(r-t)\overrightarrow{EF}$ has a unique solution given by $k-r=r-t=0 \Rightarrow k=r=t.$

Therefore, $\frac{\overrightarrow{AB}}{\overrightarrow{DE}}=\frac{\overrightarrow{AC}}{\overrightarrow{DF}}=\frac{\overrightarrow{BC}}{\overrightarrow{EF}}=k$ where k is a constant. Here, there are two cases to be considered depending on the values of the constant k.

Suppose $k \neq 1$. In this case, assume the lines $<A, D>$ and $<B, E>$ intersect at point O. Thus, $\Delta AOB \sim \Delta DOE$ by AA similarity theorem (because $\overrightarrow{AB} // \overrightarrow{DE}$ implies $\angle ODE \cong \angle OAB,\ \angle OED \cong \angle OBA$). So, $\frac{\overrightarrow{OA}}{\overrightarrow{OD}}=\frac{\overrightarrow{AB}}{\overrightarrow{DE}}$. Similarly, suppose the lines $<A, D>$ and $<C, F>$ intersect at point P. This also gives $\Delta PAC \sim \Delta PDF$ by AA similarity theorem which in turn implies $\frac{\overrightarrow{PA}}{\overrightarrow{PD}}=\frac{\overrightarrow{AC}}{\overrightarrow{DF}}$.

But from the first result, we have $\frac{\overrightarrow{AB}}{\overrightarrow{DE}}=\frac{\overrightarrow{AC}}{\overrightarrow{DF}}=\frac{\overrightarrow{BC}}{\overrightarrow{EF}}$.

Hence, $\frac{\overrightarrow{OA}}{\overrightarrow{OD}}=\frac{\overrightarrow{AB}}{\overrightarrow{DE}}=\frac{\overrightarrow{BC}}{\overrightarrow{EF}}=\frac{\overrightarrow{PA}}{PD} \Rightarrow \frac{\overrightarrow{OA}}{\overrightarrow{OD}}=\frac{\overrightarrow{PA}}{PD}$.

Besides,

$$\begin{aligned}
\frac{\overrightarrow{OA}}{\overrightarrow{OD}}=\frac{\overrightarrow{PA}}{\overrightarrow{PD}} \Rightarrow \frac{\overrightarrow{OA}}{\overrightarrow{OD}}-1=\frac{\overrightarrow{PA}}{\overrightarrow{PD}}-1 \\
\Rightarrow \frac{\overrightarrow{OA}-\overrightarrow{OD}}{\overrightarrow{OD}}=\frac{\overrightarrow{PA}-\overrightarrow{PD}}{\overrightarrow{PD}} \\
\Rightarrow \frac{\overrightarrow{AD}}{\overrightarrow{OD}}=\frac{\overrightarrow{AD}}{\overrightarrow{PD}} \Rightarrow \overrightarrow{PD}=\overrightarrow{OD} \\
\Rightarrow D-P=D-O \Rightarrow P=O
\end{aligned}$$

Hence, the lines $<A, D>, <B, E>$ and $<C, F>$ intersect at the same point and then they are concurrent.

Finally, if the ratio $k=1$, then from $\frac{\overrightarrow{AB}}{\overrightarrow{DE}}=\frac{\overrightarrow{AC}}{\overrightarrow{DF}}=\frac{\overrightarrow{BC}}{\overrightarrow{EF}}=k$, we get $\overrightarrow{AB}=\overrightarrow{DE},\ \overrightarrow{AC}=\overrightarrow{DF}$. Besides from the hypothesis, $\overrightarrow{AB} // \overrightarrow{DE},\ \overrightarrow{AC} // \overrightarrow{DF}$.

Hence, $ABED$ and $ACFD$ become parallelograms (A quadrilateral having a pair of parallel and congruent sides is said to be a parallelogram).

Thus, the lines $< A, D >, < B, E >$ and $< C, F >$ are all parallel.

Therefore, the lines $< A, D >, < B, E >$ and $< C, F >$ are concurrent.

Papu's Theorem 2.7: Let A, C, E be points on a line k and B, D, F on a line l. If $\overrightarrow{AB} // \overrightarrow{DE}$, $\overrightarrow{BC} // \overrightarrow{EF}$, then $\overrightarrow{CD} // \overrightarrow{FA}$.

Proof: Consider the diagram below. Suppose the lines k and l intersects at some point O (Refer figure 2.9b).

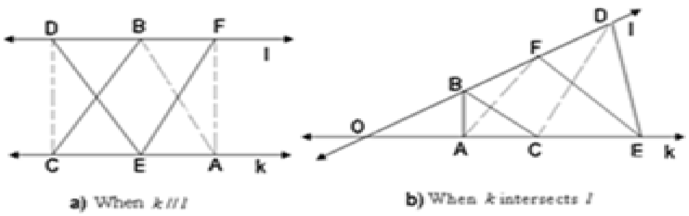


Figure 2.9

Then, using the hypothesis $\overrightarrow{AB} // \overrightarrow{DE}$, $\overrightarrow{BC} // \overrightarrow{EF}$, and by The intercept Theorem,

$$\frac{\overrightarrow{OE}}{\overrightarrow{OA}} = \frac{\overrightarrow{OD}}{\overrightarrow{OB}} \text{ and } \frac{\overrightarrow{OC}}{\overrightarrow{OE}} = \frac{\overrightarrow{OB}}{\overrightarrow{OF}}.$$

Multiplying these equations, gives $\dfrac{\overrightarrow{OC}}{\overrightarrow{OA}} = \dfrac{\overrightarrow{OD}}{\overrightarrow{OF}}$.

Therefore, by The intercept Theorem, $\overrightarrow{CD} // \overrightarrow{FA}$. Again, suppose $k // l$ (Refer figure 2.9a). In this case, $\overrightarrow{AE} // \overrightarrow{BD}$ and $\overrightarrow{AB} // \overrightarrow{ED}$.

Thus, $AEDB$ is a parallelogram. So, $\overrightarrow{AE} = \overrightarrow{BD}$ and $\overrightarrow{BF} = \overrightarrow{CE}$.

On the other hand,

$$\overrightarrow{AC} = \overrightarrow{AE} + \overrightarrow{EC} \Rightarrow \overrightarrow{AC} = \overrightarrow{AE} - \overrightarrow{CE} = \overrightarrow{BD} - \overrightarrow{BF} = \overrightarrow{FB} + \overrightarrow{BD} = \overrightarrow{FD}$$
$$\Rightarrow \overrightarrow{AC} = \overrightarrow{FD} \Rightarrow \overrightarrow{AC} // \overrightarrow{FD}$$

Thus, $\overrightarrow{AC}$ and $\overrightarrow{FD}$ are congruent and parallel.

Hence, $AFDC$ is a parallelogram.

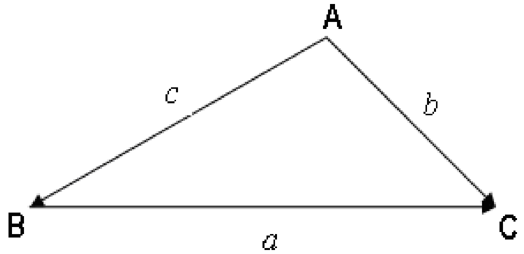
Since opposite sides of a parallelogram are parallel $\overrightarrow{CD} // \overrightarrow{FA}$.

Problem Set 2.2

1. In Affine Space, suppose the lines $l: X = t(1,1,3), m: X = (1,0,1) + r(2,3,8)$ and $n: X = (a,1,b) + k(0,1,2)$ are concurrent. Find a and b. **Answer** $: a = 3, b = 5$

2. Find the value of a for which the lines $l: y = -3x + 2, m: y = ax + 7$ and $n: y = 2x - 3$ are concurrent.

3. **The sign Law:** Given ΔABC with $\overline{AB} = c,\ \overline{BC} = a,\ \overline{AC} = b$ as shown below. Then, using *vector method*, prove that $\dfrac{a}{\sin \angle A} = \dfrac{b}{\sin \angle B} = \dfrac{c}{\sin \angle C}$.



4. Find the vector, parametric and symmetric equations of a line through the intersection of $m: y = x$ and $n: y = 2 - x$ which is parallel $l: X = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} -2 \\ 6 \end{pmatrix}$.

5. Let D, E and F be points on the sides $< B,C >, < A,C >$ and $< A,B >$ of ΔABC respectively where $\overline{AB} = 12, \overline{EA} = 3,\ \overline{BC} = 15,\ \overline{FB} = 8,\ \overline{BD} = 6$. If the lines $< A,D >, < B,E >$ and $< C,F >$ are concurrent, then calculate the length $\overline{CE}$.

Answer $: CE = 9$

6. Let D, E, F be points on the sides $\langle B,C \rangle, \langle A,C \rangle$ and $\langle A,B \rangle$ of triangle ABC such that $\dfrac{\overline{DB}}{\overline{DC}} = \dfrac{3}{4}$ and $\dfrac{\overline{CE}}{\overline{EA}} = \dfrac{-12}{5}$. Then, find $\dfrac{\overline{AF}}{\overline{FB}}$

 a) If the points D, E, F are *collinear*.

 b) If $\langle A,D \rangle, \langle B,E \rangle$ and $\langle C,F \rangle$ are *concurrent*.

7. Suppose D, E and F are collinear points on the lines $< B,C >, < A,C >$ and $< A,B >$ of ΔABC. If for any point G, $\dfrac{\overrightarrow{AG}}{\overrightarrow{GB}} \cdot \dfrac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \dfrac{\overrightarrow{CE}}{\overrightarrow{EA}} = -1$, prove that $F = G$.

8. Let $k // l$ and let A, C, E be on k and B, D, F be on l such that the lines $< A, B >$, $< C, D >$ and $< E, F >$ <E,F> are concurrent.

Then, prove that $\overrightarrow{AC} / \overrightarrow{CE} = \overrightarrow{BD} / \overrightarrow{DF}$.

9. Prove that the angle bisectors of a triangle are concurrent at a point.

10. Consider the vertices $A = (1,2,5)$, $B = (1,8,-1)$, $C = (7,2,-1)$ of an equilateral triangle ABC. Suppose the points $D = (4,5,-1), E = (4,2,2)$, $F = (1,5,2)$ are on the sides $< B, C >, < A, C >$ and $< A, B >$ of ΔABC respectively. Prove that the lines $< A, D >$, $< B, E >$ and $< C, F >$ are concurrent.

11. Suppose the medians AP, BQ, CR of ΔABC intersect at point T. Then, show that $\frac{TP}{AP} + \frac{TB}{BQ} + \frac{TR}{CR} = 1$

12. Let ABC and DEF be triangles with the condition $\overrightarrow{AB} / \overrightarrow{DE} = \overrightarrow{AC} / \overrightarrow{DF}$. Show that $\overrightarrow{BC} \parallel \overrightarrow{EF}$.

13$*$.Let D, E, F be points on sides BC, CA, AB of ΔABC respectively such that the Cevians AD, BE, CF are concurrent. Show that if M, N, P are points on EF, FD, DE respectively, then the lines AM, BN, CP are concurrent if and only if the lines DM, EN, FP are concurrent.

14. If a circle is inscribed in ΔABC such that M, N and P are points of tangency on the sides BC, CA and AC, prove that $\overleftrightarrow{AM}, \overleftrightarrow{BN}$ and $\overleftrightarrow{CP}$ are concurrent.

15. The bisector of any interior angle of a non-isosceles and the bisectors of the two exterior angles at the other vertices are concurrent.

16. In ΔABC, let $P \in AB$ and $Q \in AC$ such that $PQ // BC$. Then, prove that PC and QB concur (intersect) at a point on the median AM.

17. Let P, Q, R be points on ΔABC distinct from A, B, C respectively. Then AP, BQ, CR are concurrent if and only if $\frac{\sin\angle CAP \sin\angle ABQ \sin\angle BCR}{\sin\angle APB \sin\angle QBC \sin\angle RCA} = 1.$

18. Prove that the angle bisectors of a triangle are concurrent at a point.

CHAPTER-3

Orthogonal (Isometric)Transformations

3.1 Introductions

In order to introduce the concept of isometrics, first let's consider the notion of distance. Distance is a real valued non-negative function denoted by $d(P,Q)$ which assigns to any pair of points in the plane or space a non-negative real number satisfying the following conditions:

$i)\ d(P,Q) = d(Q,P)$

$ii)\ d(P,Q) \geq 0 d(P,Q) = 0 \Leftrightarrow P = Q$

$iii)\ d(P,R) \leq d(P,Q) + d(Q,R)$

Here, the third property is known as *triangle inequality* and equality occurs if and only if the points P, Q, R are collinear points.

Note: The notation $d(P,Q)$ stands to mean the distance from P to Q and equivalently denoted by $d(P,Q) = |PQ| = \overline{PQ}$. It is the length of the line segment between P and Q which shows that line segment is the shortest path between two points.

In Euclidean geometry, distance between two points P and Q in a plane is given by $d(P,O) = \overline{PQ} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, where $P = (x_1, y_1)$ and $Q = (x_2, y_2)$

It can be easily verified that this distance formula satisfies the above three conditions. Throughout this text, the writer uses $\overline{f(P)f(Q)}$ to mean the distance between $f(P)$ and $f(Q)$, $\overline{PQ}$ to mean the distance between P and Q.

In a plane, it is common to see the case where the norm of a vector or the distance (length) between two points to have the same measure after the vector or the points have been mapped to other vectors or points by some mappings.

In such cases, whenever $\left\| f(\vec{v}) \right\| = \left\| \vec{v} \right\|$ (That is the norm of the image vector is equal to the norm of the vector before mapping) for any vector $\vec{v}$ and $\overline{f(P)f(Q)} = \overline{PQ}$ (That is the distance between the images of two points is equal to the distance between the points) for any two points P and Q, we say that f is norm, length or distance preserving. That is really what we mean isometric or orthogonal transformations. Now let's have the formal definition.

3.2 Definition and Examples of Isometries

Definition: An isometric transformation of a plane is a transformation from a plane on to itself which preserves distances.

That is, f is an isometric transformation if for any two points P and Q in the plane, $d(f(P), f(Q)) = d(P,Q)$.

Isometrics in Euclidean space are sometimes named as orthogonal transformations or rigid motions.

Examples:

1. Verify whether the following transformations are isometries or mot.

   a) $f : R \to R$ given by $f(x) = x+5$

   b) $f : R^2 \to R^2$ given by $(y-1, x+7)$

   c) $f : R^2 \to R^2$ given by $f(x, y) = (3x, 3y)$

   d) $f : R^2 \to R^2$ given by $f(x, y) = (2y-9, 2x+9)$

Solution:

a) Since the distance between any two points x and y in R is given by

$d(x, y) = |x-y|$ and $d(f(x), f(y)) = |x+5-(y+5)| = |x-y| = d(x, y)$.

Hence, f is an isometry.

b) Here, if we take any two points $P = (x, y), Q = (z, w)$, we have

$d(P,Q) = \sqrt{(z-x)^2 + (y-w)^2}$.

Besides, $P' = f(P) = (y-1, x+7), Q' = f(Q) = (z-1, w+7)$ and thus

$$
\begin{aligned}
d(P',Q') &= d(f(P), f(Q)) \\
&= \sqrt{[(z-1)-(y-1)]^2 + [(w+7)-(x+7)]^2} \\
&= \sqrt{(z-y)^2 + (w-x)^2}
\end{aligned}
$$

This shows that $d(P',Q') = d(f(P), f(Q)) = d(P,Q)$.

Hence, it is an isometry.

c) Here, for any two points $P = (x,y), Q = (z,w)$, we have

$d(P,Q) = \sqrt{(z-x)^2 + (y-w)^2}$.

Besides,

$P' = f(P) = (3x, 3y), Q' = f(Q) = (3z, 3w)$ and thus

$$
\begin{aligned}
d(P',Q') &= d(f(P), f(Q)) \\
&= \sqrt{(3z-3z)^2 + (3w-3y)^2} \\
&= \sqrt{9(z-y)^2 + 9(w-x)^2} \\
&= 3\sqrt{(z-y)^2 + (w-x)^2}
\end{aligned}
$$

This shows that $d(P',Q') = d(f(P), f(Q)) = 3d(P,Q) \Rightarrow d(P',Q') \neq d(P,Q)$.

Hence, it is not an isometry.

d) Using similar procedures as in part (c), it is not isometric transformation.

2. If f is an isomteric transformation and $v = (a,6)$ is a vector such that

$f(a,3) = (-3,4)$, find a.

Solution: Since f is an isomtery, for any vector $\vec{v}$ we have $\left\| f(\vec{v}) \right\| = \left\| \vec{v} \right\|$.

Thus, $f(a,3) = (-3,4) \Rightarrow \| f(a,3) \| = \| (a,3) \|$

$$
\begin{aligned}
&\Rightarrow \sqrt{a^2 + 9} = \sqrt{9 + 16} \\
&\Rightarrow a^2 + 9 = 25 \Rightarrow a^2 = 16 \Rightarrow a = \pm 4
\end{aligned}
$$

3. Prove that any isometry preserves dot product and conclude that an isometry preserves angle.

Proof: Let $\vec{v}$ and $\vec{w}$ be any two vectors and f be an isometry. Since an isometry preserves length (norm), we have $\|f(\vec{v})\| = \|\vec{v}\|$, $\|f(\vec{w})\| = \|\vec{w}\|$ and

$\|f(\vec{v}) - f(\vec{w})\| = \|\vec{v} - \vec{w}\|$.

But $\|f(\vec{v}) - f(\vec{w})\| = \|\vec{v} - \vec{w}\|$

$$\Rightarrow \|f(\vec{v}) - f(\vec{w})\|^2 = \|\vec{v} - \vec{w}\|^2$$
$$\Rightarrow (f(\vec{v}) - f(\vec{w})).(f(\vec{v}) - f(\vec{w})) = (\vec{v} - \vec{w}).(\vec{v} - \vec{w})$$
$$\Rightarrow f(\vec{v}).f(\vec{v}) - f(\vec{v}).f(\vec{w}) - f(\vec{w}).f(\vec{v}) + f(\vec{w}).f(\vec{w}) = \vec{v}.\vec{v} - \vec{v}.\vec{w} - \vec{w}.\vec{v} + \vec{w}.\vec{w}$$
$$\Rightarrow \|f(\vec{v})\|^2 - 2f(\vec{v}).f(\vec{w}) + \|f(\vec{w})\|^2 = \|\vec{v}\|^2 - 2\vec{v}.\vec{w} + \|\vec{w}\|^2$$
$$\Rightarrow \|\vec{v}\|^2 - 2f(\vec{v}).f(\vec{w}) + \|\vec{w}\|^2 = \|\vec{v}\|^2 - 2\vec{v}.\vec{w} + \|\vec{w}\|^2$$
$$\Rightarrow -2f(\vec{v}).f(\vec{w}) = -2\vec{v}.\vec{w}$$
$$\Rightarrow f(\vec{v}).f(\vec{w}) = \vec{v}.\vec{w}$$

Hence, $f(\vec{v}).f(\vec{w}) = \vec{v}.\vec{w}$ shows that f preserves dot product. Besides, if θ is the angle between the non-zero vectors $\vec{v}$, $\vec{w}$ and ϕ is the angle between the image vectors $f(\vec{v})$, $f(\vec{w})$, then we have

$$\cos\phi = \frac{f(\vec{v}).f(\vec{w})}{|f(\vec{v})\|f(\vec{w})|} = \frac{\vec{v}.\vec{w}}{|\vec{v}\|\vec{w}|} = \cos\theta \Rightarrow \phi = \theta. \ (Why?)$$

3.3 Properties of Isometric (Orthogonal) Transformations

Propositions 3.1: The inverse of an isometry is an isometry.

Proof: Let f be an isometry. Now let P and Q be any two points. We need to show $\|f^{-1}(P)-f^{-1}(Q)\|=\|P-Q\|$. Since f is an isometry,

$$\|f^{-1}(P)-f^{-1}(Q)\|=\|f(f^{-1}(P))-f(f^{-1}(Q))\|$$
$$=\|(f\circ f^{-1})(P))-(f\circ f^{-1})(Q))\|=\|i(P)-i(Q)\|=\|P-Q\|$$
$$\Rightarrow\|f^{-1}(P)-f^{-1}(Q)\|=\|P-Q\|$$

Hence, for any isometry f, f^{-1} is also an isometry.

Proposition 3.2: The composition of any two isometries is again an isometry.

Proof: Let f and g be any two isometries. We need to show their composition $f\circ g$ is also an isometry. Let P and Q be any two points.

Since f is an isometry, $\|f(P)-f(Q)\|=\|P-Q\|$.

But g is also an isometry, so $\|g(f(P))-g(f(Q))\|=\|f(P)-f(Q)\|$.

Combining these two results together, we get that

$\|g\circ f(P)-g\circ f(Q)\|=\|g(f(P))-g(f(Q))\|=\|f(P)-f(Q)\|=\|P-Q\|$.

Hence, the composition $f\circ g$ is an isometry.

Proposition 3.3: An isometry maps distinct points into distinct points.

Proof: Let f be an isometry. We need to show $A\neq B\Rightarrow f(A)\neq f(B)$.

But, $A\neq B\Rightarrow A-B\neq 0\Rightarrow d(A,B)\neq 0$. Besides, since f is an isometry $d(A,B)=d(f(A),f(B))$.

So, $d(A,B)\neq 0\Rightarrow d(f(A),f(B))\neq 0\Rightarrow f(A)-f(B)\neq 0\Rightarrow f(A)\neq f(B)$.

Proposition 3.4: Any isometry preserves between ness. That means if the points A',B',C' are the image of the distinct points A,B,C under a given isometry such that B is between A and C, then B' is also between A' and C'.

Proof: Let f be an isometry such that $A'=f(A), B'=f(B), C'=f(C)$. Then, $A'B'=AB, B'C'=BC, A'C'=AC$.

Besides, as B is between A and C, $AB < AC$ (from definition of betweeness).

Now suppose B' is not between A' and C' (It may be either to the left of A' or to the right of C' refer figure 3.1).

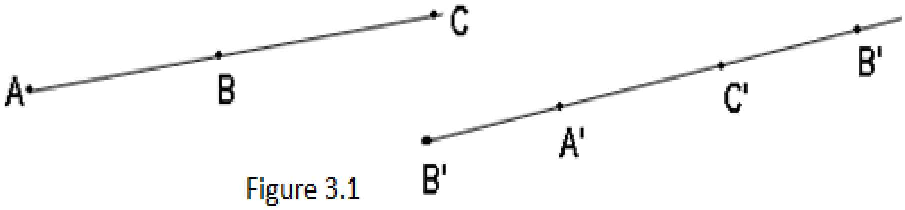


Figure 3.1

Thus, if we assume B' is to the right of C', then $A'C' < A'B' \Rightarrow AC < AB$ (Because $A'B' = AB,\ A'C' = AC$).

But this contradicts with the hypothesis $AB < AC$. Similar argument holds if B' is assumed to be on the left of A'.

Hence, any isometry preserves betweeness.

Proposition 3.5: If three points A, B, C are collinear, then their images A', B', C' are also collinear. This means any isometry preserves co linearity.

Proof: Apply Triangle inequality and proposition 3.4 to arrive at a contradiction.

Proposition 3.6: An isometry maps lines into lines and parallel lines into parallel lines.

Proof: Let l be any line and f be an isometry. Take any two distinct points A, B on l such that $f(A) = A'$ and $f(B) = B'$.

But $A \neq B \Rightarrow f(A) \neq f(B) \Rightarrow A' \neq B'$. Hence, A' and B' are distinct points and determine a unique line l'. Besides, since f preserves co-linearity for any point P on l, $f(P) = P'$ is on l'. Moreover; if $l // m$, then $f(l) // f(m)$.

Otherwise, if $f(l)$ and $f(m)$ are not parallel, they intersect at a point Q.

So, there exist two different points $A \in l$ and $B \in m$ such that $f(A) = Q$ and $f(B) = Q$ which implies $f(A) = f(B)$. But this contradicts with proposition 3.3, $A \neq B \Rightarrow f(A) \neq f(B)$. Thus, $l // m \Rightarrow f(l) // f(m)$.

Theorem 3.1 (The Three-Point Theorem):

Two isometrics with same image at three non-collinear points are equal.

Proof: Suppose f and g are two isometries such that

$f(A)=g(A),\ f(B)=g(B),\ f(C)=g(C),$ where A, B and C are non-collinear points. We need to show that $f=g$, *i.e* $f(P)=g(P)$ for any point P in the plane. Now, suppose $f \neq g$. Then, there exists at least one point P such that $f(P) \neq g(P)$. Let $f(P)=P',\ g(P)=P''$ as shown in figure 3.2.

Figure 3.2

Since f and g are isometrics, $\overline{A'P'}=\overline{AP},\ \overline{A'P''}=\overline{AP}.$

From these two equations, we get $\overline{A'P'}=\overline{A'P''}$, which shows that A' is on the perpendicular bisector of $\overline{P'P''}$ (If a point is at equal distance from the end points of a line segment, then it lies on the perpendicular bisector of that line segment).

Similarly, if we consider points B and C separately in the plane of A we get that B' and C' are also in the perpendicular bisector of $\overline{P'P''}$.

That means they all lie on the same line which yields that A', B' and C' are collinear. But, this in turn implies that A, B, C are collinear because any isometrics preserves co-linearity.

However, this contradicts the hypothesis that A, B, C are non-collinear. Consequently, $f=g$.

Problem Set 3.1

1. Show that the transformation given by $f(x,y)=\left(-\frac{4}{5}x-\frac{3}{5}y, -\frac{3}{5}x+\frac{4}{5}y\right)$ is an isometry.

2. If $f(x,y)=\left(ax+\frac{4}{5}y, \frac{4}{5}x+\frac{3}{5}y\right)$ is an isometry, find the value of a.

Answer : $a=-3/5$

3. Let P and Q be two points and α be an isometry with $\alpha(P)=P,\ \alpha(Q)=Q$. Show that $\alpha(M)=M$ where M is the mid point of P and Q.

4. Suppose ψ is a mapping with the property that $\psi(P)\psi(Q)=PQ$ for all points P and Q. Show that ψ is an isometry. (**Hint:** Simply show that ψ is a transformation)

5. Prove that any isometry preserves the Cauchy- Schwarz Inequality.

6. Using *triangle inequality*, prove that an isometry maps lines in to lines.

7. Let C be a circle with center O and radius r. Prove that if f is an isometry, then $f(C)$ is a circle with center $f(O)$ and radius r.

9. Prove that every isometry f preserves norm of a vector.

10. Prove that every isometry f preserves dot product and conclude that it also preserves angle measures.

3.4 Fundamental Types of Isometric Transformations

3.4.1 Translation

Definition: A mapping $T: R^2 \to R^2$ is called translation if there exists a vector $\vec{v}$ such that $T(P) = P + \vec{v}$ for every point P in R^2.

In other word, for $P = (x, y), v = (a, b), T(P) = T(x, y) = (x', y')$ where $\begin{cases} x' = x + a \\ y' = y + b \end{cases}$

The vector v is called **translator** vector. The translation with translator vector v is sometimes denoted by T_v. In translation problem, whenever any two of $(x, y), (x', y')$, or (a, b) (the pre image, the image or the translator vector) are given, the third can be *uniquely* determined from the translation equation.

Examples:

1. Let T be a translation that takes the point (3,4) to (5,3). Find the equation for the translation T.

Solution: Let $v = (a, b)$ be the translator vector. Then for any point $P(x, y)$, we have $T(x, y) = (x', y')$ where $\begin{cases} x' = x + a \\ y' = y + b \end{cases}$.

Particularly, for $(3,4)$, we have

$T(3,4) = (3 + a, 4 + b) = (5,3) \Rightarrow 3 + a = 5, 4 + b = 3 \Rightarrow a = 2, b = -1$.

Therefore, $T(x, y) = (x + 2, y - 1)$.

2. For any two translations T_v and T_w, if $T_v(1,-2) = 2T_w(4,-7)$ and $T_w(9,-2) = 3T_v(1,3)$, find the equations of these translations.

Solution: Let $v = (a, b)$ and $w = (c, d)$. Then, for any object (x, y), $T_v(x, y) = (x + a, y + b)$, $T_w(x, y) = (x + c, y + d)$. But,

$$T_v(1,-2) = 2T_w(4,-7) \Rightarrow (1 + a, -2 + b) = (8 + 2c, -14 + 2d)$$

$$\Rightarrow \begin{cases} a - 2c = 7 \\ b - 2d = -12 \end{cases} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (i)$$

$$T_w(9,-2) = 3T_v(1,3) \Rightarrow (9 + c, -2 + d) = (3 + 3a, 9 + 3b)$$

$$\Rightarrow \begin{cases} c - 3a = -6 \\ d - 3b = 11 \end{cases} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (ii)$$

Collecting like terms together from (i) and (ii), we get the following systems;

$$\begin{cases} a-2c=7 \\ c-3a=-6 \end{cases} \Rightarrow a=1,\ c=-3 \quad , \qquad \begin{cases} b-2d=-12 \\ d-3b=11 \end{cases} \Rightarrow b=-2,\ d=5$$

$$\Rightarrow v=(a,b)=(1,-2),\ w=(c,d)=(-3,5)$$

This implies that $v=(a,b)=(1,-2),\ w=(c,d)=(-3,5)$

Therefore, $T_v(x,y)=(x+1,y-2)$ and $T_w(x,y)=(x-3,y+5)$.

For instance, $T_v(1,-2)=(2,-4), T_w(4,-7)=(1,-2) \Rightarrow T_v(1,-2)=2(1,-2)=2T_w(4,-7)$.

Proposition 3.7 (Properties of Translations):

a) The translator vector of a translation is unique.

b) The composition of translation T_v and T_w is again a translation by $v+w$.

c) The inverse of a translation is a gain a translation with opposite vector.

d) The image of a line under a translation is a line parallel to the given line.

e) The image of a vector under a translation is an equal vector.

Proof: (Follows from the definition)

Examples:

1. If $T_v(1,2)=(3,-1), (T_v \circ T_w)(1,5)=(3,7)$, then find the equations of T_v and T_w.

Solution: Here,

$$T_v(1,2)=(3,-1) \Rightarrow v+(1,2)=(3,-1) \Rightarrow v=(2,-3)$$
$$\Rightarrow T_v(x,y)=(x+2,y-3)$$

A gain , from part (b) of the above properties,

$$(T_v \circ T_w)(1,5)=v+w+(1,5)=(3,7)$$
$$\Rightarrow (2,-3)+w+(1,5)=(3,7) \Rightarrow w=(0,5)$$

Therefore, $T_w(x,y)=(x,y+5)$.

2. If $T^2(1,3)=(7,1)$, then find the equation of T.

Solution: Here, $T^2(P)=T \circ T(P)=2v+P$. In general, $T^n(P)=nv+P$.

Hence, $T^2(1,3)=(7,1) \Rightarrow T \circ T(1,3)=(7,1) \Rightarrow 2v+(1,3)=(7,1)$
$$\Rightarrow v=(3,-1) \Rightarrow T(x,y)=(x+3,y-1)$$

Remark: The important concept that students should bear in mind about translation is that translation T_v can be applied to arbitrary shapes of object point –by-point. Each point of a given shape S is translated by v and the collection of these translated points gives the translated image S' of S. This is denoted by $S' = T_v(S)$. So, to determine the image S' when S is a polygon it is sufficient to translate its vertices. For instance, if $PQRT$ is an arbitrary four side polygon, where P, Q, R, T are its vertices, to find the image polygon under a translation T_v, it suffices to determine only the images of the vertices. i.e. $T_v(P) = P', T_v(Q) = Q', T_v(R) = R', T_V(T) = T'$. So that the image quadrilateral is $P'Q'R'T'$. This means that other points of the quadrilateral between the vertices are assumed to be translated (moved) equal distance in the same direction.

Examples: Let T be a translation by the vector $(1,2)$. Find the image of

a) ΔABC whose vertices are $A(0,0)$, $B(3,0)$ and $C(0,4)$

b) the line $l:\ x - 2y = 6$.

Solution: Let $P = (x, y)$ by any object in the plane containing ΔABC. Then, $T(P) = T(x, y) = (x+1, y+2)$ by definition of translation. Thus,

a) $A' = T(A) = (1,2), B' = T(B) = (4,2), C' = T(C) = (1,6)$. Hence, the image of ΔABC under T will be $\Delta A'B'C'$ with vertices $A' = (1,2,), B' = (4,2)$ and $C' = (1,6)$.

b) This problem can be done using two methods.

Method I: Take two points A and B on $l:\ x - 2y = 6$.

Say $A = (0,-3),\ B = (6,0)$. Then, find the images of these points under T. $T(A) = A' = (1,-1), T(B) = B' = (7,2)$.

Now form the equation of a line l' passing through A' and B'. This line will be the image of l under T. This is because from the previous explanation all other points between A and B or on the line l will be translated the same distance in the same direction. Hence, $l': y = mx + b$, where $m = \frac{2+1}{7-1} = \frac{3}{6} = \frac{1}{2}$

$\Rightarrow l': y = \frac{1}{2}x + b$. Taking either A' or B' on l' we can determine b.

Using $A', -1 = \frac{1}{2}.1 + b \Rightarrow b = -\frac{3}{2}$. Thus, $l': y = \frac{1}{2}x - \frac{3}{2} \Rightarrow x - 2y = 3$.

Method II: Take any arbitrary point $P=(x,y)$ on l. Then, find P'. Finally, substitute in the equation of l. $P'=(x',y')=T(x,y)=(x+1,y+2)$. From the general equation of translation, $x'=x+1,\ y'=y+2 \Rightarrow x=x'-1,\ y=y'-2$.

Subsisting these values in equation of l, we have

$l: x-2y=6 \Rightarrow l': (x'-1)-2(y'-20=6 \Rightarrow l': x'-2y'=3$

Note that it is advisable to use **Method II**. Because most of the time **Method I** is subjected to arithmetic errors and is a bit longer than **Method II**. In this example one can observe $m'=m=\frac{1}{2}$ which shows that l' is parallel to l.This will enable us to state the following proposition in general.

Proposition 3.8: Any translation is a dilatation.

Proof: In chapter 1, we have seen that to show a given transformation is a dilatation, first we have to show it is a collineation and then it maps any line to parallel line. Let T_v be arbitrary translation with translator vector $v=(h,k)$ and let $l: ax+by+c=0$ be any line. Now, for any point (x,y) on this line, $T_v(x,y)=(x',y')\ where \begin{cases} x'=x+h \\ y'=y+k \end{cases}$. Solving for x and y in terms of x' and y' yields $x=x'-h,\ y=y'-k$.

Then, substitute these values in the equation of l (Just using **Method II**)

$l: ax+by+c=0 \Leftrightarrow a(x'-h)+b(y'-k)+c=0 \Leftrightarrow ax'+by'+(c-ah-bk)=0$

But this is equation of a line which means that T_v is a collineation. Thus, for any line $l: ax+by+c=0$ its image under a translation T_v is a line given by $T_v(l)=l': ax'+by'+(c-ah-bk)=0$.

Besides, comparing the equations of l and l', we observe that they do have the same slope. This shows they are parallel which in turn implies that T_v maps any line into a parallel line. Consequently, any translation is a dilatation.

Proposition 3.9: For two given points, P and Q there is a unique translations that take P to Q this is usually denoted by $T_{P,Q}$. Besides, the translator vector is given by $v = Q - P = \overrightarrow{PQ}$.

Proof: Let $T_1(x,y) = (x+a, y+b), T_2(x,y) = (x+m, y+n)$ be any two translations such that $T_1(P) = Q$ and $T_2(P) = Q$. We need to show T_1 and T_2 are *identical*. But to show this it suffices to show that their translator vectors are equal. Let $P = (c,d), Q = (e,f)$. Then, $T_1(P) = (c+a, d+b) = (e,f)$ and $T_2(P) = (c+m, d+n) = (e,f)$. Thus, $(c+a, d+b) = (c+m, d+n)$ which implies, $(a,b) = (m,n)$. So, $T_1(P) = T_2(P)$ for any point P. Hence, $T_1 = T_2$.

Therefore, $T_{P,Q}$ is unique. Besides, $T_v(P) = v + P = Q \Rightarrow v = Q - P = \overrightarrow{PQ}$.

Corollary 3.1:

For any four points P, Q, R and S, if $T_{P,Q}(R) = S$, then $T_{P,Q} = T_{R,S}$.

Proof: Let v be a translator vector of the translation T. Then,

$$
\begin{aligned}
T_v(P) = v + P = Q \Rightarrow v = Q - P = \overrightarrow{PQ} \\
&\Rightarrow T_v(R) = Q - P + R = S \\
&\Rightarrow Q - P = S - R \Rightarrow \overrightarrow{PQ} = \overrightarrow{RS} \Rightarrow T_{\overrightarrow{PQ}} = T_{\overrightarrow{RS}}
\end{aligned}
$$

Problem Set 3.2

1. Let T_v be a non-identity translation in R^2. Prove that $T_v(l) = l \Leftrightarrow l // v$.

2. For any four non-collinear points A, B, C, D, $T_{\overrightarrow{AB}} = T_{\overrightarrow{DC}} \Leftrightarrow ABCD$ is a parallelogram.

3. Let f be a translation with translator vector $v = (a, b)$. If $f^4(2, -7) = (-6, 5)$, find the translator vector v and $f(4, -3)$

4. Find the image of the circle $C: x^2 + y^2 = 4y$ under a translation T which maps $(1,2)$ to $(0,7)$.

5. Let T_v be a translation with translator vector v. Then, show that $T_v^{\ n}(X) = X + nv$ and $T_v^{\ -n}(X) = X - nv$ for any object X.

6. For any two translations, T_v and T_w, their composition is commutative. i.e $T_v \circ .T_w = T_w \circ T_v$.

3.4.2 Reflection

When we look at our selves through a mirror, it is common to see our face. At this time, our true face is displayed through the mirror and the displayed face is called our image under the mirror. But in the theory of Optics one can find that this image is found to be at equal distance from the mirror as our real face. This means that the distance between the image and the mirror is the same as the distance between our face and the mirror and this is what we mean by the concept of reflection.

Definition: Given a line l and a point P. Then P' is said to be the reflection image of P on the line l if and only if $\overline{PP'}$ is perpendicular to l and $\overline{PM} = \overline{P'M}$, where M is the point of intersection of $\overline{PP'}$ and the line l. In other words, P and P' are located on different sides of l but at equal distances from the line l. In this case, P' is said to be the mirror image of P and the line l is said to be line of reflection or axis of symmetry.

Notation: Reflection on l is usually denoted by S_l.

So, $S_l(P) = \begin{cases} P, \text{ if } P \in l \\ P', \text{ if } P \notin l \text{ and } l \text{ is the perpendicular bisector of } \overline{PP'} \end{cases}$

Examples: Common reflection equations.The following simple equations of reflections can be easily derived using the definition.As they are most coomonly, used in this chapter, please bear them in mind. Let (x, y) be any point.

a) Reflection on the x-axis: $S_x(x, y) = (x, -y)$

b) Reflection on the y-axis: $S_y(x, y) = (-x, y)$

c) Reflection on a vertical line $x = a$: $S_a(x, y) = (2a - x, y)$

d) Reflection on a horizontal line $y = b$: $S_b(x, y) = (x, 2b - y)$

e) Reflection on the line $y = x$: $S_l(x, y) = (y, x)$

Example: If the image of the point $(-3,5)$ on a vertical line is $(7,5)$, find the equation of the line.

Solution: Let the line be $l: x = a$.Then, reflection on this line will have equation $S_a(x, y) = (2a - x, y)$.

Particularly, $S_a(-2,5) = (7,5) \Rightarrow (2a + 3,5) = (7,5) \Rightarrow 2a + 3 = 7 \Rightarrow a = 2$.

Hence, the line is $l: x = 2$.

Problematic situations: In solving reflection problems, there are three possible situation:

I) Given the point P and line l, we need to find the image point P' under S_l.

II) Given the image point P' and the line l, we need to find the object point P.

III) Given and the objetc point P and its image point P', we need to find the equation of l.

But, in each case, the basic definition of reflection can be used to determine any of the required values.

Examples:

1. Find the image of the point $(4,6)$ by a reflection on the line $l: y = 2x$.

Solution: Let the image point be $P'(x', y')$. Then, by definition, the midpoint of the P and P' is on the line of reflection.

That is $(\frac{x'+4}{2}, \frac{y'+6}{2}) \in l \Rightarrow \frac{y'+6}{2} = x'+4 \Rightarrow y' = 2x'+2 ....(i)$

A gain the line through P and P' is perpendicular to the given line. Hence, its slope must be $m = -\frac{1}{2}$. Thus,usin P and P', we have

$$\frac{y'-6}{x'-4} = -\frac{1}{2} \Rightarrow y' = -\frac{x'}{2} + 8 ....(ii)$$

From the two equations, we get $2x'+2 = -\frac{x'}{2} + 8 \Rightarrow x' = \frac{12}{5}, y' = \frac{34}{5}$.

Therefore, the image point is $(x', y') = (\frac{12}{5}, \frac{34}{5})$.

2. Let the image of the point $(-2,5)$ by a reflection S_l be the point $(-5,2)$. Find the equation of the line of reflection l.

Solution: Let $P=(-2,5)$, $P'=(-5,2)$ and $l: y=mx+b.$ Since $\overline{PP'}$ is perpendicular to l, the product of the slopes of l and the line through P and P' must be -1. But the slope of the line through P and P' is 1. So, slope m of l is -1. Thus, $l: y=mx+b \Rightarrow y=-x+b.$

Again, since l is the perpendicular bisector of $\overline{PP'}$, l passes through the mid point of P and P'. That is, $\frac{P+P'}{2}=\frac{(-2,5)+(-5,2)}{2}=(\frac{-7}{2},\frac{7}{2})\in l$. Thus, $(\frac{-7}{2},\frac{7}{2})$ satisfies the equation of l. So, $\frac{7}{2}=-(\frac{-7}{2})+b \Rightarrow b=0$.

Therefore, equation of l is $y=-x.$

Proposition 3.10: Properties of Reflection

a) For any two reflections, S_l and S_m, $S_l \circ S_m = -S_m \circ S_l$

b) For any reflection S_l, $S_l^{-1}=S_l$, the inverse of any reflection is itself.

c) Reflection is an involution.

Proof: (Follows from the definition)

Part (a) of the above proposition shows that composition of reflections in general is not commutative.

However, there is a special case in which $S_m \circ S_l = S_l \circ S_m$. In general, *"Under what condition, product (composition) of reflections will be commutative?"* is basic question to be answered. Since it needs some basic theorems, it is not easy to answer this question for the moment. We will get the full answer to this question later on in section 3.5 of this chapter.

Proposition 3.11: Let l and k be any two lines. Then, the following conditions are equivalent.

$a)\ S_l \circ S_k = S_k \circ S_l$ $\quad b)\ (S_l \circ S_k)^2 = i$ $\quad c)\ (S_l \circ S_k)^{-1} = S_l \circ S_k$

Proof: We need to show $a \Rightarrow b$.

$(S_l \circ S_k)^2 = (S_l \circ S_k) \circ (S_l \circ S_k) = (S_l \circ S_k) \circ (S_k \circ S_l)$
$= S_l \circ (S_k \circ S_k) \circ S_l = S_l \circ i \circ S_l = S_l \circ S_l = i$

We need to show $b \Rightarrow c$

$$(S_l \circ S_k)^2 = i \Rightarrow (S_l \circ S_k) \circ (S_l \circ S_k) = i$$
$$\Rightarrow (S_l \circ S_k) \circ (S_l \circ S_k) \circ (S_l \circ S_k)^{-1} = (S_l \circ S_k)^{-1}$$
$$= (S_l \circ S_k)^{-1} = S_l \circ S_k$$

We need to show $c \Rightarrow a$

$(S_l \circ S_k)^{-1} = S_l \circ S_k \Rightarrow S_l \circ S_k = S_k \circ S_l$, because

$(S_l \circ S_k)^{-1} = S_k^{-1} \circ S_l^{-1}$, $S_l = S_l^{-1}$, $S_k^{-1} = S_k$

The generalized Analytic Equation of Reflection

Theorem 3.2: (The Generalized Reflection Theorem): Let $l : ax + by + c = 0$ be any line and S_l be a reflection on line l.

Then, for any point (x, y), $S_l(x, y) = (x', y')$ where $\begin{cases} x' = x - \dfrac{2a(ax+by+c)}{a^2+b^2} \\ y' = y - \dfrac{2b(ax+by+c)}{a^2+b^2} \end{cases}$

Proof: From the definition of reflection, the line through $P(x, y)$ and $P'(x', y')$ is perpendicular to the given line and the midpoint of (x, y) and (x', y') is on the line l. Refer the figure 3.3.

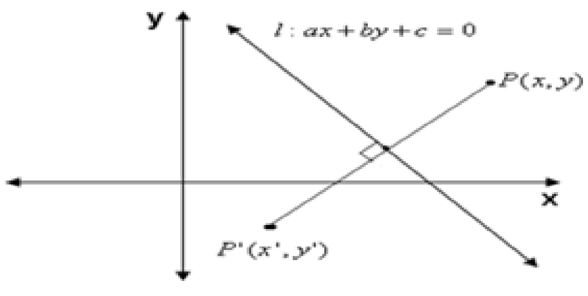


Figure 3.3: Reflection on arbitrary line l

As the slope of the given line is $m = -\dfrac{a}{b}$, the slope of the line through $P(x, y)$ and $P'(x', y')$ is $m' = \dfrac{b}{a}$.

Thus, the equation of the line through $P(x,y)$ and $P'(x',y')$ is given by

$$\frac{y'-y}{x'-x}=\frac{b}{a}\Rightarrow a(y'-y)=b(x'-x)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

Now, the midpoint of $P(x,y)$ and $P'(x',y')$ is on l means $\left(\frac{x+x'}{2},\frac{y+y'}{2}\right)$ is on l.

So, $a\left(\frac{x+x'}{2}\right)+b\left(\frac{y+y'}{2}\right)+c=0\Rightarrow ax'+by'=-2c-ax-by\ldots\ldots\ldots\ldots(ii)$

Combining these two equations gives us

$$\begin{cases} bx'-ay'=bx-ay \\ ax'+by'=-2c-ax-by \end{cases}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(iii)$$

Now, solve these equations for x' and y'. In this equation, by multiplying the first equation by $'b'$, the second by $'a'$ and adding them we obtain,

$$\begin{aligned} a^2x'+b^2x' &= b(bx-ay+a(-2c-ax-by) \\ \Rightarrow x' &= \frac{b(bx-ay)+a(-2c-ax-by)}{a^2+b^2} \\ &= \frac{b^2x+a^2x-2a^2x-2aby-2ac}{a^2+b^2} \\ &= \frac{x(a^2+b^2)-2a(ax+by+c)}{a^2+b^2} \\ &= x-\frac{2a(ax+by+c)}{a^2+b^2} \end{aligned}$$

Similarly, multiplying the first equation by $'a'$, the second by $'b'$ and adding the result gives,

$$\begin{aligned} y' &= \frac{b(-2c-ax-by)-a(bx-ay)}{a^2+b^2} \\ &= \frac{a^2y+b^2y-2b^2y-2abx-2bc}{a^2+b^2} \\ &= \frac{y(a^2+b^2)-2b(ax+by+c)}{a^2+b^2} \\ &= y-\frac{2b(ax+by+c)}{a^2+b^2} \end{aligned}$$

Examples

1. Find the image of the point $(2,3)$ by a reflection on the line $l: 3x-2y+5=0$

Solution: Given $(x,y)=(2,3)$ and from $l: 3x-2y+5=0,\ a=3, b=-2,\ c=5$.

Then,

$$\begin{cases} x' = x - \dfrac{2a(ax+by+c)}{a^2+b^2} = 2 - \dfrac{6(6-6+5)}{9+4} = -\dfrac{4}{13} \\ y' = y - \dfrac{2b(ax+by+c)}{a^2+b^2} = 3 + \dfrac{4(6-6+5)}{9+4} = \dfrac{59}{13} \end{cases}$$

Therefore, $S_l(2,3) = (-\frac{4}{13}, \frac{59}{13})$.

2. Given $S_l(a,b)=(2,5)$ where $l: x-y+1=0$. Find the value of the point (a,b).

Solution: Using the generalized reflection equation derived in the above theorem,

$$S_l(a,b)=(2,5) \Rightarrow \begin{cases} 2 = a - \dfrac{2(a-b+1)}{1+1} = a-(a-b+1) \Rightarrow b=3 \\ 5 = b + \dfrac{2(a-b+1)}{1+1} = b+a-b+1 \Rightarrow a=4 \end{cases}$$

3. Given the lines $m: y=2x+1$ and $n: y=2x-3$. Find the image of the point $(1,1)$ by a product of reflection on line m followed by line n.

Solution: We need to find $S_n \circ S_m(1,1)$

First calculate $S_m(1,1)$ using reflection equation as

$$S_m(1,1) = (x', y') \Rightarrow \begin{cases} x' = 1 - \dfrac{4(2-1+1)}{4+1} = -\dfrac{3}{5} \\ y' = 1 + \dfrac{2(2-1+1)}{4+1} = \dfrac{9}{5} \end{cases}$$

Now, $S_n \circ S_m(1,1) = S_n(S_m(1,1)) = S_n(-\frac{3}{5}, \frac{9}{5}) \Rightarrow \begin{cases} x'' = -\dfrac{3}{5} - \dfrac{4(-\frac{6}{5} - \frac{9}{5} - 3)}{4+1} = \dfrac{21}{5} \\ y'' = \dfrac{9}{5} + \dfrac{2(\frac{-6}{5} - \frac{9}{5} - 3)}{4+1} = -\dfrac{3}{5} \end{cases}$

Therefore, $S_n \circ S_m(1,1) = (\frac{21}{5}, -\frac{3}{5})$.

4 Given the line $l: y=(\tan\theta)x$.

a) Show that $S_l(x,y)=(x',y')$ where $\begin{cases} x'=x\cos 2\theta + y\sin 2\theta \\ y'=x\sin 2\theta - y\cos 2\theta \end{cases}$

b) Calculate the image of the point $(5,5)$ by a reflection on a line $l: y=3x$.

Solution:

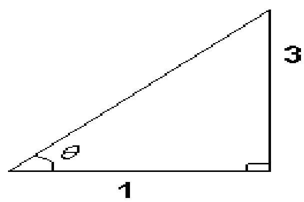
a) Here, $l: y=(\tan\theta)x \Rightarrow (\tan\theta)x - y = 0$. So, $S_l(x,y)=(x',y')$ can be calculated using direct formula

$$\begin{aligned} x' &= x - \frac{2a(ax+by+c)}{a^2+b^2}, \quad a=\tan\theta,\ b=-1,\ c=0 \\ &= x - \frac{2\tan\theta(\tan\theta\, x - y)}{\tan^2\theta+1} = \\ &= \frac{-x\tan^2\theta + x + 2y\tan\theta}{\sec^2\theta}, \quad \tan^2\theta+1=\sec^2\theta \\ &= -x\sin^2\theta + x\cos^2\theta + 2\sin\theta\cos\theta \\ &= x(\cos^2\theta - \sin^2\theta) + y(2\sin\theta\cos\theta) \\ &= x\cos 2\theta + y\sin 2\theta \end{aligned}$$

Similarly, y' can be calculated as follow:

$$\begin{aligned} y' &= y - \frac{2b(ax+by+c)}{a^2+b^2}, \quad a=\tan\theta,\ b=-1,\ c=0 \\ &= y + \frac{2(\tan\theta\, x - y)}{\tan^2\theta+1} = \\ &= \frac{y\tan^2\theta - y + 2x\tan\theta}{\sec^2\theta}, \quad \tan^2\theta+1=\sec^2\theta \\ &= y\sin^2\theta - y\cos^2\theta + 2x\sin\theta\cos\theta \\ &= -y(\cos^2\theta - \sin^2\theta) + x(2\sin\theta\cos\theta) \\ &= x\sin 2\theta - y\cos 2\theta \end{aligned}$$

b) Here, $y=3x \Rightarrow \tan\theta = 3$. Then, using trigonometric relation, we can find $\sin\theta$ and $\cos\theta$ as follow.



$$\tan\theta = 3 \Rightarrow r = \sqrt{10} \Rightarrow \sin\theta = \frac{3}{\sqrt{10}}, \cos\theta = \frac{1}{\sqrt{10}}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = -\frac{4}{5}, \sin 2\theta = 2\sin\theta\cos\theta = \frac{3}{5}$$

Thus, $\begin{cases} x' = x\cos 2\theta + y\sin 2\theta = 5(\frac{-4}{5}) + 5(\frac{3}{5}) = -1 \\ y' = x\sin 2\theta - y\cos 2\theta = 5(\frac{3}{5}) - 5(\frac{-4}{5}) = 7 \end{cases} \Rightarrow (x', y') = (-1,7)$

Here, one can check the correctness of the answer using direct formula of reflection.

$S_l(5,5) = (x', y') \Rightarrow \begin{cases} x' = 5 - \dfrac{6(15-5)}{9+1} = -1 \\ y' = 5 + \dfrac{2(15-5)}{9+1} = 7 \end{cases} \Rightarrow (x', y') = (-1,7)$ which agrees with our previous result.

Reflecting Geometric Figures: So far we saw how to find the image of a point by reflecting on a line. But it is also possible to find the images of different figures (like lines, circles, ellipse, rectangles) under a reflection. So, here under let's see some examples.

Reflecting a circle or an ellipse on a line:

To find the image of a circle under a reflection on a line l, we follow the following procedures.

First: Identify the center O and radius r of the given circle C.

Second: Find the image O' of the center O of the circle by a reflection on l.

Third: Write the equation of the image circle using the center O' and the same radius r. (Why we use the same radius? We use the same radius because reflection is an isometry that preserves length)

We also use the same procedure to reflect an ellipse. That is first identify the center of the ellipse and its minor and major axes, then reflect the center of the ellipse, finally write the equation of the ellipse using the image center and the same axes as the given ellipse. Please bear in mind that these procedure works for all types of isometries.

Examples: Find the images of the circle $C: x^2 + y^2 + 2x - 6y + 6 = 0$ and the ellipse $E: 4x^2 + 9y^2 = 36$ under a reflection on the line $l: y = x + 1$.

Solution: For clarity, let's follow the above procedure directly.

First: Identify the center and radius of the circle $C: x^2 + y^2 + 2x - 6y + 6 = 0$.

By completing square, we get $x^2 + y^2 + 2x - 6y + 6 = 0 \Rightarrow (x+1)^2 + (y-3)^2 = 4$.

Hence, the center is $O = (-1,3)$ and its radius is $r = 2$.

Second : Find the image of the center $O = (-1,3)$ by a reflection on $l: y = x + 1$

Using, reflection formula we get the image of the center to be

$O' = S_l(-1,3) = (2,0)$

Third: Write the equation of the image circle using the image center and the radius of the given circle.

That is the image circle is $C': (x-2)^2 + (y-0)^2 = 4 \Rightarrow x^2 + y^2 = 4x$

In standard form, the ellipse is written as $E: 4x^2 + 9y^2 = 36 \Rightarrow \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1.$

Thus, the major axis is $a = 9$, the minor axis is $b = 2$ and the center is $C(0,0)$.

Besides, the image of the center is $C'(-1,1)$. Therefore, the image of the ellipse becomes $E': \dfrac{(x+1)^2}{9} + \dfrac{(y-1)^2}{4} = 1 \Rightarrow 4(x+1)^2 + 9(y-1)^2 = 36.$

Reflecting a line on a line:

To find the image of a line m under a reflection on another line l, we follow the following procedures.

First: Select any two points P and Q from line m.

Second: Find the image P' and Q' of P and Q by a reflection on l.

Third: Write the equation of the line through P' and Q' which is image of m.

These procedures work for any pairs of line m and l, even when they are intersecting. Besides, these procedure works for all types of isometries.

Example: Find the images of the lines $m: y = 2x - 3$ and $n: y = x + 2$ under a reflection on the line $l: y = x - 1$.

Solution: For clarity, let's follow the above procedure directly.

First: Select two points P and Q from line m. Say, $P = (0,-3), Q = (1,-1)$.

Second: Find the image P' and Q' of P and Q by a reflection on l.

Using, the generalized reflection formula we get the images t be

$P' = (-2,-1), O' = (0,0)$

Third: Write the equation of the line through P' and Q' which is image of m.

That is the image line is $m': y = \frac{1}{2}x$. Use similar argument to find image of n

Application (Shortest Path Problem): Suppose Kumlachew wants to fetch water from a river on his way while he is going from his farm land (found at location A) to his home (found at location B) as shown in figure 3.4 below. Suggest the shortest path that takes kumlachew from his farm land to the river and then to his home and prove that the path you suggested is really the shortest path.

Solution: In this problem, any one can guess that Kumlachew should go straight to some point on the river bank, fetch water and then go to his house. But the problem is to what point of the river should he go so as to minimize the length of the road. First of all assume that the river bank to which Kumlachew wants to go forms straight line say L.

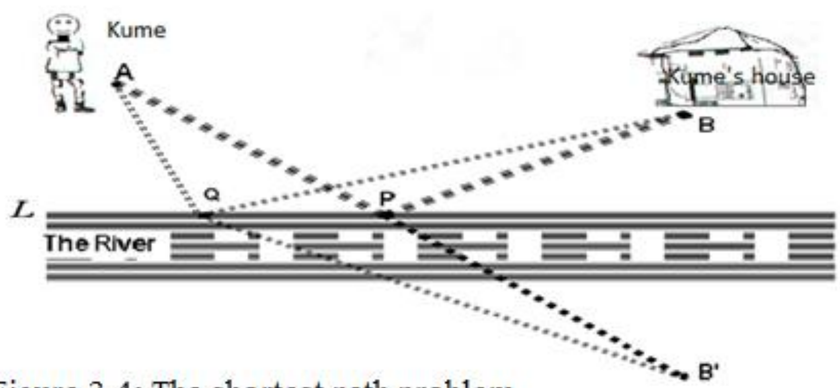


Figure 3.4: The shortest path problem

Claim: Let B' be the reflection of B in L (straight side of the river bank) and let $P = \overleftrightarrow{AB'} \cap L.$ Then, $A \to P \to B$ is the shortest path.

Proof: It suffices to show that if $Q \neq P$ is any other point on L, then $\overline{AP} + \overline{PB} < \overline{AQ} + \overline{QB}.$ Since B' is the reflection of B in L and reflection is an isometry, $\overline{QB} = \overline{QB'}.$ Thus, $\overline{AQ} + \overline{QB'} = \overline{AQ} + \overline{QB}.$ By the same argument, $\overline{PB} = \overline{PB'}$ implies $\overline{AP} + \overline{PB} = \overline{AP} + \overline{PB'}.$

Besides, as A, P, B' are collinear (do you see why?), we have

$\overline{AP} + \overline{PB'} = \overline{AB'}.$ But, by *Triangle Inequality*, $\overline{AB'} < \overline{AQ} + \overline{QB'}.$

Therefore, $\overline{AP} + \overline{PB'} < \overline{AQ} + \overline{QB'} \Rightarrow \overline{AP} + \overline{PB} < \overline{AQ} + \overline{QB}.$

This proves our claim.

Example (Application): Suppose an ant moves in straight lines from position $A = (1,5)$ to position $B = (2,4)$ by touching the line $L: y = x$ at some point P. Find the coordinate of P that minimizes the total path of the ant.

Solution: This is a particular case of the shortest path problem. As in the procedures of the above proof;

First: Find the reflection image of $B = (2,4)$ on the given line $L: y = x$. Say, B'

Second: Form the equation of the line through $A = (1,5)$ and B'. Say, $m: y = ax + b$

Third: Find the intersection of the line $L: y = x$ and $m: y = ax + b$. This is the required point P.

If we reflect the point $B = (2,4)$ on $L: y = x$, we get $B' = (4,2)$. Next find the equation of the line through $A = (1,5)$ and $B' = (4,2)$. It is found to be $m: y = -x + 6$. Finally, determine the intersection of this line with the given line. That is

$L: y = x, m: y = -x + 6 \Rightarrow x = -x + 6 \Rightarrow 2x = 6 \Rightarrow x = 3, y = 3.$

Therefore, the required point is $P = (3,3)$. (If you use Calculus, you will get the same result)

Problem Set 3.3

1. Find the image of a circle $C: x^2 + y^2 - 2x + 6y = 5$ by a reflection on the line $l: y = x - 2$. **Answer** : $(x+1)^2 + (y+1)^2 = 15$

2. Let $l: x = a$ and $m: y = b$ be two lines. If $S_m \circ S_l(2,3) = (4,7)$, find the values of a and b. **Answer** : $a = 3,\ b = 5$

3. Find the image of

   a) the circle $C: x^2 + y^2 = 1$ by a reflection on the line $y = x - 7$.

   b) the ellipse $E: 9x^2 + 4y^2 = 36$ by a reflection on the line $l: y = x + 3$

4. The equation of a reflection is given by $S_l(x,y) = \left(-\frac{4}{5}x - \frac{3}{5}y, -\frac{3}{5}x + \frac{4}{5}y\right)$.

Find the equation of the line of reflection. **Answer** : $l: y = -3x$

5. Suppose an ant moves from the position $A = (0,2)$ to the position $B = (6,1)$ by touching the x-axis at some point P. Find the coordinate of P that minimizes the total path of the ant. **Answer** : $(1/4,0)$

6. If the image of ΔABC by a reflection on the line $l: y = x + 1$ is ΔDEF where the vertices of the image are $D = (-1,1), E = (-5,1), F = (-1,-2)$.

Find the vertices of ΔABC.

Answer : $A = (0,0), B = (0,-4), C = (-3,0)$

7. If a line $y = mx + b$ is reflected along the line $y = mx + c$, show that its image is $y = mx + 2c - b$.

8. Suppose l is a line through the origin which makes an angle of θ from the positive x-axis with $\tan\theta = -3$. Find the image of the point $(-7,1)$ by a reflection on this line. **Answer** : $(5,5)$

9. Let S_l be a reflection on a line l in R^2. Prove that $S_l(v) = 2(u.v)u - v$ for all vectors v in R^2 where u is a unit vector along l.

10. Let l be a line along the vector $\vec{u}$ and let $P_{\vec{u}}(X)$ be the projection of X on l.

If $P_{\vec{u}}\begin{pmatrix}7\\3\end{pmatrix}=\begin{pmatrix}8\\6\end{pmatrix}$, find the reflection image $S_l\begin{pmatrix}7\\3\end{pmatrix}$ on l. **Answer** : $S_l\begin{pmatrix}7\\3\end{pmatrix}=\begin{pmatrix}9\\9\end{pmatrix}$

11. What is the minimum length of a flat-against the wall, full- length mirror for the Smiths who ranges in height from $170cm$ to $182cm$, if you assume eyes are $10cm$ below the top of the head. **Answer** : $72cm$

12. Prove that every reflection is its own inverse.

13. Suppose m and n are two distinct lines. Then, show that

$$S_n \circ S_m(P) = P \Rightarrow P \in m \cap n.$$

14. Can the product of two reflections ever be a reflection? Explain your answer!

3.4.3 Rotation

Definition: A rotation is a transformation in which a figure is turned about a fixed point through an angle of θ in a specific direction. In other words, rotation about a point C through directed angle θ is a transformation that fixes the point C and sends every other point P to P' such that P and P' have the same distance from the fixed point C. Here, the fixed point C is called the center of rotation and the angle θ measured from $\overrightarrow{CP}$ to $\overrightarrow{CP'}$ is called direction of the rotation. The rotation may happen either clockwise or counter clockwise direction, usually clockwise rotation will have negative measure of angle, whereas counter clockwise rotation will have positive measure of angle.

Rotation with center C through an angle of θ is usually denoted by $\rho_{C,\theta}$.

So, the image of any point P under $\rho_{C,\theta}$ is given as:

$$\rho_{C,\theta}(P) = \begin{cases} C, \text{ if } P = C \\ P', \text{ if } P \neq C, \text{ s.t. } \overline{CP} = \overline{CP'} \end{cases}$$

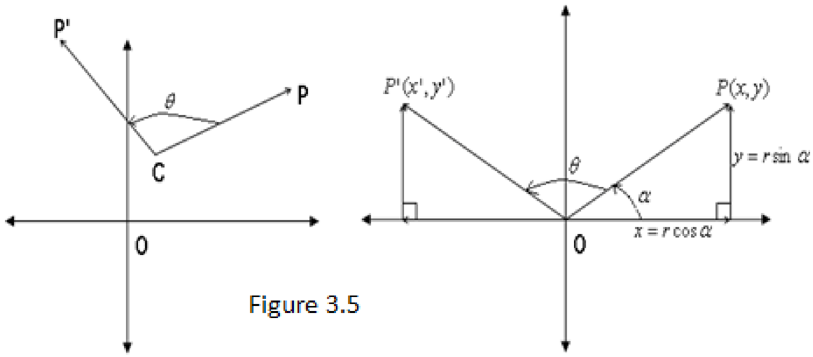


Figure 3.5

Theorem 3.3: A rotation through an angle of θ, about the origin which takes each point $P(x, y)$ in to $P'(x', y')$ is given by $\rho_{O,\theta}(x, y) = (x', y')$, where

$$\begin{cases} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \end{cases}$$

Proof: Let $\rho_{o,\theta}$ be a rotation through an angle of θ, about the origin O and let $P(x, y)$ be any point such that $\rho_{O,\theta}(x, y) = (x', y')$. As shown in figure 3.5, suppose α is the angle from the positive x-axis to the segment $\overline{OP}$.

Then the angle from the positive x-axis to the segment $\overline{OP}'$ will be $\alpha+\theta$. Let $r=|\overline{OP}|$. From the definition of rotation $|\overline{OP}|=|\overline{OP}'|$, so we have $r=|\overline{OP}'|$.

Consequently, using polar coordinates for x' and y', we get

$$\begin{aligned} x' &= r\cos(\alpha+\theta) \\ &= r\cos\alpha\cos\theta - r\sin\alpha\sin\theta, \\ &= x\cos\theta - y\sin\theta \end{aligned} \qquad \begin{aligned} y' &= r\sin(\alpha+\theta) \\ &= r\sin\alpha\cos\theta + r\cos\alpha\sin\theta \\ &= x\sin\theta + y\cos\theta \end{aligned}$$

Examples:

1. Find the image of the point $(1,1)$ and the line $l: 3x-5y=7$ by a a counter clockwise rotation through 90° about the origin.

Solution: From the above theorem**,** for any point $P(x,y)$, $\rho_{O,\theta}(x,y)=(x^{'},y')$ where $x'=x\cos\theta-y\sin\theta$ and $y'=x\sin\theta+y\cos\theta$. So, using $P(1,1)$, $\theta=90^\circ$, and $C=(0,0)$, we get $\rho_{O,\theta}(x,y)=(x^{'},y')=(-1,1)$.

To find the image of the line l, take arbitrary point (x,y) on l.

So $\rho_{O,\theta}(x,y)=(x^{'},y')$ such that $\begin{cases} x'=x\cos 90^\circ - y\sin 90^\circ = -y \\ y'=x\sin 90^\circ + y\cos 90^\circ = x \end{cases}$

Now, solving these equations for x and y and substituting in the equation of l, we get $l': 3y'-5(-x')=7 \Rightarrow 5x'+3y'=7$.

2. Find the image of the point $(\sqrt{3},1)$ by a counterclockwise rotation about the origin with $\theta=105^\circ$.

Solution: Here, we use angle sum formula to determine $\cos(105^\circ),\sin(105^\circ)$. That is

$$\cos(105^\circ)=\cos(60^\circ+45^\circ)=\cos 60^\circ\cos 45^\circ-\sin 60^\circ\sin 45^\circ$$
$$=\frac{1}{2}.\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}.\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}$$
$$\sin(105^\circ)=\sin(60^\circ+45^\circ)=\sin 60^\circ\cos 45^\circ+\cos 60^\circ\sin 45^\circ$$
$$=\frac{\sqrt{3}}{2}.\frac{\sqrt{2}}{2}+\frac{1}{2}.\frac{\sqrt{2}}{2}=\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}$$

Thus, $\begin{cases} x' = \sqrt{3}\cos(105^\circ) - \sin(105^\circ) = \sqrt{3}.(\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}) - (\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}) = -\frac{4\sqrt{2}}{4} = -\sqrt{2} \\ y' = \sqrt{3}\sin(105^\circ) + \cos(105^\circ) = \sqrt{3}.(\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}) + (\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}) = \frac{4\sqrt{2}}{4} = \sqrt{2} \end{cases}$

Therefore, $(x', y') = (-\sqrt{2}, \sqrt{2})$

3. Suppose ρ_θ is a counterclockwise rotation about the origin with $\tan\theta = \frac{2}{3}$.

Find the image of $(5\sqrt{13}, 2\sqrt{13})$ under ρ_θ.

Solution: Here, using right angle triangle, $\tan\theta = \frac{2}{3} \Rightarrow \sin\theta = \frac{2}{\sqrt{13}}, \cos\theta = \frac{3}{\sqrt{13}}$.

$$\begin{cases} x' = x\cos\theta - y\sin\theta \Rightarrow x' = 5\sqrt{13}.\frac{3}{\sqrt{13}} - 2\sqrt{13}.\frac{2}{\sqrt{13}} = 15 - 4 = 11 \\ y' = x\sin\theta + y\cos\theta \Rightarrow y' = 5\sqrt{13}.\frac{2}{\sqrt{13}} + 2\sqrt{13}.\frac{3}{\sqrt{13}} = 10 + 6 = 16 \end{cases} \Rightarrow (x', y') = (11,16)$$

Rotating a line and a circle : To find the image of a line or a circle under a given rotation we use the same procedure as we did in reflection section. For instance to find the image of a circle under a given rotation, first identify the center and radius of the given circle, second find the image of the center under the given rotation, finally write the circle using the image center and the same radius as the given circle. (Please refer the procedures we use for reflecting a line and a circle)

Theorem 3.4 (Generalized Rotation Theorem):

The image of any point $P(x, y)$ under a rotation about arbitrary center $C(h,k)$ through an angle of θ is given by $\rho_{O,\theta}(x, y) = (x', y')$ where

$$\begin{cases} x' = (x-h)\cos\theta - (y-k)\sin\theta + h \\ y' = (x-h)\sin\theta + (y-k)\cos\theta + k \end{cases}$$

Outline of the proof: The proof of this theorem is not difficult. It is simply accomplished by performing the following coordinate transformations step by step.

Step-1: *Translate* the center $C(h,k)$ to the origin by a vector $v=(-h,-k)$ so that any point $P(x,y)$ is translated to the point $(x-h,y-k)$.

Step-2: *Rotate* the result obtained in step one about the origin by an angle θ.

Step-3: *Translate* the result obtained in step two by a vector $-v=(h,k)$ to take every point back to the original position. This will complete the proof.

The equations of the generalized theorem are too complicated to memorize. Subsequently, the readers of this material are advised to remember the methods in the outline of the proof how the formulas are obtained (*Translate*- *Rotate*-*Translate)* rather than memorizing the formulas.

Example: The image of the point $(1,2)$ by a counter clockwise rotation about the center $C=(2,3)$ is $(2,3-\sqrt{2})$. Find the angle of rotation.

Solution: By the generalized rotation theorem,

$$\begin{cases} x' = (x-h)\cos\theta - (y-k)\sin\theta + h \\ y' = (x-h)\sin\theta + (y-k)\cos\theta + k \end{cases}$$

So, for $(x,y)=(1,2)$, $C=(h,k)=(2,3)$ and $(x',y')=(2,3-\sqrt{2})$, we get

$$\begin{cases} x' = -\cos\theta + \sin\theta + 2 = 2 \\ y' = -\sin\theta - \cos\theta + 3 = 3-\sqrt{2} \end{cases} \Rightarrow \begin{cases} x' = -\cos\theta + \sin\theta = 0 \\ y' = -\sin\theta - \cos\theta = -\sqrt{2} \end{cases}$$

$$\Rightarrow \begin{cases} \cos\theta = \sin\theta \\ -\sin\theta - \cos\theta = -\sqrt{2} \end{cases} \Rightarrow \sin\theta = \cos\theta = \frac{\sqrt{2}}{2}$$

Here, both $\sin\theta$ and $\cos\theta$ are positive.

But this is true if and only if θ is in the first quadrant.

Thus, an angle in the first quadrant with $\sin\theta = \cos\theta = \frac{\sqrt{2}}{2}$ is $\frac{\pi}{4}$.

Theorem 3.5 (Formula for center of a rotation):

Suppose the general equation of a rotation about any center $C=(h,k)$ with angle of rotation θ is given by $\rho_{C,\theta}(x,y)=(x',y')$ where

$$\begin{cases} x' = x\cos\theta - y\sin\theta + r \\ y' = x\sin\theta + y\cos\theta + t \end{cases}$$ with r,t,θ being real numbers.

Then, the center $C=(h,k)$ of this rotation is given

$$\begin{cases} h = \dfrac{r}{2} - \dfrac{t}{2\tan\frac{\theta}{2}} \\ k = \dfrac{t}{2} + \dfrac{r}{2\tan\frac{\theta}{2}} \end{cases}$$

Proof: From the general equation, $\begin{cases} x' = (x-h)\cos\theta - (y-k)\sin\theta + h \\ y' = (x-h)\sin\theta + (y-k)\cos\theta + k \end{cases}$

we have $\begin{cases} h(1-\cos\theta) + k\sin\theta = r \\ -h\sin\theta + k(1-\cos\theta) = t \end{cases}$

Solve these equations for h and k whenever r,t,θ are given real numbers.

This can be solved using Cramer's rules with h and k as variables.

$$\Delta = \begin{vmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{vmatrix} = 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta = 2(1-\cos\theta) = 4\sin^2\tfrac{\theta}{2}$$

$$\Delta_h = \begin{vmatrix} r & \sin\theta \\ t & 1-\cos\theta \end{vmatrix} = r(1-\cos\theta) - t\sin\theta$$

$$\Delta_k = \begin{vmatrix} 1-\cos\theta & r \\ -\sin\theta & t \end{vmatrix} = t(1-\cos\theta) + r\sin\theta$$

$$h = \frac{\Delta_h}{\Delta} = \frac{r(1-\cos\theta) - t\sin\theta}{2(1-\cos\theta)} = \frac{r}{2} - t\frac{\sin\theta}{2(1-\cos\theta)} = \frac{r}{2} - t\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}} = \frac{r}{2} - \frac{t}{2\tan\frac{\theta}{2}}$$

$$k = \frac{\Delta_k}{\Delta} = \frac{t(1-\cos\theta) + r\sin\theta}{2(1-\cos\theta)} = \frac{t}{2} + r\frac{\sin\theta}{2(1-\cos\theta)} = \frac{t}{2} + r\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}} = \frac{t}{2} + \frac{r}{2\tan\frac{\theta}{2}}$$

Example: Suppose $R_{C,\theta}$ is a counterclockwise rotation with center $C=(h,k)$

whose equations are given by $\begin{cases} x' = \dfrac{1}{2}x - \dfrac{\sqrt{3}}{2}y + 2 + 4\sqrt{3} \\ y' = \dfrac{\sqrt{3}}{2}x + \dfrac{1}{2}y + 4 - 2\sqrt{3} \end{cases}$. Find the angle and center of this rotation.

Solution: Let's use the formula for center of rotation given in the above theorem.

Here, $r = 2 + 4\sqrt{3}, t = 4 - 2\sqrt{3}$. Besides, $\cos\theta = \frac{1}{2}, \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 60^\circ$.

Hence, $$\begin{cases} h = \dfrac{r}{2} - \dfrac{t}{2\tan\frac{\theta}{2}} = \dfrac{2+4\sqrt{3}}{2} - \dfrac{4-2\sqrt{3}}{2\tan 30^\circ} = 1 + 2\sqrt{3} - \sqrt{3}(2-\sqrt{3}) = 4 \\ k = \dfrac{t}{2} + \dfrac{r}{2\tan\frac{\theta}{2}} = \dfrac{4-2\sqrt{3}}{2} + \dfrac{2+4\sqrt{3}}{2\tan 30^\circ} = 2 - \sqrt{3} + \sqrt{3}(1+2\sqrt{3}) = 8 \end{cases}$$

Therefore, the center of the rotation is $C = (h,k) = (4,8)$.

Theorem 3.6: Let R be a counter clock wise rotation by a given angle θ about the origin. Then,

a) $R_\theta \circ R_\beta = R_{\theta+\beta}$, for any two angles

b) $R_\theta^{-1} = R_{-\theta}$, the inverse of a rotation by θ is a rotation by $-\theta$

c) $R_\theta = i \Leftrightarrow \theta = 2n\pi,\ n \in Z$ (Where i is identity rotation)

Proof: a) Let $P(x,y)$ be any point. Then,

$$\begin{aligned} R_\theta R_\beta(x,y) &= R_\theta(R_\beta(x,y)) \\ &= R_\theta(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta) = (x',y'), \text{ where} \end{aligned}$$

$$\begin{cases} x' = (x\cos\beta - y\sin\beta)\cos\theta - (x\sin\beta + y\cos\beta)\sin\theta \\ y' = (x\cos\beta - y\sin\beta)\sin\theta + (x\sin\beta + y\cos\beta)\sin\theta \end{cases}$$

Rearranging these equations and using angle sum theorem, we get

$$\begin{cases} x' = x\cos(\theta+\beta) - y\sin(\theta+\beta) \\ y' = x\sin(\theta+\beta) + y\cos(\theta+\beta) \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

On the other hand, using $\varphi = \theta + \beta$ as angle of rotation, we get

$$R_{\theta+\beta}(x,y) = (x\cos(\theta+\beta) - y\sin(\theta+\beta), x\sin(\theta+\beta) + y\cos(\theta+\beta)) = (x',y') \ldots\ldots\ldots\ldots(ii)$$

Comparing equations (i) and (ii), one can conclude that $R_\theta R_\beta = R_{\theta+\beta}$.

The proof of part (b) and (c) is left as an exercise.

The above theorem reveals that the composition of two rotations R_θ and R_β about the point C is the same as a single rotation by an angle of $\theta+\beta$ about the same point C. Besides, the order of the rotation is immaterial and we write this relation as $R_{c,\theta}\circ R_{c,\beta}=R_{c,\beta}\circ R_{c,\theta}=R_{c,\theta+\beta}$. On the other hand part (b) of the theorem tells us that the inverse of a rotation about center C by an angle of θ is the same as a rotation about the same center but by an angle of $-\theta$ and this is written as $R^{-1}_{c,\theta}=R_{c,-\theta}$. This means that the inverse of a counter clock wise rotation is the same as a rotation in clock wise direction by the same angle through the same center. Besides, part (c)
shows that a rotation is identity rotation if and only if the angle of rotation is a multiple of 2π, that is $\theta=2n\pi,\ n\in Z$.

Examples:

1. Find the image of the point $(2,5)$ by a product of rotations through an angle of $\theta=15^\circ$ and $\beta=75^\circ$ in counterclockwise direction about the same center $C=(7,2)$.

Solution: From the above theorem, $\rho_{C,\theta}\circ\rho_{C,\beta}(x,y)=\rho_{C,\theta+\beta}(x,y)$ where $(x,y)=(2,5)$, $\theta=15^\circ$, $\beta=75^\circ$ and $C=(7,2)$.

Thus, $\rho_{C,15^\circ}\circ\rho_{C,75^\circ}(2,5)=\rho_{C,90^\circ}(2,5)\Rightarrow\begin{cases}x'=(2-7)\cos 90^\circ-(5-2)\sin 90^\circ+7=4\\ y'=(2-7)\sin 90^\circ+(5-2)\cos 90^\circ+2=-3\end{cases}$

2. Suppose R_θ is a counterclockwise rotation about the origin whose equations are given by $\begin{cases}x'=\dfrac{\sqrt{3}}{2}x-\dfrac{1}{2}y\\ y'=\dfrac{1}{2}x+\dfrac{\sqrt{3}}{2}y\end{cases}$. Find the equations for the inverse, $R^{-1}{}_\theta$, of this rotation.

Solution: Here, $\cos\theta=\dfrac{\sqrt{3}}{2},\sin\theta=\dfrac{1}{2}\Rightarrow\theta=30^\circ$.

These, by part (b) of the above theorem, we have

$$R^{-1}{}_{\theta}(x,y)=(x',y') \text{ where } \begin{cases} x'=x\cos(-30^{\circ})-y\sin(-30^{\circ})=\dfrac{\sqrt{3}}{2}x+\dfrac{1}{2}y \\ y'=x\sin(-30^{\circ})+y\cos(-30^{\circ})=-\dfrac{1}{2}x+\dfrac{\sqrt{3}}{2}y \end{cases}$$

3. If $R_{\theta}=R_{-\theta}$, then what should be the possible values of θ? Particularly for $0<\theta<2\pi$.

Solution: Using part (c) of the above theorem, we have

$$\begin{aligned} &R_{\theta}=R_{-\theta}\Leftrightarrow R_{\theta}(P)=R_{-\theta}(P)\Leftrightarrow R_{\theta}\circ R_{\theta}(P)=R_{\theta}\circ R_{-\theta}(P) \\ &\Leftrightarrow R_{2\theta}(P)=R_{0}(P)=i(P)=P\Leftrightarrow R_{2\theta}=i \\ &\Leftrightarrow 2\theta=2\pi n,\ n\in Z\Leftrightarrow \theta=\pi n,\ n\in Z \end{aligned}$$

Particularly for $0<\theta<2\pi$, $\theta=\pi$, when $n=1$.

4. Suppose R_{θ} and R_{β} are counterclockwise rotations about the origin such that $R_{\theta}\circ R_{\beta}(x,y)=(-y,x)$ and $\beta=35^{\circ}$. Find angle θ.

Solution: Here, using part (a) of the above theorem, we have

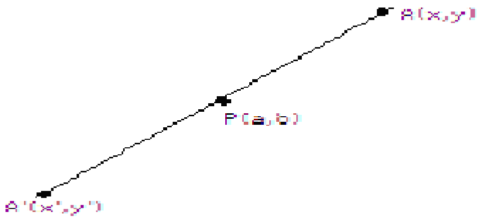
$$R_{\theta}\circ R_{\beta}(x,y)=(-y,x)\Rightarrow \begin{cases} x'=x\cos(\theta+35^{\circ})-y\sin(\theta+35^{\circ})=-y \\ y'=x\sin(\theta+35^{\circ})+y(\theta+35^{\circ})=x \end{cases}$$

$$\Rightarrow \begin{cases} \cos(\theta+35^{\circ})=0 \\ \sin(\theta+35^{\circ})=1 \end{cases} \Rightarrow \theta+35^{\circ}=90^{\circ}\Rightarrow \theta=55^{\circ}$$

Half-turns

Definition: *A half turn* is a rotation by 180°. A half turn about a point P is denoted by H_P. If A is rotated by 180° about point $P(a,b)$, then $A'P=AP$. In other words P is the mid point of A and A'.

Thus using the midpoint formula, $\dfrac{A'+A}{2}=P\Rightarrow \begin{cases} \dfrac{x'+x}{2}=a \\ \dfrac{y'+y}{2}=b \end{cases} \Rightarrow x'=-x+2a,\ y'=-y+2b$.



This will lead us to the formal definition of a half turn. Let $P=(a,b)$. Then, half turn H_P about a point P is a transformation whose equation is given by

$$H_P(x,y)=(x',y'), \text{ where } \begin{cases} x'=-x+2a \\ y'=-y+2b \end{cases}$$

Examples:

1. Find the image of a point $(2,-7)$ by a half-turn about the point $P=(5,-3)$.

Solution: By definition,

$$H_P(x,y)=(x',y') \text{ where} \begin{cases} x'=-x+10 \\ y'=-y-6 \end{cases}. \text{ Therefore, } H_P(2,-7)=(8,1).$$

2.If the image of $(-2,3)$ by a half-turn is $(10,11)$, find the center of the half-turn.

Solution: Let the center be $P=(a,b)$. Then, using the definition, we have

$$\begin{aligned} H_P(-2,3)=(10,11) &\Rightarrow (2a+2,2b-3)=(10,11) \\ &\Rightarrow 2a+2=10, 2b-3=11 \\ &\Rightarrow a=4, b=7 \Rightarrow P=(4,7) \end{aligned}$$

3. Let H_P be a half turn about $P=(-3,2)$. Find,

a) The image of the line $l: y=5x+7$

b) The pre-image of the line $m: y=2x+17$

Solution:

a) Let (x,y) be any point on the given line. Then, $(x',y')=H_P(x,y)=(-x-6,-y+4)$. Solving for x and y from this equation gives us $x=-x'-6,\ y=-y'+4$. So, substitute these values in the equation of the line so that the equation will be in terms of x', y' and that will be the image line. $l: y=5x+7 \Rightarrow l': -y'+4=5(-x'-6)+7 \Rightarrow l': y'=5x'+27$

Therefore, $H_P(l)=l': y=5x+27$. Here, the line $l': y=5x+27$ is called the image of l under H_P and the line $l: y=5x+7$ is called the pre-image of l'.

b) In this case the image of some line m is given we are required to find the pre image m of this line. To do this problem we can use different methods. For instance first calculate the inverse H_P^{-1} of H_P and then find the image of the given line under H_P^{-1}. That will be the pre-image of the given line. Since half turn is an involution, every half turn is its own inverse.

So, $H_P^{-1}(x,y) = (-x-6, -y+4) = (x', y')$.

Using this equation we get the pre-image of the given line to be $m: y = 2x - 1$.

Proposition 3.12(Characterization Theorem of a half turn):

a) A Half turn H_P fixes a line l if and only if $P \in l$.

b) A Half turn H_P fixes a point A if and only if $A = P$.

c) $H_P(Q) = R$ if and only if P is the mid point of Q and R.

That is $H_P(Q) = R \Leftrightarrow P = \dfrac{Q+R}{2}$.

d) Half turn is an involution. That is $H^2{}_P = H_P \circ H_P = i$.

Proof:

a) Let $l: ax + by + c = 0$ be any line and let $P = (h,k)$. Let (x,y) be arbitrary point on line l. Then, $H_P(x,y) = (x', y') = (-x+2h, -y+2k)$. Thus, solving for x and y and substituting in the equation of l, we get,

$ax + by + c = 0 \Leftrightarrow ax' + by' + c - 2(ah + bk + c) = 0.$

But, the last equation defines equation of a line. Thus, the image of the line l under H_P is also a line $l' = H_P(l)$.

Hence, H_P fixes a line l if and only if $l = l'$. But from equation of l and l', they will be the same line if and only if

$$-2(ah + bk + c) = 0 \Leftrightarrow ah + bk + c = 0 \Leftrightarrow (h,k) \in l \Leftrightarrow P \in l.$$

Thus, H_P fixes a line l if and only if $P \in l$.

b) Let $A = (x,y)$ and $P = (h,k)$ such that $H_P(A) = A$. We need to show $A = P$.

But,

$$H_P(A) = A \Leftrightarrow (-x+2h, -y+2k) = (x,y) \Leftrightarrow x = h,\ \ y = k \Leftrightarrow (x,y) = (h,k) \Leftrightarrow A = P$$

Proposition 3.13: The composition of two half turns H_P and H_Q is a translation by a vector $2\overrightarrow{PQ}$ in the direction from P to Q. That is $H_Q \circ H_P = T_{2\overrightarrow{PQ}}$

Proof: Let $A = (x,y)$ be any object point. Then,

$$\begin{aligned}(H_Q \circ H_P)(A) &= H_Q(H_P(A)) = H_Q(-A+2P)\\ &= (-(-A+2P)+2Q) = (A-2P+2Q)\\ &= (A+2(Q-P)) = A+2\overrightarrow{PQ}\\ &= T_{2\overrightarrow{PQ}}(A) = T^2{}_{P,Q} = T_{2PQ}\end{aligned}$$

The composition of two half turns H_P and H_Q is a translation by a vector $2\overrightarrow{PQ}$ in the direction from P to Q. That is $H_Q \circ H_P = T_{2\overrightarrow{PQ}}$

Proposition 3.14: Let P, Q and R be any three points. Then, Q is the midpoint of the P and R if and only if $H_R \circ H_Q = T_{P,R} = H_Q \circ H_P$.

Proof: $(\Rightarrow)$ Suppose Q is the midpoint of P and R. In the above theorem, we have proved that the composition of any two half turns is a translation. Thus, both $H_R \circ H_Q$ and $H_Q \circ H_P$ are translations. Besides, by proposition 3.12, since Q is the midpoint of P and R, $H_R \circ H_Q(P) = H_R(R) = R$ and $H_Q \circ H_P(P) = H_Q(P) = R$. Here, both $H_R \circ H_Q$ and $H_Q \circ H_P$ takes P to R. But, from the previous discussion of translation (in proposition 3.9), there is a unique translation $T_{P,R}$ that takes P to R. Therefore, $H_R \circ H_Q = T_{P,R} = H_Q \circ H_P$.

Conversely, suppose $H_R \circ H_Q = T_{P,R} = H_Q \circ H_P$. We need to show $Q = \frac{1}{2}(P+R)$.

But, for any point X,

$$\begin{aligned}(H_R \circ H_Q)(X) &= H_Q \circ H_P(X)\\ &\Leftrightarrow H_R(-X+2Q) = H_Q(-X+2P)\\ &\Leftrightarrow -(-X+2Q)+2R = -(-X+2P)+2Q\\ &\Leftrightarrow X-2Q+2R = X-2P+2Q\\ &\Leftrightarrow 4Q = 2P+2R\\ &\Leftrightarrow Q = \tfrac{1}{2}(P+R)\end{aligned}$$

Here one can observe that not only the composition of two half turns is a translation but also any translation is the composition of two half turns and this is left for the reader to justify.

Corollary 3.2: Let Q be the midpoint of the points P and R. If S is the midpoint of the points P and Q, then $H_R \circ H_Q = T_{4PS} = H_Q \circ H_P$

Proposition 3.15: If $A \neq C$, then the following conditions are equivalent.

a) B is the midpoint of A and C

b) $H_B \circ H_A = H_C \circ H_B$

c) $H_B \circ H_A \circ H_B = H_C$

Proof: $(a) \Rightarrow (b)$. Suppose B is the mid point of A and C. Then,

$2\overrightarrow{AB} = \overrightarrow{AC} = 2\overrightarrow{BC} \Rightarrow T_{2\overrightarrow{AB}} = T_{2\overrightarrow{BC}} \Rightarrow H_B \circ H_A = H_C \circ H_B$ (From proposition 3.14)

$(b) \Rightarrow (c)$. Suppose $H_C \circ H_B = H_B \circ H_A$. Then,

$$\begin{aligned} H_B \circ H_A = H_C \circ H_B &\Rightarrow H_B \circ H_A \circ H_B = H_C \circ H_B \circ H_B \\ &\Rightarrow H_B \circ H_A \circ H_B = H_C \circ i, \quad H_B \circ H_B = i \\ &\Rightarrow H_B \circ H_A \circ H_B = H_C \end{aligned}$$

$(c) \Rightarrow (a)$. Suppose $H_B \circ H_A \circ H_B = H_C$. Then,

$$\begin{aligned} H_B \circ H_A \circ H_B = H_C &\Rightarrow H_B \circ H_A = H_C \circ H_B \\ &\Rightarrow T_{2\overrightarrow{AB}} = T_{2\overrightarrow{BC}} \\ &\Rightarrow 2\overrightarrow{AB} = 2\overrightarrow{BC} \\ &\Rightarrow \overrightarrow{AB} = \overrightarrow{BC} \\ &\Rightarrow \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = 2\overrightarrow{AB} = 2\overrightarrow{BC} \end{aligned}$$

Therefore, B is the midpoint of A and C. From this, we can infer that for any three collinear points A, B, C, B is the midpoint of A and C if and only if $H_B \circ H_A = H_C \circ H_B$ if and only if $T_{2\overrightarrow{AB}} = T_{2\overrightarrow{BC}}$.

Example: If H_A and H_B are half-turns such that $H_B \circ H_A = H_C \circ H_B$ where $A = (1,3), B = (-1,5)$, find the coordinates of C.

Solution: By proposition 3.15, we have that

$$H_B \circ H_A = H_C \circ H_B \Rightarrow B = \tfrac{1}{2}(A + C) \Rightarrow C = 2B - A = (-2,10) - (1,3) = (-3,7)$$

Proposition 3.16: The composition of three half turns is a half turn.

Proof: Let H_P, H_Q and H_R be three half turns about the points $P = (a,b)$, $Q = (c,d)$ and $R = (e,f)$ respectively. Then, for any object $X = (x,y)$, $H_R \circ H_Q \circ H_P(x,y) = (-x + 2[a + e - c], -y + 2[b + f - d])$. But this is an equation of a half turn with center at the point $(a + e - c, b + f - d)$.

Proposition 3.17: If points P, Q, R and T are non-collinear points, then $H_R \circ H_Q \circ H_P = H_T$ whenever $PQRT$ is a parallelogram.

One of the applications of this corollary is that in any parallelogram $PQRT$, from the knowledge of the three vertices of the parallelogram $PQRT$, any one of the fourth vertex can be determined from the relation $H_R \circ H_Q \circ H_P = H_T$.

Example: Let $PQRT$ be a parallelogram with vertices $P(1,2), Q(6,2), T(1,4)$. Find the vertex R.

Solution: From the above corollary, $PQRT$ is a parallelogram if and only if $H_R \circ H_Q \circ H_P = H_T$. Let $R = (a,b)$ and $X = (x,y)$ be any point.

Then,

$$\begin{aligned}
& H_R \circ H_Q \circ H_P = H_T \\
& \Leftrightarrow H_R \circ H_Q \circ H_P(X) = H_T(X) \\
& \Leftrightarrow H_R \circ H_Q(-x+2,-y+4) = (-x+2,-y+8) \\
& \Leftrightarrow H_R(x+10,y) = (-x+2,-y+8) \\
& \Leftrightarrow (-x-10+2a,-y+2b) = (-x+2,-y+8) \Leftrightarrow (2a-10,2b) = (2,8) \\
& \Leftrightarrow (2a,2b) = (12,8) \Leftrightarrow (a,b) = (6,4)
\end{aligned}$$

Thus, the unknown vertex is $R = (a,b) = (6,4)$.

Problem Set 3.4

1. If a rotation takes the vector $\begin{pmatrix}10\\5\end{pmatrix}$ to $\begin{pmatrix}a\\11\end{pmatrix}$, then find the value(s) of a.

Answer : $a = \pm 2$

2. *Suppose an $X'Y'$-coordinate axis is obtained by rotating an $XY-$ coordinate axis CCW through an angle of $45°$ about the origin. Find the equation of

   a) the curve $3x'^2+y'^2=6$ in the $XY-$coordinate axis

   b) the curve $2x^2+2y^2-2xy=3$ in the $X'Y'-$ coordinate axis.

   Answer : $a)\ x^2+xy+y^2=3$ $\qquad b)\ x'^2+3y'^2=3$

3. Suppose ρ_θ is a counterclockwise rotation about the origin with $\tan\theta=\frac{2}{3}$. Find the image of $(\sqrt{13},\sqrt{13})$ under ρ_θ. **Answer** : $(1,5)$

4. Find the image of the line $l: y=5x+7$ under a half turn H_P with center $P(-3,2)$. **Answer** : $y=2x+27$

5. If a half turn $H_P(x,y)=(4-x,12-y)$ fixes the line $l: y=mx$, find the slope m and give the equation of the line l. **Answer** : $l: y=3x$

6. If $A=(1,2),\ B=(-3,6)$, find C such that $H_B\circ H_A(C)=(4,-2)$.

   Answer : $C=(12,-10)$

7. If $E=(2,3)$, then find D such that $H_E\circ H_D(1,-2)=(3,5)$.

8. For any point (x,y), if $H_A\circ H_B(x,y)=(x+2,y-6)$, find the vector $\overrightarrow{AB}$.

   Answer : $(-1,3)$

9. Let $P=(2,3),\ Q=(-1,5), R=(0,-2)$. Find a point T so that the product of the half turns $H_P\circ H_Q\circ H_R$ is equal to a single half turn about T.

   Answer : $T=(3,-4)$

10. Let P and Q be any two points. Then, show that

    $a)\ T_{\overrightarrow{PQ}}=H_Q\circ H_P \Leftrightarrow P=Q \qquad b)\ H_P\circ H_Q=H_Q\circ H_P \Leftrightarrow P=Q$.

11. Show that rotation preserves parallelism of lines.

12. Show that a translation can be written as a product of two rotations.

13. Prove that if $Q=T_{A,B}(P)$, then $T_{A,B}\circ H_P\circ T_{A,B}^{-1}=H_Q$.

14. Show that for any two points P,Q, $T_{\overrightarrow{PQ}}=H_Q\circ H_P \Leftrightarrow P=Q$.

15. Suppose H_P and H_Q are half turns about the points P and Q respectively.

If $\overrightarrow{PQ} = (-2,3)$, find $H_P \circ H_Q(2,4)$. **Answer** :$(6,-2)$

16. Suppose P, Q, R and T are non-collinear points.

a) Prove that $H_R \circ H_Q \circ H_P = H_T$ whenever $PQRT$ is a parallelogram.

b) If $P(1,2), Q(6,2), T(1,4)$, find the vertex R such that $PQRT$ forms a parallelogram.

3.3.4 Glide Reflection

Definition: A glide reflection g is the composition of a reflection S_l over a line l followed by a translation T_v with *non-zero* vector v where the line l is parallel to the direction of the translation or parallel to the translator vector v. The vector v in this case is called *glide vector* and the line l is called *axis of the*

glide reflection. Here, the vector v is required to be non-zero otherwise translation by a zero vector will be identity map and the composition also will be the usual reflection but not glide reflection. We can easily justify that the same result is obtained by first reflecting and then translating or vice versa. As a result, the order of the two transformations
(Translation and reflection) is immaterial. So, $g = T_v \circ S_l = S_l \circ T_v$.

Example : Let S_l be a reflection on the line $y = x+1$ and T_v be a translation by $v = (1,1)$. Here, $S_l(x,y) = (y-1, x+1)$ and $T_v(x,y) = (x+1, y+1)$ for any point (x,y). So, $g(x,y) = T_v \circ S_l(x,y) = (y, x+2)$ and $g(x,y) = S_l \circ T_v(x,y) = (y, x+2)$ which verifies that $g = T_v \circ S_l = S_l \circ T_v$.

General Equations of Glide-Reflections:

Let g be a glide reflection with axis $l: ax+by+c=0$ and glide vector $\vec{v} = (\mathbf{d}, \mathbf{e})$ with the condition $a.\mathbf{d} + b.\mathbf{e} = 0$ .Then, the general equation of g is given by

$$g(x,y) = T_v \circ S_l(x,y) = S_l(x,y) + \vec{v} = (x', y') \text{ where } \begin{cases} x' = x - \dfrac{2a(ax+by+c)}{a^2+b^2} + \mathbf{d} \\ y' = y - \dfrac{2b(ax+by+c)}{a^2+b^2} + \mathbf{e} \end{cases}$$

Conversely, if g is a glide reflection given by $g(x,y) = (x', y')$ where

$$\begin{cases} x' = ax+by+c \\ y' = bx-ay+d \end{cases}$$

Then, the axis of g is given by $l: 2bx - 2(a+1)y + ad - bc + d = 0$ or $2x = c$.

Examples:

1. Let g be a glide reflection with axis $l: 3x-4y-2=0$ and glide vector $\vec{v} = (-4,-3)$. Find the equation of g and calculate the image of the point $(0,0)$

Solution: Here, $g(x,y) = T_v \circ S_l(x,y) = S_l(x,y) + \vec{v} = (x', y')$ where

$$\begin{cases} x' = x - \dfrac{2a(ax+by+c)}{a^2+b^2} + \mathbf{d} = x - \dfrac{6(3x-4y-2)}{9+16} - 4 = \dfrac{7x}{25} + \dfrac{24y}{25} - \dfrac{88}{25} \\ y' = y - \dfrac{2b(ax+by+c)}{a^2+b^2} + \mathbf{e} = y + \dfrac{8(3x-4y-2)}{9+16} - 3 = \dfrac{24x}{25} - \dfrac{32y}{25} - \dfrac{91}{25} \end{cases}$$

Therefore the image of the point $(0,0)$ is given by

$g(0,0) = (x', y')$ where $x' = \dfrac{7(0)}{25} + \dfrac{24(0)}{25} - \dfrac{88}{25} = -\dfrac{88}{25}, y' = \dfrac{24(0)}{25} - \dfrac{32(0)}{25} - \dfrac{91}{25} = -\dfrac{91}{25}$

Hence, $g(0,0) = (-\dfrac{88}{25}, -\dfrac{91}{25})$.

2. Suppose g is a glide reflection with axis $l: 5x - ky + 7 = 0$ and glide vector $\vec{v} = (6,10)$. Then, find the value of the constant k.

Solution: The condition for g to be a glide reflection with axis $l: ax + by + c = 0$ and glide vector $\vec{v} = (\mathbf{d}, \mathbf{e})$ is that $a.\mathbf{d} + b.\mathbf{e} = 0$. In our case, g to be a glide reflection with axis $l: 5x - ky + 7 = 0$ and glide vector $\vec{v} = (6,10)$, we have $5.(6) - k(10) = 0 \Rightarrow 30 - 10k = 0 \Rightarrow k = 3$

3. Let g be a glide reflection with axis $l: 2x + 7y - 9 = 0$ and glide vector $\vec{v} = (d,4)$. Then, find the value of the constant d.

Solution: Similarly, as in example (2), $2d + 28 = 0 \Rightarrow 2d = -28 \Rightarrow d = -14$.

4. Let g be a glide reflection with axis $l: 2x + y - 5 = 0$ and glide vector $\vec{v} = (d,e)$. If $g(0,0) = (6,-2)$, then find the glide vector $\vec{v}$.

Solution: Here, the general equations of g are given by

$$\begin{cases} x' = x - \dfrac{2a(ax+by+c)}{a^2+b^2} + d = x - \dfrac{4(2x+y-5)}{5} + d = -\dfrac{3}{5}x - \dfrac{4}{5}y + 4 + d \\ y' = y - \dfrac{2b(ax+by+c)}{a^2+b^2} + e = y - \dfrac{2(2x+y-5)}{5} + e = -\dfrac{4}{5}x + \dfrac{3}{5}y + 2 + e \end{cases}$$

That is for any point (x,y), $g(x,y) = \left(-\dfrac{3}{5}x - \dfrac{4}{5}y + 4 + d, -\dfrac{4}{5}x + \dfrac{3}{5}y + 2 + e\right)$

Particularly, $g(0,0) = (6,-2) \Rightarrow (4+d, 2+e) = (6,-2) \Rightarrow 4 + d = 6, 2 + e = -2$

$$\Rightarrow d = 2, e = -4 \Rightarrow \vec{v} = (2,-4)$$

5. Let g be a glide reflection with axis $l: x-y+1=0$ and glide vector $\vec{v}=(3,3)$. If $g(p,q)=(5,8)$, then find the point (p,q).

Solution: Using the general equations of glide- reflection, we have

$$\begin{cases} x' = p - \dfrac{2(p-q+1)}{2} + 3 = q+2 = 5 \Rightarrow q = 3 \\ y' = q + \dfrac{2(p-q+1)}{2} + 3 = p+4 = 8 \Rightarrow p = 4 \end{cases} \Rightarrow (p,q) = (3,4)$$

Proposition 3.19: The square of a glide reflection (composition of a glide reflection with itself) is a translation. That is $g^2(P)=(g\circ g)(P)=T(P)$ for some translation T. Any glide reflection fixes exactly one line, its axis. The midpoint of any point P and its image always lies on the axis of the glide reflection.

Proof: We will prove here the first part of the theorem the others follow from the above discussions. Let g be a glide reflection. Then, by definition of glide reflection, $g=S_l\circ T_v=T_v\circ S_l$ where S_l is a reflection and T_v is a translation with vector v parallel to the line of reflection l.

Thus,

$$\begin{aligned}
& g^2(X) = g\circ g(X) \\
& = (S_l\circ T_v)\circ(S_l\circ T_v)(X) \\
& = (T_v\circ S_l)\circ(S_l\circ T_v)(X) \\
& = T_v\circ(S_l\circ S_l)\circ T_v(X) \\
& = T_v\circ i\circ T_v(X),\quad S_l\circ S_l = i \\
& = T_v\circ T_v(X) \\
& = T_v{}^2(X) \\
& = T_{2v}(X) \\
& \Rightarrow g^2(X) = g\circ g(X) = T_{2v}(X),\ \forall X \\
& \Rightarrow g^2 = T_{2v}
\end{aligned}$$

Hence, composition of a glide reflection with itself is a translation by twice its original glide vector. We know that in any reflection S_l, the mid point of any point P and its image $S_l(P)$ lies on the line of reflection or axis of reflection.

Thus, for any glide reflection g, P and its image $g(P)$ always lies on the axis of the glide reflection, too.

Examples:

1. Suppose g is a glide reflection such that $g^2(3,-5)=(7,1)$.Then, find the glide vector of g.

Solution: Let the glide vector be v. Then, by the above proposition, $g^2=T_{2v}$.

So, for any point P, we have

$$g^2(P)=T_{2v}(P)=2v+P \Rightarrow g^2(3,-5)=(7,1)$$
$$\Rightarrow 2v+(3,-5)=(7,1)$$
$$\Rightarrow 2v=(7,1)-(3,-5) \Rightarrow v=(2,3)$$

2. Suppose g is a glide reflection such that $g(1,2)=(-3,4)$ and $g(-1,3)=(5,7)$. Then, find the axis of g.

Solution: For any glide reflection g, the mid point of any point P and its image always lies on the axis of g. In particular, the mid point of $(1,2)$ and $(-3,4)$, the midpoint of $(-1,3)$ and $(5,7)$ lie on the axis of the glide reflection g. Hence, the axis of g passes through the points $(-1,3)$ and $(2,5)$. Let the axis of g be the line given by $l: y=mx+b$. Then, the slope is $m=\frac{5-3}{2+1}=\frac{2}{3}$.

So, $l: y=\frac{2}{3}x+b$.

Taking one of the above points say $(-1,3)$, $y=\frac{2}{3}x+b \Rightarrow -\frac{2}{3}+b=3 \Rightarrow b=\frac{11}{3}$.

Hence, the axis of the glide reflection is $l: y=\frac{2}{3}x+\frac{11}{3} \Rightarrow 2x-3y+11=0$.

Proposition 3.20: The product (composition) of two glide reflections about the same axis is a translation with translation vector of the sum of the two glide vectors. That is if g and h are glide reflections with the same axis l and glide vectors v and w respectively, then $g \circ h = T_{\vec{v}+\vec{w}}$.

Proof: Suppose g and h are glide reflections with the same axis l and glide vectors v and w respectively. That is $g = S_l \circ T_{\vec{v}}$, $h = S_l \circ T_{\vec{w}}$. We need to show $g \circ h$ is a translation. Here, using associativiy of composition and the fact that $g = S_l \circ T_{\vec{v}} = T_{\vec{v}} \circ S_l$, $h = S_l \circ T_{\vec{w}} = T_{\vec{w}} \circ S_l$, for any point P, we have

$$
\begin{aligned}
(g \circ h)(P) = g(h(P)) &= (S_l \circ T_{\vec{v}}) \circ (S_l \circ T_{\vec{w}}(P)) \\
&= (T_{\vec{v}} \circ S_l) \circ (S_l \circ T_{\vec{w}}(P) \\
&= T_{\vec{v}} \circ (S_l \circ S_l) \circ T_{\vec{w}}(P) \\
&= T_{\vec{v}} \circ i \circ T_{\vec{w}}(P) \\
&= (T_{\vec{v}} \circ T_{\vec{w}})(P) = \vec{v} + \vec{w} + P = T_{\vec{v}+\vec{w}}(P) = T_{\vec{u}}(P),\ \vec{u} = \vec{v} + \vec{w} \Rightarrow g \circ h = T_{\vec{u}}
\end{aligned}
$$

Example: Suppose g and h are glide reflections with the same axis such that $h^2(x, y) = (x-6, y+2)$ and $(g \circ h)(3,-7) = (9,6)$.Then, determine the glide vectors of g and h.

Solution: Let the glide vector of g be $\vec{v}$ and that of h be $\vec{w}$.Then, using proposition 3.19, we have

$$
\begin{aligned}
h^2(P) = T_{2\vec{w}}(P) \Rightarrow h^2(x, y) &= T_{2\vec{w}}(x, y) \\
&\Rightarrow 2\vec{w} + (x, y) = (x-6, y+2) \\
&\Rightarrow 2\vec{w} = (-6,2) \Rightarrow \vec{w} = (-3,1)
\end{aligned}
$$

A gain, using proposition 3.20, we have

$$
\begin{aligned}
g \circ h = T_{\vec{v}+\vec{w}} &\Rightarrow (g \circ h)(P) = T_{\vec{v}+\vec{w}}(P) = \vec{v} + \vec{w} + P, \text{ for any point } P \\
&\Rightarrow (g \circ h)(3,-7) = (9,6), \text{ particularly for } P = (3,-7) \\
&\Rightarrow \vec{v} + \vec{w} + (3,-7) = (9,6) \\
&\Rightarrow \vec{v} + (-3,1) + (3,-7) = (9,6) \\
&\Rightarrow \vec{v} = (9,6) - (0,-6) = (9,12)
\end{aligned}
$$

Problem Set 3.5

1. Let g be a glide reflection with axis $l : x - y + 3 = 0$ and glide vector $\vec{v} = (4,4)$ Then, find the general equation of g and calculate the image of the point $(0,0)$

Answer : $g(x, y) = (y+1, x+7), g(0,0) = (1,7)$

2. Suppose g is a glide reflection with fixed (invariant) line $y = x+1$. If it maps the point $(0,1)$ to $(3,4)$, find its equation. **Answer** : $g(x, y) = (y+2, x+4)$

3. Let g be a glide reflection with axis $l: x+y-7=0$. If $g^2(13,-14) = (3,-4)$, then find its equation. **Answer** : $g(x, y) = (2-y, 12-x)$

4. Let $g = S_l \circ T_v$ be a glide reflection where S_l is a reflection on line l passing through the point $(2,7)$. If $g^2(0,0) = (4,12)$, find the equation of the line l.

Answer : $y = 3x+1$

5. For what value of the constant k will an isometry g be a glide reflection with axis $l: 3x+ky-17=0$ and glide vector $\vec{v} = (-6,2)$? **Answer** : $k = 9$

6. Let g be a glide reflection with axis $l: 2x+7y-9=0$ and glide vector $\vec{v} = (-14, e)$. Then, what must be the value of the constant e? **Answer** : $e = 4$

7. Let g be a glide reflection with axis $l: 2x+y-15=0$ and glide vector $\vec{v}$. If $g(-3,-4) = (20,0)$, then find the glide vector $\vec{v}$. **Answer** : $\vec{v} = (3,-6)$

8. Let g be a glide reflection with axis $l: y = x+1$ and glide vector $\vec{v} = (1,1)$ Then, find the image of the point $(3,2)$. **Answer** : $(2,5)$

9. Let g be a glide reflection with axis $l: x-y+6=0$ and glide vector $\vec{v} = (2,2)$ Then, find the general equation of g and calculate the image of the point $(0,0)$

Answer : $g(x, y) = (y-4, x+8)$

10. Find the equation of a translation T of $2\sqrt{2}$ units at 45° into the first quadrant and a glide reflection g of $2\sqrt{2}$ units at 45° with axis $l: x-y=4$. Find the image of ΔABC with $A=(0,0),\ B=(1,0),\ C=(0,2)$ under each isometgry.

Answer :$T(x,y)=(x+2,y+2),\ g(x,y)=(y+6,x-2)$

11. Consider the line $l:2x+3y-2=0$ and the points $P=\begin{pmatrix}3\\-1\end{pmatrix}, Q=\begin{pmatrix}0\\1\end{pmatrix}$.

Then, give the equations of a glied reflection $g=S_l\circ T_{P,Q}$.

Answer : $g(x,y)=(\frac{5}{13}x-\frac{12}{13}y-\frac{31}{13},-\frac{12}{13}x-\frac{5}{13}y+\frac{38}{13})$

12. Let g be a glide reflection with axis $l:ax-by+c=0$ and glide vector $\vec{v}=(d,e)$. Then, show that $\frac{b}{a}=\frac{d}{e}$.

3. 5 Orientation and Orthogonal Transformations

3.5.1 Orientation of Vectors

Consider a pair of vectors $X=\begin{pmatrix}x\\y\end{pmatrix}$ and $Y=\begin{pmatrix}z\\w\end{pmatrix}$ regarded as order pair (X,Y).

Denote the angle measured from X to Y in the counter clockwise direction by θ where $\theta\neq 0,\pi$ as shown in the diagram below.

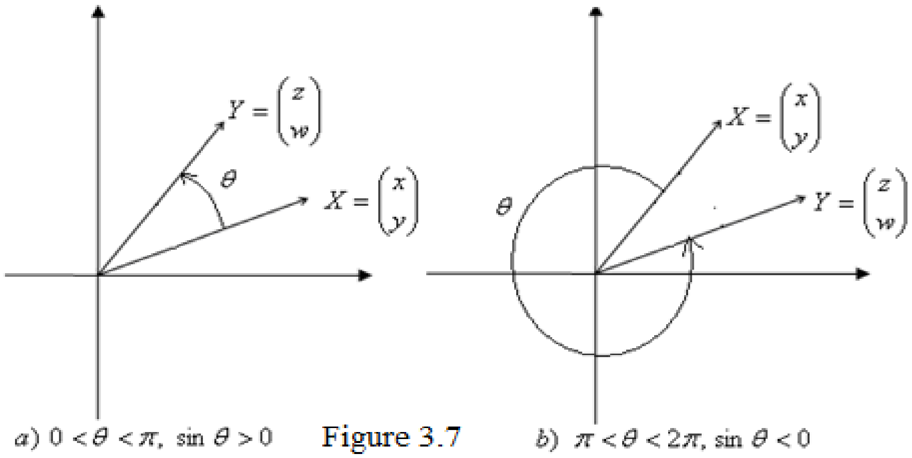


Figure 3.7

Then, with the help of these diagrams we will give the following definitions.

Definition:

a) The pair of vectors (X,Y) is said to be positively oriented if and only if $\sin\theta > 0$. In this case, we say that the vectors X and Y have positive orientation. The first diagram above shows how positively oriented vectors are placed.

b) The pair of vectors (X,Y) is said to be negatively oriented if and only if $\sin\theta < 0$. In this case, we say that the vectors X and Y have negative orientation. The second diagram shows how negatively oriented vectors are placed.

Now, if we are given any two vectors $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $Y = \begin{pmatrix} z \\ w \end{pmatrix}$, how can we determine whether they are positively oriented or negatively oriented simply by using their coordinates x, y, z, w?

The method how can we determine whether a pair of vectors is positively oriented or negatively oriented from their coordinates is given below.

Theorem 3.7 (Orientation Test) :

Let $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $Y = \begin{pmatrix} z \\ w \end{pmatrix}$ be any two non-zero and non- parallel vectors. Then,

a) X and Y are positively oriented if and only if $\det(X,Y) = \begin{vmatrix} x & z \\ y & w \end{vmatrix} > 0$

b) X and Y are negatively oriented if and only if $\det(X,Y)=\begin{vmatrix} x & z \\ y & w \end{vmatrix}<0$

This theorem is known as *Orientation Test Theorem.*

Proof: Consider the following diagram (figure 3.6)

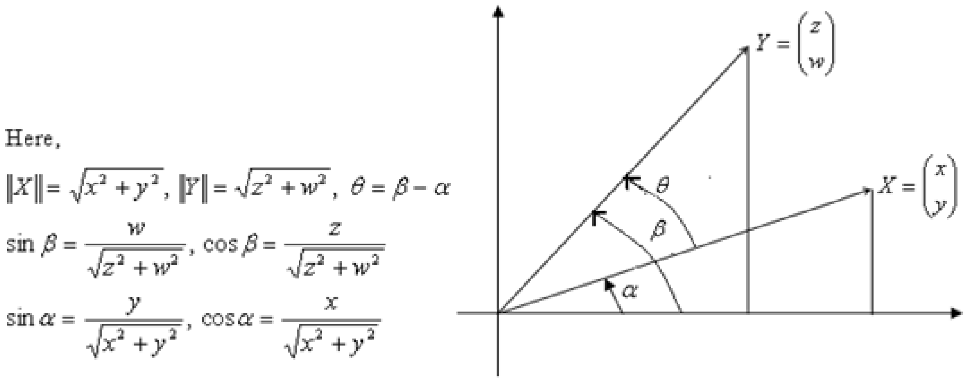


Figure 3.6

From these values,

$$\begin{aligned}\sin\theta &= \sin(\beta-\alpha)\\ &= \sin\beta\cos\alpha - \cos\beta\sin\alpha\\ &= \frac{w}{\sqrt{z^2+w^2}}\frac{x}{\sqrt{x^2+y^2}} - \frac{z}{\sqrt{z^2+w^2}}\frac{y}{\sqrt{x^2+y^2}}\\ &= \frac{xw-yz}{\sqrt{x^2+y^2}\sqrt{z^2+w^2}}\end{aligned}$$

Now, from the above definition the pair (X,Y) is positively oriented if and only if $\sin\theta>0$. But,

$$\begin{aligned}\sin\theta>0 &\Leftrightarrow \frac{xw-yz}{\sqrt{x^2+y^2}\sqrt{z^2+w^2}}>0\\ &\Leftrightarrow xw-yz>0;\ \because \sqrt{x^2+y^2}.\sqrt{z^2+w^2}>0\\ &\Leftrightarrow \begin{vmatrix} x & z \\ y & w \end{vmatrix}>0\end{aligned}$$

On the other hand, the pair (X,Y) is negatively oriented if and only if $\sin\theta<0$. But,

$$\sin\theta < 0 \Leftrightarrow \frac{xw - yz}{\sqrt{x^2 + y^2}\sqrt{z^2 + w^2}} < 0$$

$$\Leftrightarrow xw - yz < 0; \ \because \sqrt{x^2 + y^2}.\sqrt{z^2 + w^2} > 0$$

$$\Leftrightarrow \begin{vmatrix} x & z \\ y & w \end{vmatrix} < 0$$

Hence, the proof is complete.

In general, the vectors $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $Y = \begin{pmatrix} z \\ w \end{pmatrix}$ are

a) Positively oriented if and only if $\sin\theta > 0 \Leftrightarrow \det(X, Y) = \begin{vmatrix} x & z \\ y & w \end{vmatrix} > 0$.

b) Negatively oriented if and only if $\sin\theta < 0 \Leftrightarrow \det(X, Y) = \begin{vmatrix} x & z \\ y & w \end{vmatrix} < 0$.

Examples: Determine whether the following pair of vectors are positively or negatively oriented.

a) $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ b) $X = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $Y = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

Solution:

a) For $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, $\det(X, Y) = \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = 3 > 0$. So, the pair (X, Y) is positively oriented. That means X and Y have positive orientation.

b) For $X = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $Y = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$, $\det(X, Y) = \begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix} = -17 < 0$. So, the pair (X, Y) is negatively oriented. That means X and Y have negative orientation.

Remarks:

i. Orientation is not defined for parallel vectors. Because for parallel vectors, if $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $Y = \begin{pmatrix} z \\ w \end{pmatrix}$, then $Y = tX = \begin{pmatrix} tx \\ ty \end{pmatrix}$ for some scalar t (definition of parallel vectors). Thus, $\det(X, Y) = \begin{vmatrix} x & tx \\ y & ty \end{vmatrix} = 0$. But zero is neither positive nor negative. In this case, we say that X and Y have zero orientation.

ii. Orientation is not defined for three collinear points. For any three collinear points A, B, C, their orientation is determined from the orientation of the vectors $\overrightarrow{AB}, \overrightarrow{AC}$. But if the points A, B, C are collinear, then the vectors $\overrightarrow{AB}, \overrightarrow{AC}$ will be parallel and hence from the first remark their orientation is not defined.

iii. A pair of vectors (X,Y) is assumed to have the same orientation with a pair of vectors (Z,W) if and only if $\det(X,Y)$ and $\det(Z,W)$ have the same sign.

If these determinants $\det(X,Y)$ and $\det(Z,W)$ have opposite sign, then we say that the pairs (X,Y) and (Z,W) have opposite orientation.

Examples:

1. Let $X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $Y = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$, $Z = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $W = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$. Determine whether (Y,W) and (Z, X) have the same or opposite orientation.

Solution: Here, $\det(Y,W) = \begin{vmatrix} -2 & 3 \\ -3 & 5 \end{vmatrix} = -1 < 0$, $\det(Z, X) = \begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} = -2 < 0$.

So, (Y,W) and (Z, X) have the same orientation. But if we take (X,Y) and (Z,W), $\det(X,Y) = \begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} = 1 > 0$, $\det(Z,W) = \begin{vmatrix} 0 & 3 \\ 2 & 5 \end{vmatrix} = -6 < 0.$

Thus, (X,Y) and (Z,W) have the opposite orientation.

2. Let X , Y be any two vectors and $t > 0$ be a scalar. Then,

   a) (X,Y) and (tX,Y) have the same orientation.

   b) (X,Y) and (Y, X) have opposite orientation.

Solution: Let $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $Y = \begin{pmatrix} z \\ w \end{pmatrix}$ and $t > 0$. Then, $\det(tX,Y) = \begin{vmatrix} tx & z \\ ty & w \end{vmatrix} = t\begin{vmatrix} x & z \\ y & w \end{vmatrix}$.

Since $t > 0$, the sign of $\det(tX,Y)$ depends on the sign of $\det(X,Y) = \begin{vmatrix} x & z \\ y & w \end{vmatrix}$.

Hence, (X,Y) and (tX,Y) have the same orientation.

$\det(X,Y)=\begin{vmatrix} x & z \\ y & w \end{vmatrix}$ and $\det(Y,X)=\begin{vmatrix} z & x \\ w & y \end{vmatrix}$. But, from properties of determinants,

$\begin{vmatrix} z & z \\ w & w \end{vmatrix} = -\begin{vmatrix} x & z \\ y & w \end{vmatrix} \Rightarrow \det(Y,X)=(-1)\det(X,Y).$

Thus, (X,Y) and (Y,X) have opposite orientation.

3.5.2 Orientation of Plane Figures

Here, we will see how to determine the orientation of a triangle and then the orientation of other plane figures can be defined in the same way.

Definition (Orientation of Triangles): For any triangle ABC, there are two possibilities for its orientation. If we move from A to B to C again to A in counterclockwise direction (figure 3.7a), then ΔABC is assumed to have *positive orientation.* If A, B, C follow the clockwise direction (figure 3.7b), then ΔABC is assumed to have *negative* orientation.

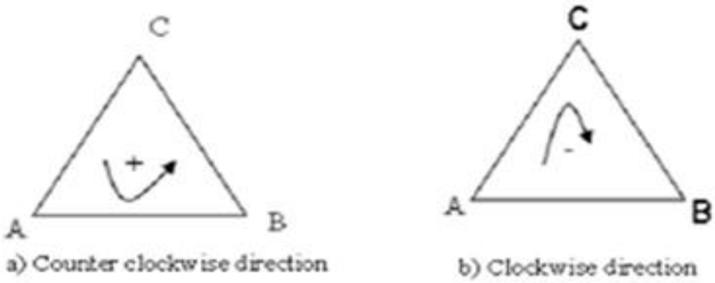


Figure 3.7

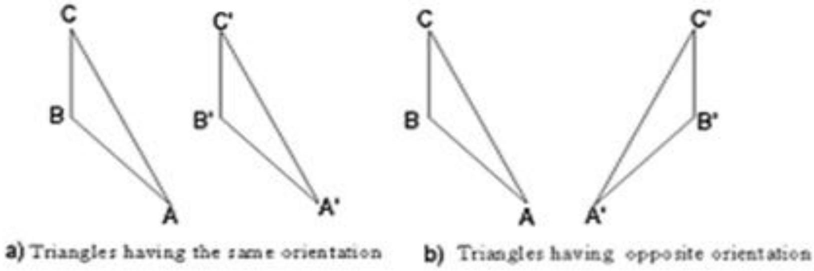


Figure 3.8

Let A, B, C be vertices of ΔABC. Then, the orientation of ΔABC is determined from the orientation of the vectors $\overrightarrow{AB}, \overrightarrow{AC}$. Thus, for

$$A=\begin{pmatrix} x \\ y \end{pmatrix},\ B=\begin{pmatrix} z \\ w \end{pmatrix},\ C=\begin{pmatrix} u \\ v \end{pmatrix}, \overrightarrow{AB}=\begin{pmatrix} z-x \\ w-y \end{pmatrix},\ \overrightarrow{AC}=\begin{pmatrix} u-x \\ v-y \end{pmatrix}$$

So, from *Orientation Test Theorem*, the orientation of the pair of vectors

$(\overrightarrow{AB}, \overrightarrow{AC})$ depends on the sign of $\det(\overrightarrow{AB}, \overrightarrow{AC}) = \begin{vmatrix} z-x & u-x \\ w-y & v-y \end{vmatrix}$.

a) If $\det(\overrightarrow{AB}, \overrightarrow{AC}) > 0,$ then $(\overrightarrow{AB}, \overrightarrow{AC})$ will have positive orientation and so is ΔABC b) If $\det(\overrightarrow{AB}, \overrightarrow{AC}) < 0,$ then $(\overrightarrow{AB}, \overrightarrow{AC})$ will have negative orientation and so is ΔABC

Example: Determine the orientation of ΔABC with vertices

$$A=\begin{pmatrix}1\\3\end{pmatrix},\ B=\begin{pmatrix}3\\2\end{pmatrix},\ C=\begin{pmatrix}4\\6\end{pmatrix}.$$

Solution: Here, $\overrightarrow{AB}=\begin{pmatrix}2\\-1\end{pmatrix},\ \overrightarrow{AC}=\begin{pmatrix}3\\3\end{pmatrix}\Rightarrow \det(\overrightarrow{AB},\overrightarrow{AC})=\begin{vmatrix}2&3\\-1&3\end{vmatrix}=9>0$.

This means the pair of vectors $(\overrightarrow{AB},\overrightarrow{AC})$ has positive orientation which is the orientation of ΔABC .

3.5.3 Orientation Preserving and Orientation Reversing Isometries

Definitions: Let g be any orthogonal transformation. Then, we say that g preserves orientation if and only if for any positively oriented vectors X and Y , their images $X'=g(X),\ Y'=g(Y)$ are again positively oriented vectors. In this case, g is said to be orientation preserving orthogonal transformation. In general, if the pair (X,Y) and the pair $(g(X),g(Y))$ have the same orientation, then g preserves orientation. But, if they have opposite orientation, then g reverses (changes) orientation. In this case, g is said to be orientation reversing (changing) orthogonal transformation.

Examples:

1. Determine whether the following isometries preserve or reverse orientation.

$$a)\ g\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-y\\x\end{pmatrix}\qquad b)\ g\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\-y\end{pmatrix}\qquad c)\ g\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+3\\y-2\end{pmatrix}$$

Solution: Let $X=\begin{pmatrix}x\\y\end{pmatrix}$ and $Y=\begin{pmatrix}z\\w\end{pmatrix}$ be positively oriented vectors. Then,

$$\det(X,Y)=\begin{vmatrix}x&z\\y&w\end{vmatrix}>0$$

a) From the given formula,

$$X'=g(X)=g\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-y\\x\end{pmatrix},\ Y'=g(Y)=g\begin{pmatrix}z\\w\end{pmatrix}=\begin{pmatrix}-w\\z\end{pmatrix}$$

$$\Rightarrow \det(X',Y')=\begin{vmatrix}-y&-w\\x&z\end{vmatrix}=xw-yz=\begin{vmatrix}x&z\\y&w\end{vmatrix}>0$$

Thus, the pair (X',Y') is positively oriented, has the same orientation to the pair (X,Y), which shows that g preserves orientation.

b) Similarly as in part (a), $X' = g(X) = \begin{pmatrix} x \\ -y \end{pmatrix}$, $Y' = g(Y) = \begin{pmatrix} z \\ -w \end{pmatrix}$

$$\Rightarrow \det(X',Y') = \begin{vmatrix} x & z \\ -y & -w \end{vmatrix} = -xw + yz = yz - xw = -\begin{vmatrix} x & z \\ y & w \end{vmatrix} < 0$$

Thus, the pair (X',Y') is negatively oriented, has opposite orientation to the pair (X,Y), which implies that g reverses or changes orientation.

c) Similarly, we get that g preserves orientation.

Theorem 3.8 (General Orientation Test for Transformations):

i) Let $g: R^2 \to R^2$ be any transformation given by $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$. Then, g preserves orientation if and only if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0$.

ii) In general, if g is given by $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by+h \\ cx+dy+k \end{pmatrix}$.

Then, g preserves orientation if and only if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0$.

Proof: Suppose g preserves orientation and let $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $Y = \begin{pmatrix} z \\ w \end{pmatrix}$ be positively oriented vectors. Then,

$$X' = g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix},\ Y' = \begin{pmatrix} az+bw \\ cz+dw \end{pmatrix} \Rightarrow \det(X',Y') = \begin{vmatrix} ax+by & az+bw \\ cx+dy & cz+dw \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{vmatrix} x & z \\ y & w \end{vmatrix}$$

Since g preserves orientation, for any positively oriented vectors $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $Y = \begin{pmatrix} z \\ w \end{pmatrix}$, $\det(X,Y) = \begin{vmatrix} x & z \\ y & w \end{vmatrix} > 0$ implies X' and Y' are also positively oriented.

Hence, $\det(X',Y') = \begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{vmatrix} x & z \\ y & w \end{vmatrix} > 0 \Leftrightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0$ because $\begin{vmatrix} x & z \\ y & w \end{vmatrix} > 0$.

Conversely, suppose $\begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0$. Then, for any positively oriented vectors $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $Y = \begin{pmatrix} z \\ w \end{pmatrix}$, we need to show X' and Y' are also positively oriented.

But, (X, Y) is positively oriented implies that $\det(X, Y) = \begin{vmatrix} x & z \\ y & w \end{vmatrix} > 0$.

On the other hand, from $\begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0$ and $\det(X, Y) = \begin{vmatrix} x & z \\ y & w \end{vmatrix} > 0$, we have that

$$\det(X', Y') = \begin{vmatrix} ax + by & az + bw \\ cx + dy & cz + dw \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{vmatrix} x & z \\ y & w \end{vmatrix} > 0 \Rightarrow \det(X', Y') > 0.$$

Therefore, g preserves orientation.

In general, whether a given transformation g preserves or reverses orientation is determined from its effect on the orientation of a triangle. This means if g preserves the orientation of any triangle ABC, then it is orientation preserving and if g reverses the orientation of ΔABC, then it is orientation reversing transformation.

Now, having this fact as basis, let's see the prove of the second part. Let ΔABC be arbitrary triangle. Then its orientation is determined from the orientation of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Suppose ΔABC has positive orientation. That means $\det(\overrightarrow{AB}, \overrightarrow{AC}) > 0$. On the other hand, let $\Delta A'B'C'$ be the image of ΔABC under g. Then, the orientation of $\Delta A'B'C'$ is determined from the orientation of the pair $(\overrightarrow{A'B'}, \overrightarrow{A'C'})$.

Now suppose g is orientation preserving.

But from $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by + h \\ cx + dy + k \end{pmatrix}$, using $A = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} z \\ w \end{pmatrix}, C = \begin{pmatrix} u \\ v \end{pmatrix}$

$$A' = g(A) = \begin{pmatrix} ax + by + h \\ cx + dy + k \end{pmatrix},\ B' = g(B) = \begin{pmatrix} az + bw + h \\ cz + dw + k \end{pmatrix}, C' = g(C) = \begin{pmatrix} au + bv + h \\ cu + dv + k \end{pmatrix}$$

$$\Rightarrow \overrightarrow{A'B'} = \begin{pmatrix} a(z - x) + b(w - y) \\ c(z - x) + d(w - y) \end{pmatrix},\ \overrightarrow{A'C'} = \begin{pmatrix} a(u - x) + b(v - y) \\ c(u - x) + d(v - y) \end{pmatrix}$$

Thus,

$$\det(\overrightarrow{A'B'},\overrightarrow{A'C'})=\det\begin{pmatrix} a(z-x)+b(w-y) & a(u-x)+b(v-y) \\ c(z-x)+d(w-y) & c(u-x)+d(v-y)\end{pmatrix}$$

$$=\det\left[\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} z-x & u-x \\ w-y & v-y\end{pmatrix}\right] \quad \text{(Using matrix multiplication property)}$$

$$=\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}\det\begin{pmatrix} z-x & u-x \\ w-y & v-y\end{pmatrix} \quad \text{(Using the property, det (AB) = det A.detB)}$$

$$=\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}\det\det(\overrightarrow{AB},\overrightarrow{AC}), \qquad (\det\begin{pmatrix} z-x & u-x \\ w-y & v-y\end{pmatrix}=\det(\overrightarrow{AB},\overrightarrow{AC})$$

$$=\begin{vmatrix} a & b \\ c & d\end{vmatrix}\det(\overrightarrow{AB},\overrightarrow{AC})$$

Hence, g preserves orientation if and only if

$$\det(\overrightarrow{A'B'},\overrightarrow{A'C'})>0 \Leftrightarrow \begin{vmatrix} a & b \\ c & d\end{vmatrix}\det(\overrightarrow{AB},\overrightarrow{AC})>0 \Leftrightarrow \begin{vmatrix} a & b \\ c & d\end{vmatrix}>0 \text{ because } \det(\overrightarrow{AB},\overrightarrow{AC})>0$$

from our assumption.

Similarly, g reverses orientation if and only if

$$\det(\overrightarrow{A'B'},\overrightarrow{A'C'})<0 \Leftrightarrow \begin{vmatrix} a & b \\ c & d\end{vmatrix}\det(\overrightarrow{AB},\overrightarrow{AC})<0 \Leftrightarrow \begin{vmatrix} a & b \\ c & d\end{vmatrix}<0.$$

Examples: Determine whether the following transformations are orientation preserving or orientation reversing.

a) $g: R^2 \to R^2$ given by $g\begin{pmatrix} x \\ y\end{pmatrix}=\begin{pmatrix} 3x-2y+7 \\ x+5y-11\end{pmatrix}$

b) $\alpha: R^2 \to R^2$ given by $\alpha\begin{pmatrix} x \\ y\end{pmatrix}=\begin{pmatrix} x+1 \\ y-11\end{pmatrix}$

Solution:

a) Here, $g\begin{pmatrix} x \\ y\end{pmatrix}=\begin{pmatrix} 3x-2y+7 \\ x+5y-11\end{pmatrix} \Rightarrow a=3, b=-2, c=1, d=5 \Rightarrow \begin{vmatrix} a & b \\ c & d\end{vmatrix}=\begin{vmatrix} 3 & -2 \\ 1 & 5\end{vmatrix}=17>0$

Hence, by the above theorem g preserves orientation.

b) Here, $\alpha\begin{pmatrix} x \\ y\end{pmatrix}=\begin{pmatrix} x+1 \\ 2-y\end{pmatrix} \Rightarrow a=1, b=0, c=0, d=-1 \Rightarrow \begin{vmatrix} a & b \\ c & d\end{vmatrix}=\begin{vmatrix} 1 & 0 \\ 0 & -1\end{vmatrix}=-1<0$

Hence, by the above theorem α reverses orientation.

3.5.4 Orientation and the Fundamental Types of Isometries

In our discussion of isometries, we have seen as there are four basic types of isometries: Translation, Rotation, Reflection and Glide- reflections. Now, using the above general orientation tests, let’s see which of those orthogonal transformations are orientation preserving and orientation reversing (changing) isometries.

I. Rotations:

Let $R_{C,\theta}$ be a counter clockwise rotation with center C through an angle of θ. Then, for any vectors $X=\begin{pmatrix}x\\y\end{pmatrix}$ and $Y=\begin{pmatrix}z\\w\end{pmatrix}$, the isometer $R_{C,\theta}$ is given by

$$X'=R_{C,\theta}(X)=\begin{pmatrix}x\cos\theta-y\sin\theta\\x\sin\theta+y\cos\theta\end{pmatrix} \text{ and } Y'=R_{\theta}(Y)=\begin{pmatrix}z\cos\theta-w\sin\theta\\z\sin\theta+w\cos\theta\end{pmatrix}$$

Here, using the orientation test for vectors, we have

$$\det(X',Y')=\begin{vmatrix}x\cos\theta-y\sin\theta & z\cos\theta-w\sin\theta\\x\sin\theta+y\cos\theta & z\sin\theta+w\cos\theta\end{vmatrix}$$

$$=(x\cos\theta-y\sin\theta)(z\sin\theta+w\cos\theta)-[(x\sin\theta+y\cos\theta)(z\cos\theta-w\sin\theta)]$$

$$=xw\cos^2\theta-yz\sin^2\theta+xw\sin^2\theta-yz\cos^2\theta$$

$$=xw(\sin^2\theta+\cos^2\theta-yz(\sin^2\theta+\cos^2\theta)$$

$$=xw-yz$$

$$=\begin{vmatrix}x & z\\y & w\end{vmatrix}>0\Rightarrow\det(X',Y')>0$$

Thus, the pair (X',Y') is positively oriented for any positively oriented vectors (X,Y). This means $R_{C,\theta}$ preserves orientation.

Alternatively, using the above theorem,

$$X'=R_{\theta}(X)=\begin{pmatrix}x\cos\theta-y\sin\theta+h\\x\sin\theta+y\cos\theta+k\end{pmatrix}$$

$$\Rightarrow a=\cos\theta, b=-\sin\theta, c=\sin\theta, d=\cos\theta$$

$$\Rightarrow\begin{vmatrix}a & b\\c & d\end{vmatrix}=\begin{vmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{vmatrix}$$

$$=\cos^2\theta+\sin^2\theta=1>0$$

Thus, $R_{C,\theta}$ is orientation preserving isometry.

II. Translations: Let $T : R^2 \to R^2$ be a translation with translation vector $v = \begin{pmatrix} a \\ b \end{pmatrix}$. Then, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+h \\ y+k \end{pmatrix}$. Is T orientation preserving or orientation reversing?

Here, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+h \\ y+k \end{pmatrix} \Rightarrow a = 1, b = 0, c = 0, d = 1 \Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 > 0$

Hence, T is orientation preserving isometry.

III. Reflections:

Let $l : ax + by + c = 0$ be any line and S_l be a reflection on line l.

For any vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$ in a plane, $S_l\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ where $\begin{cases} x' = x - \dfrac{2a(ax+by+c)}{a^2+b^2} \\ y' = y - \dfrac{2b(ax+by+c)}{a^2+b^2} \end{cases}$

After some rearrangement, we get

$$\begin{cases} x' = \dfrac{(b^2-a^2)}{a^2+b^2}x - \dfrac{2ab}{a^2+b^2}y - \dfrac{2ac}{a^2+b^2} \\ y' = \dfrac{-2ab}{a^2+b^2}x + \dfrac{(a^2-b^2)}{a^2+b^2}y - \dfrac{2bc}{a^2+b^2} \end{cases}$$

From, these equations we obtain the determinant of the coefficients as

$$\begin{vmatrix} \dfrac{b^2-a^2}{a^2+b^2} & -\dfrac{2ab}{a^2+b^2} \\ \dfrac{-2ab}{a^2+b^2} & \dfrac{a^2-b^2}{a^2+b^2} \end{vmatrix} = \frac{(b^2-a^2)(a^2-b^2)-4a^2b^2}{(a^2+b^2)^2} = \frac{-(b^2-a^2)^2-4a^2b^2}{(a^2+b^2)^2}$$

$$= \frac{-(b^4-2a^2b^2+a^4)-4a^2b^2}{(a^2+b^2)^2} = \frac{-[b^4+2a^2b^2+a^4)}{(a^2+b^2)^2}$$

$$= \frac{-[b^4+2a^2b^2+a^4)}{(a^2+b^2)^2} = \frac{-(b^2+a^2)^2}{(a^2+b^2)^2} = -1 < 1$$

Thus, a reflection, S_l on any line l is orientation preserving isometry.

IV. Glide-Reflections

Any glide reflection is a composition of a reflection S_l over a line l and a translation T_v with *non-zero* vector v where the line l is parallel to the direction of the translator vector v. But from (I) and (III) cases above, we saw that translations and reflections are orientation preserving and orientation reversing isometries respectively. Therefore, their composition will be orientation reversing isometry (we will prove this later on as a theorem in *Chapter 5*) and thus glide reflection is orientation reversing.

To sum up our discussion, let's summarize our results above as conclusion.

Conclusion: Translations and rotations (including identity) are the only types of isometries preserving orientation. Reflections and glide-reflections are the only types of isometries reversing (changing) orientation.

Example: Suppose α is an isometry which maps ΔPQR into $\Delta P'Q'R'$ where the vertices of the triangles are

$$P=\begin{pmatrix}4\\2\end{pmatrix},\ Q=\begin{pmatrix}1\\-2\end{pmatrix},\ R=\begin{pmatrix}12\\-4\end{pmatrix},\ P'=\begin{pmatrix}2\\3\end{pmatrix},\ Q'=\begin{pmatrix}-1\\-1\end{pmatrix},\ R'=\begin{pmatrix}10\\-3\end{pmatrix}.$$

Determine whether α is a translation, rotation, glide reflection or reflection and find its equation.

Solution: To apply the above test, first determine the orientation of ΔPQR and $\Delta P'Q'R'$. The orientation of ΔPQR is determined from $\det\left(\overrightarrow{PQ},\overrightarrow{PR}\right)$ where

$\det\left(\overrightarrow{PQ},\overrightarrow{PR}\right)=\begin{vmatrix}-3 & 8\\-4 & -6\end{vmatrix}=50>0$. Hence, ΔPQR has positive orientation.

Similarly, the orientation of $\Delta P'Q'R'$ is determined from $\det\left(\overrightarrow{P'Q'},\overrightarrow{P'R'}\right)$ where

$\det\left(\overrightarrow{P'Q'},\overrightarrow{P'R'}\right)=\begin{vmatrix}-3 & 8\\-4 & -6\end{vmatrix}=50>0$ which shows that $\Delta P'Q'R'$ also has positive orientation. So, we have got that ΔPQR and $\Delta P'Q'R'$ have the same orientation.

As a result, α is orientation preserving isometry. Therefore, α is either a translation or a rotation.

Now, to determine whether α is a translation or a rotation, find the vectors $\overrightarrow{PP'}, \overrightarrow{QQ'}, \overrightarrow{RR'}$.

Here, $\overrightarrow{PP'}=\begin{pmatrix}-2\\1\end{pmatrix}$, $\overrightarrow{QQ'}=\begin{pmatrix}-2\\1\end{pmatrix}$, $\overrightarrow{RR'}=\begin{pmatrix}-2\\1\end{pmatrix}$. Thus, $\overrightarrow{PP'}=\overrightarrow{QQ'}=\overrightarrow{RR'}$.

So, α is a translation given by $\alpha\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-2 \\ y+1 \end{pmatrix}$.

(What happens if $\overrightarrow{PP'} \neq \overrightarrow{QQ'} \neq \overrightarrow{RR'}$? We will see the genral case in section 3.9)

3.6 Fixed Points of Isometries

Classification of isometries based on their fixed points:

So far we have seen about fixed points of transformations. Now, among the fundamental types of isometries, we are going to see which isometrics are with fixed points and without fixed points.

i) Exacltly one fixed point: Isometries that have *exactly one* fixed point are only **Rotations:** Any rotation has exactly one fixed point and the fixed point is exactly the center of the rotation.

Example: Show that that the only fixed point of a rotation R_θ about the origin is the origin itself.

Solution: Let $P=(x,y)$ be arbitrary fixed point of R_θ. Then we need to find the coordinates of P. From the definition of fixed point, we have that

$R_\theta(P) = R_\theta(x,y) = P = (x,y) \Rightarrow \begin{cases} x = x\cos\theta - y\sin\theta \\ y = x\sin\theta + y\cos\theta \end{cases}$. Now, we have to solve this system for x and y.

From $x = x\cos\theta - y\sin\theta$, solving for y we get $y = \dfrac{x\cos\theta - x}{\sin\theta}$. Substitute this value of y in the second equation $y = x\sin\theta + y\cos\theta$.

$$\frac{x\cos\theta - x}{\sin\theta} = x\sin\theta + \frac{x\cos\theta - x}{\sin\theta}\cos\theta \qquad (*)$$

$$\Rightarrow \frac{x\cos\theta - x}{\sin\theta} = \frac{x\sin^2\theta + x\cos^2\theta - x\cos\theta}{\sin\theta}$$

$$\Rightarrow x\cot\theta - \frac{x}{\sin\theta} = \frac{x}{\sin\theta} - x\cot\theta$$

$$\Rightarrow x\cot\theta = \frac{x}{\sin\theta} \Rightarrow x(\frac{\cot\theta - 1}{\sin\theta}) = 0 \Rightarrow x = 0$$

So the only solution for $(*)$ is $x=0$ and substituting this value of x in

$y = \dfrac{x\cos\theta - x}{\sin\theta}$, we get $y = 0$.

Therefore, the only fixed point of a rotation R_θ about the origin is $P = (0,0)$ which is the origin itself.

ii) Two or more fixed points:

Any isometry that has two fixed points but not identity is **a reflection** over a line and the whole points on the line of reflection are also fixed points.

As a result the line of reflection is a fixed line.

iii) Three non-collinear fixed points:

An isometry that has three non-collinear fixed points is an **identity**.

iv) No fixed poin:

Isometries that have *no* fixed point at all- This category includes **translation** and **glide reflection.**

Examples:

1. An isometry ρ has exactly one fixed point $(5,1)$ and maps the point $(7,2)$ into $(3,0)$. Then, find the equation of ρ.

Solution: An isometry with exactly one fixed point is a rotation where the fixed point is the center itself.So, ρ is a rotation with center $(5,1)$.

Hence, its equation at any point (x, y) becomes $\rho_{C,\theta}(x, y) = (x', y')$ where

$$\begin{cases} x' = (x-5)\cos\theta - (y-1)\sin\theta + 5 \\ y' = (x-5)\sin\theta + (y-1)\cos\theta + 1 \end{cases}$$

In particular, $\rho_{C,\theta}(7,2) = (3,0) \Rightarrow \begin{cases} x' = 2\cos\theta - \sin\theta + 5 = 3 \\ y' = 2\sin\theta + \cos\theta + 1 = 0 \end{cases} \Rightarrow \begin{cases} 2\cos\theta - \sin\theta = -2 \\ 2\sin\theta + \cos\theta = -1 \end{cases}$

Here, adding twice the second on the second gives $5\cos\theta = -5 \Rightarrow \cos\theta = -1$.

Similarly, subtracting twice the second from the first gives

$-5\sin\theta = 0 \Rightarrow \sin\theta = 0$. Thus, $\cos\theta = -1, \sin\theta = 0 \Rightarrow \theta = \pi$.

Therefore, the equation of ρ is given by $\rho_{(5,1),\pi}(x, y) = (10 - x, 2 - y)$.

2. A non identity isometry α fixes the points $(-3,-5)$ and $(1,4)$. Then, find the equation of α.

Solution: A non identity two given fixed points is a reflection on aline through the fixed points. So, α is a reflection on a line through the points $(-3,-5)$ and $(1,4)$. Thus, the line of reflection has slope $m = \frac{9}{4}$. Hence, $l: y = \frac{9}{4}x + b$ but $(-3,-5) \in l$.

That is $y = \frac{9}{4}x + b \Rightarrow -5 = -\frac{27}{4} + b \Rightarrow b = \frac{7}{4} \Rightarrow y = \frac{9}{4}x + \frac{7}{4} \Rightarrow 9x - 4y + 7 = 0$.

Hence, you can find the equations of α using general equations of a reflection.

3.7 Linear and Non-linear Isometries

Any given isometry is said to be linear or non-linear based on whether the origin is its fixed point or not. In what follows, we are going to see linear isometry and decomposition of an isometry using linear isometry.

Definition: Any isometry β is said to be linear isometry if and only if it fixes the origin. That means β is linear if and only if $\beta(0) = 0$.

Examples:

a) $\beta : R \to R$ given by $\beta(x) = -x$ is a linear isometry because $\beta(0) = 0$.

b) $\beta : R^2 \to R^2$ given by $\beta(x, y) = (-x, -y)$ is a linear isometry because $\beta(0,0) = (0,0)$.

c) $\beta : R^2 \to R^2$ given by $\beta(x, y) = (\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x + \frac{1}{2}y)$ is a linear isometry because $\beta(0,0) = (0,0)$.

d) $\beta : R^2 \to R^2$ given by $\beta(x, y) = (x+1, y-2)$ is not linear isometry. Because it is an isometry but it does not fix the origin, $\beta(0,0) = (1,-2) \neq (0,0)$.

Proposition 3.20: The composition of any two linear isometries is linear isometry.

Proof: Let α and β be any two linear isometries on the same plane. Since the composition of any two isometries is an isometry, $\alpha \circ \beta$ is also an isometry. Besides, as α and β are linear, $\alpha(0) = 0,\ \beta(0) = 0$.

Therefore, $\alpha \circ \beta(0) = \alpha(\beta(0)) = \alpha(0) = 0$.

Hence the composition $\alpha \circ \beta$ preserves the origin and it is a linear isometry.

Proposition 3.21: Any given isometry can be expressed as a composition of a translation and a linear isometry.

Proof: Let α be any isometry. Define a vector v by $v = \alpha(0)$ and consider $\beta = T_{-v} \circ \alpha.$

Claim: β is a linear isometry. Since T_{-v} and α are isometries and so is their composition (as the composition of any two isometries is again an isometry) . Hence, $\beta = T_{-v} \circ \alpha$ is an isometry. Beside,

$\beta(0) = T_{-v} \circ \alpha(0) = T_{-v}(\alpha(0)) = T_{-v}(v) = v + -v = 0$. This means β fixes the origin.

Therefore, β is a linear isometry. Thus, $\beta = T_{-v} \circ \alpha \Rightarrow \alpha = T_v \circ \beta$.

Hence, for any isometry α, $\alpha = T_v \circ \beta$ where T_v is a translation and β is a linear isometry. But as α is an arbitrary isometry, any isometry can be expressed as a composition of a translation and a linear isometry.

Example: Let $\alpha : R^2 \to R^2$ be an isometry given by $\alpha(x,y) = (-x+2, -y+14)$. Express α as a composition of a translation and a linear isometry.

Solution: Following the proof of the above proposition, take the vector $v = \alpha(0,0) = (2,14)$.

Then, for any object (x,y), take $\beta(x,y) = T_{-v} \circ \alpha(x,y) = (-x,-y)$

Here, clearly β is an isometry. Besides, $\beta(0,0) = (0,0)$ which means β is a linear isometry and T_v is a translation with translator vector $v = (2,14)$.

Therefore, $\alpha = T_v \circ \beta$ where $T_v(x,y) = (x+2, y+14),\ \beta(x,y) = (-x,-y).$

Proposition 3.22: Every isometry f in R has a unique decomposition of the form $f = T_c \circ \beta$ where $c = f(0)$ and $\beta(x) = \pm x$.

Proof: Let f be an isometry in R such that $c = f(0)$ and let T_{-c} be a translation by $-c$ in R such that $\beta = T_{-c} \circ f$.

Claim: β is a linear isometry. Since T_{-c} and f are isometries and so is their composition (as the composition of any two isometries is again an isometry). Hence, $\beta = T_{-c} \circ f$ is an isometry.

A gain, $\beta(0) = T_{-c} \circ f(0) = T_{-c}(f(0)) = T_{-c}(c) = c + -c = 0$. This means β fixes the origin and thus β is a linear isometry. Besides, in R, $d(x,y) = |x-y|$.

Hence, $|\beta(x)| = |\beta(x) - \beta(0)| = |x-0| = |x| \Rightarrow \beta(x) = \pm x \Rightarrow \dfrac{\beta(x)}{x} = \pm 1,\ \forall x \neq 0,\ x \in R.$

If $\dfrac{\beta(x)}{x} \neq \pm 1$ at least for some x_1, x_2, we get $\beta(x_1) \neq \pm x_1$ and $\beta(x_2) \neq \pm x_2$ which implies $|\beta(x_2) - \beta(x_1)| \neq |x_2 - x_1|$.

But this is a contradiction because β is an isometry.

Therefore, $\beta(x) = \pm x,\ \forall x \in R.$

Hence, from $\beta = T_{-c} \circ f$, composing both sides on the left by T_c gives $T_c \circ \beta = T_c \circ (T_{-c} \circ f) \Rightarrow f = T_c \circ \beta$ which means $f(x) = T_c \circ \beta(x) = \pm x + c$ where $c = f(0)$ and $\beta(x) = \pm x$. Complete the uniqueness.

Example: Let $f: R \to R$ be an isometry given by $f(3)=6$. Find a translation T and a linear isometry β with $\beta(2)=-2$.

Solution: Any isometry in R is of the form $f(x)=\pm x+c, \forall x \in R$ for a fixed constant c.

But $f(3)=6 \Rightarrow \pm 3+c=6 \Rightarrow c=3$ or $c=9 \Rightarrow f(x)=x+3$ or $f(x)=-x+9$. Then, following the proof of the above proposition,

$f(x)=x+3 \Rightarrow \beta(x)=(T_{-c} \circ f)(x)=(T_{-3} \circ f)(x)=x+3-3 \Rightarrow \beta(x)=x$ **or**
$f(x)=-x+9 \Rightarrow \beta(x)=(T_{-c} \circ f)(x)=(T_{-9} \circ f)(x)=-x+9-9=-x \Rightarrow \beta(x)=-x$

Here, we have two options for β, that is $\beta(x)=x$ or $\beta(x)=-x$.

But $\beta(2)=-2 \Leftrightarrow \beta(x)=-x$.

Hence, we have got a linear isometry β and a translation T_9 such that $f=T_9 \circ \beta$ where $T_9(x)=x+9,\ \beta(x)=-x$.

Problem Set 3.6

1. Determine the orientation of ΔABC whose vertices are
$A=\begin{pmatrix}-4\\1\end{pmatrix}, B=\begin{pmatrix}-1\\1\end{pmatrix}, C=\begin{pmatrix}-1\\-3\end{pmatrix}$

2. Let $f: R \to R$ be an isometry with $f(2)=10$. Find a translation T and a linear isometry β with $\beta(-2)=2$.

Answer : $f=T_{12} \circ \beta$ where $T_{12}(x)=x+12,\ \beta(x)=-x$

3. Let α be a half turn about $C=(1,0)$. Find a vector v and a linear isometry β such that $\alpha=T_v \circ \beta$. **Answer** : $v=(2,0),\ \beta(x,y)=(-x,-y)$

4. Let f be a CCW rotation about $C=(4,2)$ by an angle of $\theta=\frac{\pi}{3}$. Find a translation T_v and a linear isometry β such that $f=T_v\circ\beta$.

Answer :$T_v(x,y)=(x+2+\sqrt{3},y+1-2\sqrt{3}),\ \beta(x,y)=(\frac{1}{2}x-\frac{\sqrt{3}}{2}y,\frac{\sqrt{3}}{2}x+\frac{1}{2}y)$

5. Let $\alpha:R^2\to R^2$ be given by $\alpha(x,y)=(\frac{1}{2}x-\frac{\sqrt{3}}{2}y+1,\frac{\sqrt{3}}{2}x+\frac{1}{2}y-1)$. Show that α is an isometry but not linear isometry. Find a translation T_v and a linear isometry β such that $\alpha=T_v\circ\beta$.

Answer :$T_v(x,y)=(1,-1),\ \beta(x,y)=(\frac{1}{2}x-\frac{\sqrt{3}}{2}y,\frac{\sqrt{3}}{2}x+\frac{1}{2}y)$

6. If S_l fixes the points $(1,4)$ and $(-5,0)$,then find the equation of line l.

Answer : $l:2x-3y+10=0$

7. Suppose f is a non-identity isometry such that $f(\sqrt{5},4)=(\sqrt{5},4)$ and $f(0,-1)=(0,-1)$. Find the value of $f(1,0)$ and explain how you know $f(1,0)$ must have that value. Determine the general equation of this isometry.

Answer :It a reflection on a line $l:y=\sqrt{5}x-1$

8. If an isometry α is involution, prove that for any point P, the mid point of P and $\alpha(P)$ is a fixed point of α.

9. Suppose f is a non-identity isometry which fixes $(1,1)$ and $(2,2)$. Find the equation of f and calculate $f(1,0)$.

10. Prove that the inverse of any linear isometry is also linear isometry.

3.8 Representations of Orthogonal (Isometric)

Transformation as a Product of Reflections

So far, we have discussed that the product of two isometrics is a gain an isometry. Besides, we have encountered four types of isometries: reflections, translations, rotations and glide reflections. From now onwards, our concern is to investigate: "*Are these the only isometries or do any other exist*?" In other words, from the fact that the product of two isometries is again an isometry, it is natural to ask ourselves whether the product of any of two isometries we have seen so far results among one of those or a new one that we did not discussed. By investigating, all of the possible combination products of reflection, rotation and translation, we will find out that the result is any one of these but not a new form of isometry. Finally, we will generalize that there are no other isometries or our bag of isometries is complete. Besides, we will conclude that reflections are the building blocks for plane isometries from the result that every isometry is the product of three or fewer reflections. Further more; if it is a rotation, then through what angle and with what center? If it is a translation, what is the translation vector? If it is a reflection, what is the line of reflection? and so on will be answered.

3.8.1 Product of Reflections on Two Lines

When we say product of reflections on two lines , we mean $S_n \circ S_m$, reflection on line m followed by a reflection on line n. Here, m and n may be intersecting or parallel. From our next discussion, we will point out that the composition of two reflections $S_n \circ S_m$ is a rotation, a translation or a reflection itself. So, later on after the discussion, you should be able to describe in detail in what situation it will be a *rotation, translation* or *reflection*.

Case I: When the two Lines are Intersecting

Theorem 3.7: The composition of two reflections on two intersecting lines l and m is a rotation about the point of intersection through twice the directed angle between the lines.

Restatement: Let l and m be any two lines intersecting at $c=(a,b)$ and the angle measured from l to m being θ. Then $S_m \circ S_l = \rho_{c,2\theta}$

Proof: Consider the figure 3.9a: To prove this theorem, it suffices to show that $\angle PQP'' = 2\angle RQT$ and $\overline{QP} = \overline{QP''}$. In order to show these two conditions, let's consider two different cases:

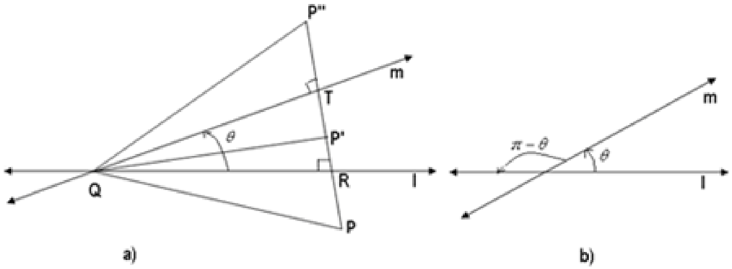


Figure 3.9

Case-1: Suppose $P \notin m$. Then, $\overline{PR} = \overline{P'R}$ by definition of reflection and $\overline{QR} = \overline{QR}$ by reflexive property. Again, $\overline{QR}$ is perpendicular to $\overline{PP'}$ as m is the line of reflection of P to P'. Hence, $\Delta PQR \cong \Delta P'QR$ by SAS which implies that $\angle PQR \cong \angle P'QR$.

Similarly, $\Delta P'QT \cong \Delta P''QT$ by SAS and hence $\angle P'QT \cong \angle P''QT$. But, $m(\angle PQP'') = m\angle PQP') + m(\angle P'QP'') = 2m(\angle RQT) = 2\theta.$ (Do you see how?).

Besides, as $\Delta PQP'$ and $\Delta P'QP''$ are isosceles, $\overline{PQ} \equiv \overline{P'Q}$ and $\overline{P'Q} \equiv \overline{P''Q}$, then by transitivity $\overline{PQ} \equiv \overline{P''Q}$.

Case-2: Suppose either $P \in m$ or $P' \in n$. For $P \in m$, $S_m(P) = P$ and $S_n(P) = P'$. Yet, with similar argument as in the first case, $\angle PQR \cong \angle P'QR$ by SAS, so, $\angle PQR \cong \angle P'QR$ and thus

$m(\angle PQP') = m\angle PQR) + m(\angle P'QR) = 2m(\angle PQR) = 2\theta$ and $\overline{QP} = \overline{QP'}$.

The same reasoning holds when $P' \in n$. Therefore, the theorem follows.

Examples:

1. Let $l : 2x - y - 3 = 0$ and $m : 3x + y - 7 = 0.$ Then find the image of the point $(5,-7)$ by the composite reflection $S_m \circ S_l$ using single rotation.

Solution: Since the two lines have different slopes, they must be intersecting lines. So, to apply theorem 3.7, first let's find the point of intersection and the angle between them. The point of intersection is obtained as

$\begin{cases} 2x - y - 3 = 0 \\ 3x + y - 7 = 0 \end{cases} \Rightarrow x = 2,\ y = 1 \Rightarrow c = (2,1)$. Let θ be the angle between l and m.

Then, from coordinate geometry, $\tan\theta = \pm\dfrac{m_1 - m_2}{1 + m_1 m_2}$ where m_1 and m_2 are slopes of the non-vertical lines l and m. Thus,

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2} = \pm\frac{-3-2}{1+2(-3)} = \pm\frac{-5}{-5} = \pm 1$$
$$\Rightarrow \tan\theta = \pm 1 \Rightarrow \theta = \tan^{-1}(\pm 1) = 45^\circ \ or\ 135^\circ$$

To calculate $S_m \circ S_l(5,-7)$ determine which angle will be used. (Be careful here! On how to determine the angle). Hence, $S_m \circ S_l(5,-7) = R_{(2,1),90^\circ}(5,-7) = (10,4)$.

2. If $m: y = x - 1$ and $S_m \circ S_l(1,-1) = \rho_{(3,2),90^\circ}(1,-1)$, find the equation of l.

Solution: By Theorem 3.7, the equation $S_m \circ S_l(1,-1) = \rho_{(3,2),90^\circ}(1,-1)$ gives us the two lines are intersecting at $(3,2)$ and the angle between them is $\theta = 45^\circ$. Then, if m is the slope of l, we have that $\tan 45^\circ = \dfrac{1-m}{1+m} = 1 + m = 1 - m \Rightarrow m = 0$. This means that the line l is a horizontal line passing through the point $(3,2)$. Therefore, its equation is $l: y = 2$.

Corollary 3.2: Let l and m be two lines intersecting at $c = (a,b)$. Then,

$S_l \circ S_m = \rho_{c,-2\theta}$ where θ is the angle measured from l to m.

Proof: Consider figure 3.9b. Since the directed angle from l to m is θ, the directed angle from m to l (in the same direction) is $\pi - \theta$.

So, by theorem 3.7, $S_l \circ S_m(x,y) = \rho_{c,2\alpha}(x,y)$, where

$\alpha = 2(\pi - \theta)$. Then, by using the generalized rotation theorem,

$$S_l \circ S_m(x,y) = \rho_{c,2\alpha}(x,y)$$
$$= \begin{pmatrix} (x-a)\cos(2\pi-2\theta)-(y-b)\sin(2\pi-2\theta)+a \\ (x-a)\sin(2\pi-2\theta)+(y-b)\cos(2\pi-2\theta)+b \end{pmatrix}$$
$$= \begin{pmatrix} (x-a)\cos(-2\theta)-(y-b)\sin(-2\theta)+a \\ (x-a)\sin(-2\theta)+(y-b)\cos(-2\theta)+b \end{pmatrix}$$
$$= \rho_{c,-2\theta}(x,y)$$

Hence, $S_l \circ S_m = \rho_{c,-2\theta}$ whenever $S_m \circ S_l = \rho_{c,2\theta}$. This result shows that if we interchange the order of the composition the sign of the angle will also be changed. So, $S_m \circ S_l = \rho_{c,2\theta}$ if and only if $S_l \circ S_m = \rho_{c,-2\theta}$.

Corollary 3.3: Let l and m be any two lines intersecting at $c=(a,b)$ and the angle measured from l to m being θ.

Then, $S_m \circ S_l(x,y) = (x',y')$ where $\begin{cases} x' = (x-a)\cos 2\theta - (y-b)\sin 2\theta + a \\ y' = (x-a)\sin 2\theta + (y-b)\cos 2\theta + b \end{cases}$

Proof: (Expand the result of theorem 3.7 using theorem 3.4)

Theorem 3.8 (Converse of theorem 3.7): Given a rotation $\rho_{C,\theta}$. Let m and n be any two lines intersecting at C so that the angle between them is $\frac{\theta}{2}$. Then, $\rho_{C,\theta} = S_n \circ S_m$.

Proof: Given point C, lines m and n with $\angle(m,n) = \frac{\theta}{2}$. Consider figure 3.10.

Let M and N be points on m and n so that $d(C,M) = d(C,N)$. Clearly, C, M and N are non-collinear points. Now, let $S_m(N) = N'$ and $S_n(M) = M'$.

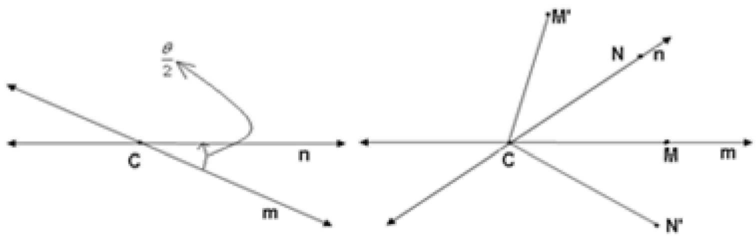


Figure 3.10

Since center of rotation is fixed under any rotation and bedsides C is on both lines m and n, $S_m(C) = C$ and $S_n(C) = C$ which implies $S_n \circ S_m(C) = C$.

Thus, C is fixed both under $\rho_{C,\theta}$ and $S_n \circ S_m$. Again, as S_n is an isometry, $d(C,M) = d(C,M')$. Besides, as isometry preserves angles,

$$m\angle(M-C, M'-C) = m\angle(M-C, N-C) + m\angle(N-C, M'-C) = \frac{\theta}{2} + \frac{\theta}{2} = \theta.$$

So, from $d(C,M) = d(C,M')$ and $m\angle(M-C, M'-C) = \theta$, we get $\rho_{C,\theta}(M) = M'$.

On the other hand, we have $S_n \circ S_m(M) = S_n(M) = M'$ which results in $\rho_{C,\theta}(M) = S_n \circ S_m(M) = M'$. Similarly, from $d(C,N) = d(C,N')$ and $m\angle(N-C, N'-C) = \theta$, we have $\rho_{C,\theta}(N') = N$ and then $\rho_{C,\theta}(N') = S_n \circ S_m(N') = S_n(N) = N$.

But, as C, N' and M are non-collinear, C, N and M are also non-collinear besides $\rho_{C,\theta}(C) = S_n \circ S_m(C)$, $\rho_{C,\theta}(M) = S_n \circ S_m(M)$ and $\rho_{C,\theta}(N') = S_n \circ S_m(N')$. This shows that $\rho_{C,\theta}$ and $S_n \circ S_m$ agrees on three non-collinear points. But, by the three point theorem (Theorem 3.1), any two isometries are equal if they agree on three non-collinear points. Therefore, $\rho_{C,\theta} = S_n \circ S_m$.

Theorem 3.9: The product of two reflections on perpendicular lines m and n at a point P is a half turn with center P. That means $S_n \circ S_m = H_P$. Conversely, a half turn about a point is the product of two reflections on two perpendicular lines intersecting at the center of the half turn.

Proof: Let m and n be perpendicular lines intersecting at P. By theorem 3.7, the product of two reflections on two intersecting lines is a rotation with center at the point of intersection by twice the angle between them.

In this case, since the two lines are perpendicular, $\theta = \frac{\pi}{2}$.

Hence, $S_n \circ S_m = \rho_{P,2\theta} = \rho_{P,\pi} = H_P$. From this theorem, we can generalize that the product of two reflections on intersecting lines is a half turn about their point of intersection if and only if the two lines are perpendicular.

Examples:

1. Find the equations of two lines m and n such that $(2,-4)$ is on m and $S_n \circ S_m = H_P$ where H_P is a half turn with center $P = (3,0)$.

Solution: From theorem 3.9, the product of two reflections is equal to a half turn with center P if and only if the two lines are perpendicular at P. So, we have to assume that the lines m and n are perpendicular at $P = (3,0)$ so as $S_n \circ S_m$ to be a half turn about P. Let the equation of m be $y = ax + b$.

From the given, the points $(3,0)$ and $(2,-4)$ are on m which implies that slope of m is $a = \frac{-4-0}{2-3} = 4$.

So, $m: y = ax + b \Rightarrow y = 4x + b$ and $(3,0) \in m \Rightarrow 0 = 12 + b \Rightarrow b = -12.$

Therefore, $m: y = 4x - 12$.

Now, $m \perp n \Rightarrow n: y = -\frac{1}{4}x + c$. Besides, $(3,0) \in n \Rightarrow 0 = -\frac{3}{4} + c \Rightarrow c = \frac{3}{4}$.

Therefore, $n: y = -\frac{1}{4}x + \frac{3}{4}.$

2. Let $m: y = 2x + 1$. Find the equation of line n such that $S_n \circ S_m$ is a half turn about $P = (1,3)$.

Solution: By Theorem 3.9, $S_n \circ S_m = H_P$ if and only if the two lines are perpendicular at P. Now since the slope of m is 2, the slope of n must be $-\frac{1}{2}$.

Thus, $n: y = -\frac{1}{2}x + b$.

Besides as $P = (1,3)$ is on n. $y = -\frac{1}{2}x + b \Rightarrow -\frac{1}{2} + b = 3 \Rightarrow b = \frac{7}{2}$.

Hence its equation n is $y = -\frac{1}{2}x + \frac{7}{2}$.

Remark (Condition for product of reflections to be Commutative):

Let l and m be two lines intersecting at point C. We need to investigate the condition at which $S_m \circ S_l = S_l \circ S_m$.

First recall from theorem 3.6, $\rho_\theta \circ \rho_\beta = \rho_{\theta+\beta}$, $\rho_\theta^{-1} = \rho_{-\theta}$ and $\rho_\theta = i \Leftrightarrow \theta = 2n\pi,\ n \in Z$. Now, from theorem 3.7 and 3.8, we have $S_m \circ S_l = \rho_{C,2\theta}$ where θ is the directed angle from l to m.

On the other hand, from corollary 3.2 , $S_l \circ S_m = \rho_{C,-2\theta}$.

Consequently,

$$\begin{aligned} S_m \circ S_l = S_l \circ S_m &\Leftrightarrow R_{C,2\theta} = R_{C,-2\theta} \\ &\Leftrightarrow R_{C,2\theta} \circ R_{C,2\theta} = R_{C,2\theta} \circ R_{C,-2\theta}, \quad R_{C,2\theta} \circ R_{C,-2\theta} = i \\ &\Leftrightarrow R_{C,4\theta} = i \Leftrightarrow 4\theta = 2\pi k,\ k \in Z \end{aligned}$$

If $k = 0,\ \theta = 0$, which is impossible since the lines are intersecting and the angle between any two intersecting lines is non zero.

If $k = 1,\ \theta = \frac{\pi}{2}$ which means the lines are perpendicular.

If $k = -1,\ \theta = -\frac{\pi}{2}$ which means again the lines are perpendicular.

Since the smaller of the two angles between two intersecting lines is less than or equal to $\frac{\pi}{2}$, there is no need to consider any other values of k. Otherwise if we consider $k \geq 2$, we get $\theta \geq \pi$ which is a contradiction with the fact that if θ is the angle between any two distinct intersecting lines, then $0 < \theta < \pi$.

From this analysis, we can conclude that the product of reflections on two distinct intersecting lines is commutative if and only if the two lines are perpendicular. Hence, $S_m \circ S_l = S_l \circ S_m \Leftrightarrow l \perp m$.

Example: Let l and m be intersecting lines such that $S_m \circ S_l = S_l \circ S_m$ where $m: x + 5y - 17 = 0$. If $S_m \circ S_l(-1,1) = (5,5)$, find the equation of line l.

Solution: Our previous discussion tells us that $S_m \circ S_l = S_l \circ S_m \Leftrightarrow l \perp m$. Besides, $S_m \circ S_l = \rho_{P,2\theta}$ where P is the point of intersection say (a,b) and θ is the angle between the lines which is $\frac{\pi}{2}$ in this case.

Thus, $S_m \circ S_l(x,y) = \rho_{P,\pi}(x,y) = (-x+2a, -y+2b)$ for any point (x,y) (The aim is to determine the intersection point of the lines). But,

$S_m \circ S_l(-1,1) = (5,5) \Rightarrow (1+2a, -1+2b) = (5,5) \Rightarrow a = 2,\ b = 3 \Rightarrow P = (2,3)$.

Hence, the point of intersection is determined to be $(2,3)$.

On the other hand, $m \perp l$ $m \perp n$ implies slope of line l is 5. Thus, $l: y = 5x + b$ and $(2,3) \in l \Rightarrow 3 = 10 + b \Rightarrow b = -7$.

So, the equation of l is found to be $l: y = 5x - 7$.

Case II: When the two Lines are Parallel

Theorem 3.10: The composition of two reflections on two parallel lines l and m is a translation by a vector $2v$ where v is a vector perpendicular to both l and m . Conversely, if T_v is a translation, then there are parallel lines l and m such that $T_v = S_l \circ S_m$, where l and m are parallel lines perpendicular to the translator vector v and the distance between them is half the length of v .

Note that for any given vector v and any line l, there is a unique line m parallel to l such that $T_v = S_l \circ S_m$. This can be justified as follow: If v is perpendicular to both l and m, the relation holds true. Now, if l is perpendicular to v, then take line m to be $m: (x, y) = l + \frac{1}{2}v = \{M : M = p + \frac{1}{2}v, \forall p \in l\}$.

This is the translation of line l by $\frac{1}{2}v$.

So, from the previous theorem, we can easily show that $S_l \circ S_m = T_{2(\frac{1}{2}v)} = T_v$

Proof: (Left as an exercise)

Remarks: This theorem ensures that the product of reflections on two parallel lines is a translation and any translation is also a product of reflections on two parallel lines. If $S_m \circ S_l = T_{2v}$, then $S_l \circ S_m = T_{-2v}$ where l and m are parallel lines and v is a vector perpendicular to both l and m . If L is on l and M is on m where the line through L and M is perpendicular to both l and m , then

$S_m \circ S_l = T_{2\overline{LM}} = T^2{}_{\overline{LM}} = H_M \circ H_L$ and $S_l \circ S_m = T_{-2\overline{LM}} = T_{2\overline{ML}} = T^2{}_{\overline{ML}} = H_L \circ H_M$.

The composition reflections $S_m \circ S_l$ and $S_l \circ S_m$ are inverse of each others.

Examples:

1. Given the lines $m: y = 2x+1$ and $n: y = 2x-3$. Find the image of the point $(1,1)$ by a product of reflection on line m followed by line n using direct formula, translation, half turn and compare the answers.

Solution: We need to find $S_n \circ S_m(1,1)$. This problem can be done using three methods:

Method I: Using reflection (Direct) Formula: First calculate $S_m(1,1)$ using reflection equation as

$$S_m(1,1) = (x', y') \Rightarrow \begin{cases} x' = 1 - \dfrac{4(2-1+1)}{4+1} = -\dfrac{3}{5} \\ y' = 1 + \dfrac{2(2-1+1)}{4+1} = \dfrac{9}{5} \end{cases}$$

Now, $S_n \circ S_m(1,1) = S_n(S_m(1,1)) = S_n(-\frac{3}{5}, \frac{9}{5}) \Rightarrow \begin{cases} x'' = -\dfrac{3}{5} - \dfrac{4(-\frac{6}{5} - \frac{9}{5} - 3)}{4+1} = \dfrac{21}{5} \\ y'' = \dfrac{9}{5} + \dfrac{2(\frac{-6}{5} - \frac{9}{5} - 3)}{4+1} = -\dfrac{3}{5} \end{cases}$

Therefore, $S_n \circ S_m(1,1) = (\frac{21}{5}, -\frac{3}{5})$.

Method II: Using Translation:

Since the given lines are parallel, the product of reflections on these lines is the same as a translation by the double of the perpendicular vector pointing from m to *n*. That means, $S_n \circ S_m(1,1) = T_{2\overrightarrow{MN}}$ where $M \in m$ and $N \in n$ such that $\overrightarrow{MN} \perp m, n$. Here emphasis should be given on how to find such points.

This is accomplished by taking any line that is perpendicular to the given lines and it is simple to find the intersections of this line with the lines m and *n.*

Take $l: y=-\frac{1}{2}x$. The intersection of this line with the lines m and n is found as follow:

Intersection of l and m:

$$y=2x+1=-\frac{1}{2}x \Rightarrow \frac{5}{2}x=-1 \Rightarrow x=-\frac{2}{5},\ y=\frac{1}{5}$$

$$\Rightarrow M=(x,y)=(-\frac{2}{5},\frac{1}{5})$$

Intersection of l and n:

$$y=2x-3=-\frac{1}{2}x \Rightarrow \frac{5}{2}x=3 \Rightarrow x=\frac{6}{5},\ y=-\frac{3}{5}$$

$$\Rightarrow N=(x,y)=(\frac{6}{5},-\frac{3}{5})$$

Thus, the vector $\overrightarrow{MN}$ is found to be $\overrightarrow{MN}=N-M=(\frac{8}{5},-\frac{4}{5})$

Therefore, $S_n \circ S_m(1,1)=T_{2\overrightarrow{MN}}(1,1)=(1+\frac{16}{5},1-\frac{8}{5})=(\frac{21}{5},-\frac{3}{5})$

Method III: Using Half turns: This problem can also be done using product of half turns as $S_n \circ S_m(1,1)=H_N \circ H_M(1,1)$ where the centers of the half turns M and N are as explained above.

So,

$$S_n \circ S_m(1,1)=H_N \circ H_M(1,1)=H_{(\frac{6}{5},-\frac{3}{5})}(H_{(-\frac{2}{5},\frac{1}{5})}(1,1))=H_{(\frac{6}{5},-\frac{3}{5})}(-\frac{9}{5},-\frac{3}{5})=(\frac{21}{5},-\frac{3}{5}).$$

The use of different methods in solving the above example will help us to grasp the relations among reflections, translations and half turns.

2. Suppose the line $l: y=2x-1$ is parallel to the line m. If the point $(9,2)$ is on line m, then find $S_m \circ S_l(2,5)$ and $S_l \circ S_m(2,5)$ using a single translation.

Solution: Since, l and m are parallel, to do the problems, we need only two point L on l and M on m where the line through L and M is perpendicular to both l and m. But, $M=(9,2)\in m$ (Given).

Here, $L=(a,b)$ on $l \Rightarrow b=2a-1$.

Again, $LM \perp l$ implies slope of a line a long $\overline{LM}$ is $\frac{-1}{2}$ which gives $\frac{b-2}{a-9} = \frac{-1}{2} \Rightarrow 2b+a=13$. Combining these two equations, we get $\begin{cases} b = 2a-1 \\ 2b+a=13 \end{cases} \Rightarrow a=3, b=5 \Rightarrow L=(a,b)=(3,5)$

So, $S_m \circ S_l(2,5) = T^2{}_{L,M}(2,5)$ where T is a unique translation that takes L to M. Let this translator vector be $v=(c,d)$. But,

$T(L)=M \Rightarrow T(3,5)=(9,2) \Rightarrow 3+c=9,\ 5+d=2 \Rightarrow c=6,\ d=-3$. Hence, the translator vector of T is $v=(6,-3)$ so that $T(x,y)=(x+6, y-3)$ for any point (x,y). So, $S_m \circ S_l(2,5) = T^2{}_{L,M}(2,5) = T_{L,M}(T_{L,M}(2,5) = T_{L,M}(8,2) = (14,-1)$.

Besides, using the relation $S_m \circ S_l = T_{2\overline{LM}} \Leftrightarrow S_l \circ S_m = T_{-2\overline{LM}}$ in the remark, we get, $S_l \circ S_m(2,5) = T_{-\overline{LM}}(T_{-\overline{LM}}(2,5) = T_{-\overline{LM}}(-4,8) = (-10,11)$. Once the points L and M are determined, this problem can also be done using half turns about L and M using proposition 3.13 and Theorem 3.10 as follow.

$S_m \circ S_l(2,5) = H_M \circ H_L(2,5) = H_M(4,5) = (14,-1)$ and

$S_l \circ S_m(2,5) = H_L \circ H_M(2,5) = H_L(16,-1) = (-10,11)$.

This means that from proposition 3.13 and Theorem 3.10 we can relate reflections, translations and half turns as $S_l \circ S_m = T_{2\overline{ML}} = H_L \circ H_M$ where $l // m$ and the points L and M are on the lines l and m respectively such that the vector $\overrightarrow{LM}$ is perpendicular to the lines.

3. Let l and m be two parallel lines. If $S_m \circ S_l(2,1) = (-8,7)$, calculate $S_m \circ S_l(3,4)$ and $S_l \circ S_m(3,4)$.

Solution: Since the lines l and m are parallel, by Theorem, 3.10, we have above theorem, we have $S_l \circ S_m = T_{2v}$ where v is a perpendicular vector directed from m to l. So, for any point P,

$P' = S_l \circ S_m(P) = T_{2v}(P) \Rightarrow 2v + P = P'$.

Particularly, for $P=(2,1)$,

$S_l \circ S_m(2,1)=T_{2v}(2,1)\Rightarrow 2v+(2,1)=(-8,7)\Rightarrow 2v=(-10,6)\Rightarrow v=(-5,3)$.

Hence, $S_m \circ S_l(3,4)=T_{2v}(3,4)=2(-5,3)+(3,4)=(-7,10)$. On the other hand,

$S_l \circ S_m = T_{2v} \Leftrightarrow S_m \circ S_l = T_{-2v}$.Thereofe,

$S_l \circ S_m(3,4)=T_{-2v}(3,4)=-2(-5,3)+(3,4)=(13,-2)$.

4. Given $l: y=2x-1$and a vector $v=(12,-6)$. Find a line m parallel to $l$such that $S_m \circ S_l = T_v$.

Solution: By the converse of the above theorem, the line $m$is obtained by translating the given line by $\frac{1}{2}v$. For any point (x,y)on l,

$T_{\frac{1}{2}v}(x,y)=(x+6,y-3)=(x',y')\Rightarrow x=x'-6, y=y'+3$.

So, $l: y'+3=2(x'-6)-1\Rightarrow l: y'=2x'-16$.

5. Let$l: y=x$ and $m: y=x+4$. Then, show that $S_m \circ S_l$ is a translation and find its equation.

Solution: Since the two lines are parallel, by the above thorem $S_m \circ S_l$ is a translation. Besides, $S_l(x,y)=(y,x),\ S_m(x,y)=(y-4,x+4)$.

Hence, $S_m \circ S_l(x,y)=S_m(y,x)=(x-4,y+4)$.

Therefore, $S_m \circ S_l = T_v$ where $v=(-4,4)$.

6. Let $l: y=3x+20$ and $m: y=3x-10$. Then, find a translation T with translator vector $v$such that $S_m \circ S_l = T_v$and calculate $S_m \circ S_l(0,0)$.

Solution: By using similar procedures, as in example 5, we get

$T_v(x,y)=(x+18,y-6),\ S_m \circ S_l(0,0)=(18,-6)$.

Note: If the lines m and $n$are parallel and the points $M$and $N$are as indicated in figure below, we always have the relation

$S_n \circ S_m = T_{2\overrightarrow{MN}} = H_N \circ H_M$.

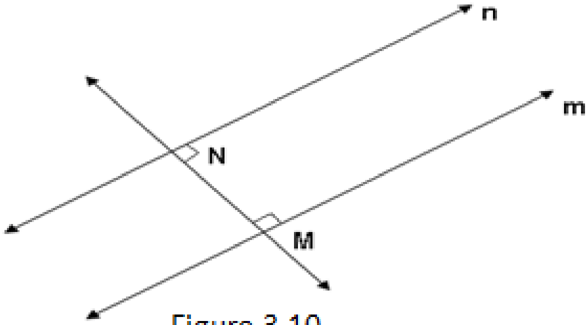


Figure 3.10

Here, we must give attention how to write the order of the subscripts in the translation and half turns with the order of the reflections. For instance, if we interchange the order of the product of the reflections, the order of the half turns and the direction of the vector in the translation will also be changed. It becomes $S_m \circ S_n = T_{2\overline{NM}} = H_M \circ H_N$ and the resulting image is also different.

Example: Given the line $l: y = 2x - 3$ and a vector $v = (-6,3)$. Find a line m such that $T_v = S_l \circ S_m$.

Solution: Apply the above result.

Remark (Characterizing products of rotations): Prior to this, we have seen that the product of rotations about the same center is a gain a rotation about that center. Besides, the product of rotations is commutative as far as the individual rotations are rotations about the same center. Here, one may ask that what will happen if the rotations are performed about different centers with different angles of rotations as indicated in figure 3.11. In figure 3.11a, we can see that $R_{B,\beta} \circ R_{A,\theta}(P) = P''$.

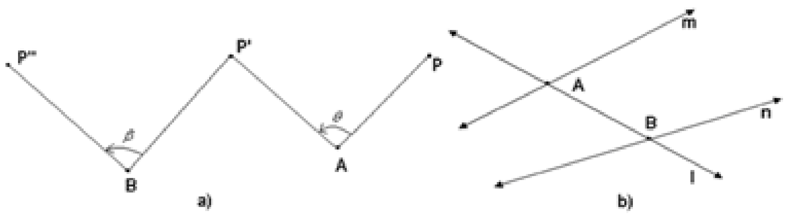


Figure 3.11

Now we want to find a single isometry that will have the same effect as the product $R_{B,\beta} \circ R_{A,\theta}$. To investigate this, consider the two rotations $R_{A,\theta}$ and $R_{B,\beta}$ where A and B are different centers. Once we have two different points A and B, we can determine a unique line $\overleftrightarrow{AB}$. So, by the previous theorem there are lines m and n through A and B respectively (Refer figure 3.11b) such that the rotations $R_{A,\theta}$ and $R_{B,\beta}$ are expressed as $R_{A,\theta} = S_l \circ S_m$ and $R_{B,\beta} = S_n \circ S_l$.

Thus, from the product $R_{B,\beta} \circ R_{A,\theta}$ is also expressed as

$R_{B,\beta} \circ R_{A,\theta} = (S_n \circ S_l) \circ (S_l \circ S_m) = S_n \circ S_m$. This means if $m // n$, then the product $S_n \circ S_m$ will be a translation and so is $R_{B,\beta} \circ R_{A,\theta}$. If the lines m and n intersect at some point, then $S_n \circ S_m$ will be a rotation about their intersection point and so is $R_{B,\beta} \circ R_{A,\theta}$.

Therefore, this observation is summarized by the following theorem.

Proposition 3.18 (Characterization of products of rotations):

Let $R_{A,\theta}$ and $R_{B,\beta}$ be two rotations about different centers A and B. Then, the product $R_{B,\beta} \circ R_{A,\theta}$ is a rotation if and only if $\theta + \beta$ is not a multiple of 2π and it is a translation whenever $\theta + \beta$ is a multiple of 2π. Since the proof of this theorem needs some sketch pad construction, it is omitted.

Any way, the important result that the theorem tells us the product of two rotations is either a rotation or a translation and in general the product is not commutative.

Proof: (Follows from the general formula of rotation)

Problem Set 3.7

1. Let $l: y = 2x + 1$, $m: y = 2x - 3$ and $n: y = -\frac{1}{2}x$. Find the image of $(1,1)$ by the composite reflections $S_m \circ S_l$ and $S_n \circ S_l$. State the procedure you followed in each case. **Answer** $: S_m \circ S_l(1,1) = (\frac{21}{5}, -\frac{3}{5}),\ S_n \circ S_l(1,1) = (-\frac{9}{5}, -\frac{3}{5})$

2. Let l be a line through the points $(-1,-9), (4,1)$ and m be a line through the points $(6,2), (5,5)$. Find the center and angle of a rotation $\rho_{C,\theta}$ such that $S_m \circ S_l = \rho_{C,\theta}$. **Answer** $: C = (\frac{27}{5}, \frac{19}{5}),\ \theta = \frac{\pi}{2}$

3. Let l be $y = x$ and v be vector $v = (2,-2)$. Find line m such that $T_v = S_l \circ S_m$.

4. Let $T(x, y) = (x, y + 2)$. Find two lines l, m such that $T = S_l \circ S_m$.

5. Let $l: y = 2$ and $m: y = 5$. Show that $S_l \circ S_m = T_{2v}$ where $v = (0,-3)$.

6. Let l and m be any two parallel lines. If $S_m \circ S_l(1,2) = (7,-8)$, calculate $S_l \circ S_m(3,4)$. **Answer** $: (-3,14)$

7. Let l be the line $y = 5$. Find line m such that $S_m \circ S_l = \rho_{(3,5),\frac{\pi}{2}}$.

Answer $: m: y = x + 2$

8. Given the lines $l: x + y = 3$ and $m: y = x + 1$. Express $S_l \circ S_m$ using a single rotation $\rho_{C,\theta}$. **Answer** $: S_l \circ S_m = \rho_{C,\theta}$ where $C = (1,2),\ \theta = \pi$

9. If $m \perp n$ such that $S_m \circ S_n(1,2) = (5,6)$, then find $S_m \circ S_n(3,-4)$.

Answer $: S_m \circ S_n(3,-4) = (3,12)$

10. Let l and m be any two perpendicular lines intersecting at the point $(2,3)$, calculate $S_l \circ S_m(-2,7)$ and $S_m \circ S_l(-2,7)$. Are the results equal? Why?

Answer $: (6,-1)$

11. If $A = (2,-4)$, find equations of lines m and n such that $(0,-3)$ is on m and $H_A = S_n \circ S_m$. **Answer** $: n: 2x - y - 8 = 0,\ m: x + 2y + 3 = 0$

12. Let l be the line $x = 3$ and $P \in R^2$ be the point $(5,4)$. Show that $S_l \circ H_P$ is a glide reflection by finding the axis of the glide reflection and the glide vector.

13. Given a line $m: y = x + 3$. Find the equation of line l such that

$a)\ \rho_{C,\theta} = S_m \circ S_l$ where $C = (7,10),\ \theta = \pi/2$ $\qquad b)\ S_m \circ S_l = H_P$ where $P = (2,5)$

$c)\ S_m \circ S_l = T_{\vec{v}}$ where $\vec{v} = (3,-3)$ $\qquad d)\ S_m(l) = l$ with $P = (2,7) \in l$

Answer $: a)\ l: y = 10 \quad b)\ l: x + y - 7 = 0 \quad c)\ l: x - y + 6 = 0 \quad d)\ l: x + y - 9 = 0$

3.8.2 Product of Reflections on Three Lines

Case-I: When the three lines are concurrent

So far, we discussed that the product of reflections on two intersecting lines is a rotation. But what will be the result if the product of reflection is on three concurrent lines (on three lines intersecting at a single point). This is what we are going to address in the next theorem.

Theorem 3.11: Let m, n, p be lines intersecting at a point C. Then, there is a line q through C so that $S_n \circ S_m \circ S_p = S_q$.

Proof: Since m and n are intersecting lines at a point C, $S_n \circ S_m$ is a rotation $\rho_{C,\theta}$ where the angle between m and n is $\frac{\theta}{2}$.

Now, suppose q is a line through C such that the angle between p and q is $\frac{\theta}{2}$ (this is supposition or assumption of line q is possible by angle construction postulate). Then, $S_q \circ S_p = \rho_{C,\theta}$.

So, $\rho_{C,\theta} = S_n \circ S_m$ and $\rho_{C,\theta} = S_q \circ S_p$ imply that

$$\begin{aligned}
& S_n \circ S_m = S_q \circ S_p \\
& \Rightarrow S_n \circ S_m \circ S_p = S_q \circ S_p \circ S_p \\
& \Rightarrow S_n \circ S_m \circ S_p = S_q \circ i \\
& \Rightarrow S_n \circ S_m \circ S_p = S_q
\end{aligned}$$

From this theorem, we can conclude that the composition of reflections on three concurrent lines is again a reflection on a line through the same point.

Case-II: When the three lines are parallel

So far, we discussed that the product of reflections on two parallel lines is a translation. But what will be the result if the product of reflection is on three parallel lines.

Theorem 3.12: Let $l, m,$ and n be three parallel lines in a plane. Then there is a unique line p parallel to the given lines $l, m,$ and n such that $S_m \circ S_l \circ S_n = S_p$.

Equivalently $S_m \circ S_l = S_p \circ S_n$.

Proof: Since l and m are parallel lines, from Theorem 3.10, $S_m \circ S_l = T_v$, where v is perpendicular to both l and m. Similarly, as n is parallel to l and m, v is also perpendicular to n.

Again, by the converse of theorem 3.10, there exists line p parallel to the given line n such that $T_v = S_p \circ S_n$. Thus combining the two results, we get

$$S_m \circ S_l(Q) = S_p \circ S_n(Q), \text{ for any point } Q$$
$$\Rightarrow (S_m \circ S_l) \circ S_n(Q) = (S_p \circ S_n) \circ S_n(Q)$$
$$\Rightarrow (S_m \circ S_l) \circ S_n(Q) = S_p \circ (S_n \circ S_n)(Q), \text{ because composition is associative}$$
$$\Rightarrow (S_m \circ S_l) \circ S_n(Q) = S_p(Q), \text{because } S_n \circ S_n = id, \text{ for any reflection } S_n$$
$$\Rightarrow S_m \circ S_l \circ S_n = S_p$$

From this theorem by considering one additional line q like that of p, we can state a useful corollary.

Corollary 3.4: Let $l, m,$ and n be three parallel lines in a plane. Then there exist unique lines p and q parallel to the given lines $l, m,$ and n such that

$S_m \circ S_l = S_n \circ S_p = S_q \circ S_n$

Proof: Consider the diagram below and look for the lines p and q from the equation $S_m \circ S_l = S_n \circ S_p = S_q \circ S_n$ (This equation actually has unique solutions for p and q).

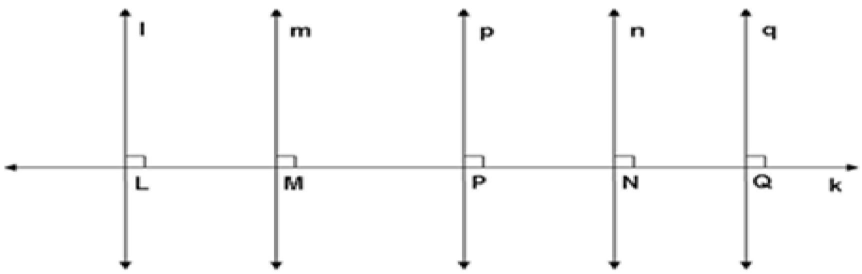


Figure 3.12

Let P and Q be unique points on line k such that the products of the half turns with centers L, M, N, P, Q are equal as follow:

$H_M \circ H_L = H_N \circ H_P = H_Q \circ H_N$. But, we know that a unique line can be drawn through a point on a given line perpendicular to the line. So, let's drop line p through P and line q through Q both perpendicular to line k.

Now, $l // m \Rightarrow S_m \circ S_l = H_M \circ H_L$,

$n // p \Rightarrow S_n \circ S_p = H_N \circ H_P$ and $q // n \Rightarrow S_q \circ S_n = H_Q \circ H_N$.

Combining these and our previous relations $H_M \circ H_L = H_N \circ H_P = H_Q \circ H_N$, we get $S_m \circ S_l = S_n \circ S_p = S_q \circ S_n$. From the first equality, $S_m \circ S_l = S_n \circ S_p$ we have $S_n \circ S_m \circ S_l = S_p$ and from the second equality

$S_m \circ S_l = S_q \circ S_n$, we get $S_m \circ S_l \circ S_n = S_q$.

Remarks: The uniqueness of p and q is shown from the fact that reflections on two lines are equal if and only if the two lines are equal.

That means $S_n = S_m \Leftrightarrow n = m.$

So, if we assume there is another line h for which $S_n \circ S_p = S_n \circ S_h$. Then,

$S_n \circ S_p = S_n \circ S_h \Rightarrow S_n \circ S_n \circ S_p = S_n \circ S_n \circ S_h \Rightarrow S_p = S_h \Rightarrow p = h.$

Examples: Given the lines $l: y = 2x + 5,\ m: y = 2x + \frac{1}{2},\ n: y = 2x - 5$. Find the lines p and q parallel to the given lines such that $S_m \circ S_l = S_n \circ S_p = S_q \circ S_n$.

Solution: To do this problem, we use the analysis followed in the proof of the above corollary. Since the given lines are parallel, take any line k perpendicular to the given lines but the line k should be simple for analysis. Let k be the line $k: y = -\frac{1}{2}x$.

(Note the problem can also be done using any other line without affecting the solution for p and q). Now find the points L, M, N on the lines $l, m,$ and n respectively which are the intersections of line k with these lines.

Intersection of l **and** k **:**

$$y = 2x + 5 = -\frac{1}{2}x \Rightarrow \frac{5}{2}x = -5 \Rightarrow x = -2,\ \ y = 1 \Rightarrow L = (x, y) = (-2,1)$$

Intersection of m **and** k **:**

$$y = 2x + \frac{1}{2} = -\frac{1}{2}x \Rightarrow \frac{5}{2}x = -\frac{1}{2} \Rightarrow x = -\frac{1}{5},\ \ y = \frac{1}{10} \Rightarrow M = (x, y) = (-\frac{1}{5}, \frac{1}{10})$$

Intersection of n **and** k **:**

$$y = 2x - 5 = -\frac{1}{2}x \Rightarrow \frac{5}{2}x = 5 \Rightarrow x = 2,\ \ y = -1 \Rightarrow N = (x, y) = (2,-1)$$

Now determine two points P and Q on line k which are the intersection of the lines p and q with k respectively (as labeled in figure 3.12) such that $H_M \circ H_L = H_N \circ H_P = H_Q \circ H_N$.

Let $P = (a,b)$ and $Q = (c,d)$. But, for any point (x, y),

$$H_M \circ H_L(x, y) = (x + \frac{18}{5}, y - \frac{9}{5})$$
$$H_N \circ H_P(x, y) = (x - 2a + 4, y - 2b - 2)$$
$$H_Q \circ H_N(x, y) = (x + 2c - 4, y + 2d + 2)$$

Thus,

$$H_M \circ H_L(x, y) = H_N \circ H_P(x, y) \Rightarrow (x + \frac{18}{5}, y - \frac{9}{5}) = (x - 2a + 4, y - 2b - 2)$$
$$\Rightarrow x + \frac{18}{5} = x - 2a + 4,\ \ y - \frac{9}{5} = y - 2b - 2$$
$$\Rightarrow a = \frac{1}{5},\ \ b = -\frac{1}{10}$$
$$\Rightarrow P = (a,b) = (\frac{1}{5}, -\frac{1}{10})$$

On the other hand, using the second equality $H_M \circ H_L = H_Q \circ H_N$,

$$H_M \circ H_L(x, y) = H_Q \circ H_N(x, y) \Rightarrow (x + \frac{18}{5}, y - \frac{9}{5}) = (x + 2c - 4, y + 2d + 2)$$
$$\Rightarrow x + \frac{18}{5} = x + 2c - 4,\ \ y - \frac{9}{5} = y + 2d + 2$$
$$\Rightarrow c = \frac{19}{5},\ \ d = -\frac{19}{10}$$
$$\Rightarrow Q = (c,d) = (\frac{19}{5}, -\frac{19}{10})$$

Since the lines p and q are parallel to the given lines, then they are also perpendicular to line k.

Thus, we are required to determine lines p and q through points P and Q which are perpendicular to the line k at these points of intersections.

$p // l \Rightarrow p: y = 2x + b$. But p passes through $P = (\frac{1}{5}, -\frac{1}{10})$, then

$y = 2x + b \Rightarrow -\frac{1}{10} = \frac{2}{5} + b \Rightarrow b = -\frac{1}{2}$. Hence, line p is found to be $p: y = 2x - \frac{1}{2}$

Similarly, q is a line through Q and perpendicular to line k and parallel to the lines. Thus, it has the form $q: y = 2x + c \Rightarrow -\frac{19}{10} = \frac{38}{5} + c \Rightarrow c = -\frac{19}{2}$.

Hence, line q is found to be $p: y = 2x - \frac{19}{2}$. In this solution the choice of line k is arbitrary, one can choose any other line perpendicular to the given lines and gets the same result but the choice of line k should be in such a way that one can calculate the points of intersection easily.

Case-III: When the three lines are neither parallel nor concurrent

Theorem 3.13: Let l, m, n be neither concurrent nor parallel lines (two may be parallel). Then, $S_n \circ S_m \circ S_l$ is a glide reflection.

Proof: To prove that $S_n \circ S_m \circ S_l$ is a glide reflection, it suffices to find a line p and a vector v with $v // p$ such that $S_n \circ S_m \circ S_l = T_v \circ S_p$. Since all the three lines are not parallel assume that m and n intersect at P. (Refer figure 3.13a).

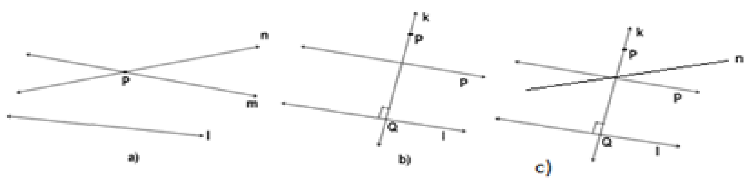


Figure 3.13

Besides, l, m, n are not all concurrent, l does not pass through P. So, let k be a line perpendicular to l through P. Here, m, n, k becomes concurrent at P.

Thus, by theorem 3.11, there is a line t through P such that $S_k \circ S_n \circ S_m = S_t$ which in turn implies $S_n \circ S_m = S_t \circ S_k$. Again let Q be the intersection of k and l. As $k \perp l$, we have $S_k \circ S_l = H_Q$ (because composition of reflections on two perpendicular lines is a half turn about their intersection). Now, construct line p perpendicular to t through Q and line q through Q parallel to t. Since $t // q$, there is a vector v perpendicular to lines t and q such that $S_t \circ S_q = T_v$. Hence, combining these results above

$S_n \circ S_m = S_t \circ S_k$, $S_k \circ S_l = H_Q$ and $S_t \circ S_q = T_v$, we get,

$$\begin{aligned} S_n \circ S_m \circ S_l &= S_t \circ S_k \circ S_l \\ &= S_t \circ H_Q \\ &= S_t \circ S_q \circ S_p \\ &= T_v \circ S_p \end{aligned}$$

But, from our construction, $v \perp t$ and $p \perp t$ implies $v // p$ which shows that $T_v \circ S_p$ is a glide reflection. Therefore, $S_n \circ S_m \circ S_l = T_v \circ S_p$ implies that $S_n \circ S_m \circ S_l$ is a glide reflection. In this theorem, the lines l, m, n are neither concurrent nor parallel (of course two may be parallel).

If $l // m$ and n is perpendicular to l and m, then $S_m \circ S_l$ is a translation and S_n is a reflection, so the product $S_n \circ S_m \circ S_l$ is called glide reflection and line n is known as the axis of the glide reflection.

Corollary 3.5: Let l be any line and P be a point not on l. Then, $g = H_P \circ S_l$ is a glide reflection.

Proof: Since $P \notin l$, consider line n through P perpendicular to l and line m perpendicular to line n at P. See figure 3.13

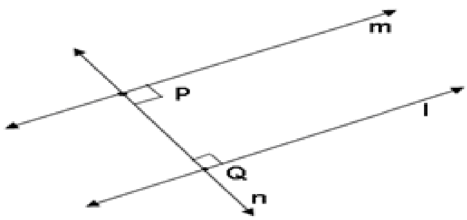


Figure 3.13

Now, $m \perp n$ and $n \perp l$ gives $S_n \circ S_m = H_P$ (Because product of two reflections on perpendicular lines is a half turn about their intersection). So,

$g = H_P \circ S_l = S_n \circ S_m \circ S_l$. But l, m, n are neither concurrent nor all parallel from theorem 3.11 $S_n \circ S_m \circ S_l$ is a glide reflection and so is $H_P \circ S_l$ as they are equal. From figure 3.13, one can also easily verify that

$g = S_n \circ S_l \circ S_m = H_Q \circ S_m = S_l \circ S_m \circ S_n = S_l \circ H_P$ using the fact that product of two reflections on perpendicular lines commute and is a half turn through their intersection. In general, for $l // m$, $g = H_Q \circ S_m = S_l \circ H_P$ is a glide reflection if and only if $Q \in l$ and $P \in m$. This shows that glide reflection is the product of three reflections on three neither parallel nor concurrent lines,

Or the product of a half turn and a reflection in either order where the center of the half turn is not on the line like as P or Q above, Or the product of a translation and a reflection in any order where the translation vector is parallel to the line of reflection.

Examples:

1. Let m and n be parallel lines through M and N where $M = (1,8)$ and n is the line $y - x - 1 = 0$. If $g = S_n \circ H_M = H_N \circ S_m$ is a glide reflection, find the equation of H_N, the axis of g and line m.

Solution: From corollary 3.5, $g = S_n \circ H_M = H_N \circ S_m$ is a glide reflection whenever $M \in m$, $N \in n$ and the axis of g passes through M and N perpendicular to the lines m and n. Now, $m // n$ implies the slope of m is 1 so then $m: y = x + b$. But, $M \in m \Rightarrow 8 = 1 + b \Rightarrow b = 7 \Rightarrow m: y = x + 7$.

On the other hand, the axis of g is perpendicular to m implies the slope of the axis of g is -1 and then $l: y = -x + c$. But, the axis also passes through $M = (1,8)$ which implies that $l: y = -x + c \Rightarrow 8 = -1 + c \Rightarrow c = 9 \Rightarrow l: y = -x + 9$.

Similarly, this axis $l : y = -x + 9$ which is perpendicular to m at M is also perpendicular to n at N. That means N is the intersection point of $n : y - x - 1 = 0$ and $l : y = -x + 9$.

So, $y = x + 1 = -x + 9 \Rightarrow x = 4 \Rightarrow N = (x, y) = (4,5)$.

Hence, $H_N(x, y) = (8 - x, 10 - y)$.

2. Let l be the line $y = 0$ and $A \in R^2$ be the point $(0,-3)$. Show that $H_A \circ S_l$ is a glide reflection by finding the axis of the glide reflection and the glide vector.

Solution: Let (x, y) be any point so that $H_A(x, y) = (-x, -y - 6)$, $S_l(x, y) = (x, -y)$

Then, $H_A \circ S_l(x, y) = H_A(x, -y) = (-x, y - 6)$ .... (i)

Here, take the line $m : x = 0$ and the vector $v = (0,-6)$.

Thus, $S_m \circ T_v(x, y) = S_m(x, y - 6) = (-x, y - 6) = T_v \circ S_m(x, y)$ .... (ii)

Combining (i) and (ii), we get $H_A \circ S_l = T_v \circ S_m$. Besides, $m // v$ and then $T_v \circ S_m$ is a glide reflection by definition.

Therefore, $H_A \circ S_l = T_v \circ S_m$ implies $H_A \circ S_l$ itself is a glide reflection with axis $m : x = 0$ and a glide vector $v = (0,-6)$.

Theorem 3.14: If f is an isometry with exactly one fixed point C, then $f = S_n \circ S_m$ where m and n are lines intersecting at the point C.

Proof: Suppose f is an isometry that fixes only the point C. Let P be a point different from C, such that $f(P) = P'$ and let m the perpendicular bisector of the segment $\overline{PP'}$.

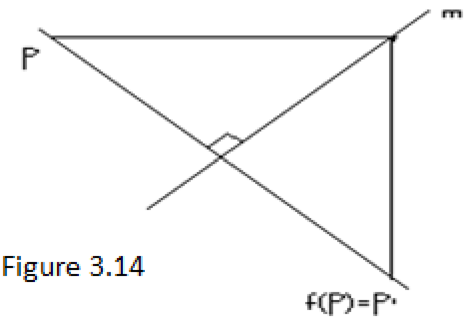


Figure 3.14

Since f is an isometry $\overline{f(C)f(P)} = \overline{C'P'} = \overline{CP'} \Rightarrow \overline{CP} = \overline{CP'}$.

This means C is at equal distance from the two end point of the segment $\overline{PP'}$. So, C is on line m (because if a point is at equal distance from the end points of a line segment, then it is on the perpendicular bisector of the segment.). Thus, $S_m(C) = C$ and $S_m(P') = P.$ Then, $S_m \circ f(C) = S_m(C) = C$ and $S_m \circ f(P) = S_m(P') = P$.

This shows that $S_m \circ f$ fixes two different points C and P.

So, $S_m \circ f = i$ or $S_m \circ f = S_n$ where n is the line through the points C and P. But, $S_m \circ f \neq i$ because if $S_m \circ f = i$, then f will have more than one fixed point which contradicts from the hypothesis that f has exactly one fixed point. Thus, $S_m \circ f = S_n \Rightarrow f = S_m \circ S_n$ where m and n are lines intersecting at the point C. This theorem guarantees that whenever an isometry has exactly one fixed point, it is a composition of two reflections by lines intersecting at the fixed point of the isometry. In what follows we are going to consider one of the fundamental theorems which is useful in the classification of isometrics.

3.8.3 The Fundamental Theorems of Isometries

Theorem 3.14 (The First Fundamental Theorem of Isometries):

Every isometry can be expressed as a composition of three or fewer reflections.

Proof: The identity map is one of the isometrics which is the composition of two reflections, any reflection with itself. That is $i = S_l \circ S_l$. Now, let f be any isometry different from the identity. Then, there is some point P which is not fixed by f. Let $f(P) = Q,\ P \neq Q$. Let m be the perpendicular bisector of $\overline{PQ}$.

Then, by definition of a reflection S_m, $S_m(Q) = P.$

Then, $Q = f(P) \Rightarrow S_m \circ f(P) = P$. So, P is the fixed point of $S_m \circ f(P).$

If $S_m \circ f(P)$ has other fixed points besides P, then it must be either a reflection S_n which in turn implies $S_m \circ f(P) = S_n \Rightarrow S_m \circ S_m \circ f = S_m \circ S_n \Rightarrow f = S_m \circ S_n$ which is a product of two reflections or the identity in which case $S_m \circ f = i \Rightarrow f = S_m$ which is a reflection.

On the other hand, if $S_m \circ f(P)$ has only P as a fixed point, then by theorem 3.12, there are lines l and n intersecting at P such that

$S_m \circ f = S_l \circ S_n \Rightarrow S_m \circ S_m \circ f = S_m \circ S_l \circ S_n \Rightarrow f = S_m \circ S_l \circ S_n$ which is the composition of three reflections. Besides, $f = S_m \circ S_l \circ S_n$ will be reflection or glide reflection based on the relation of the lines. Thus, in any of the above cases, any isometry can be written as a composition of one, two, or three reflections. As we see from Theorem 3.14, reflection is the building blocks of all isometries. That means every isometry can be expressed as a product of one, two or three reflections.

Theorem 3.15: Any isometry of the plane is a reflection, rotation, translation or a glide reflection.

Proof: Apply the above Theorems

Theorem 3.16 (**The second Fundamental Theorem of Isometries**):

If A, B, C are three non-collinear points, and A', B', C' are also non-collinear such that $A'B' = AC,\ A'C' = AC,\ B'C' = BC$, then there exists a unique isometry that maps A to A', B to B' and C to C'. Besides, this isometry can be expressed as a product of at most three reflections.

Proof : This theorem has two parts to be proved: **Existence** and **Uniqueness**.

Existence: Construct line k as the perpendicular bisector of A and A'. Then $S_k(A) = A'$ and $S_k(B) = P$. If $P \neq B'$, consider a line l which is the perpendicular bisector of P and B'. Then, $S_l \circ S_k(B) = S_l(P) = B'$.

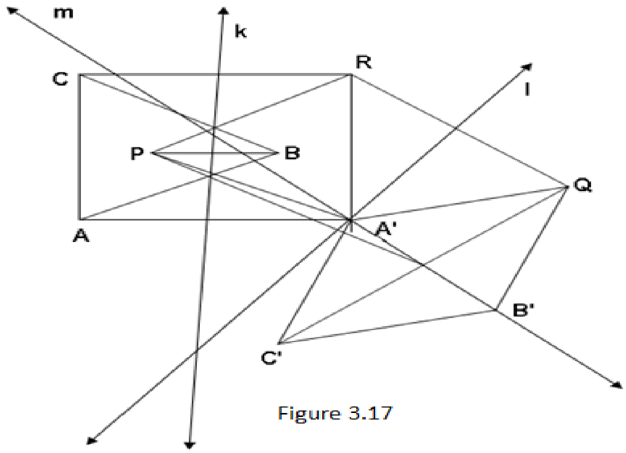


Figure 3.17

Claim I: $S_l \circ S_k(A) = A'$. Here, $S_k(B) = P$ and $S_k(A) = A'$ which implies that $A'P = AB$. Besides from the hypothesis, $B'A' = BA$. Combining these two relations we get $B'A' = PA'$. This implies that B' and P are equidistant from A' which means A' is on the perpendicular bisector of P and B'. But this line is l and hence A' is on l. Thus, $S_l \circ S_k(A) = S_l(A) = A'$ and $S_l \circ S_k(B) = S_l(P) = B'$.

Now consider, $S_k(C) = R$ and $S_l \circ S_k(C) = S_l(R) = Q$. If $Q = C'$, the proof is complete. However, suppose $Q \neq C'$. Consider line m through A' and B'

Claim II: $S_m(Q) = C'$. From the hypothesis, $A'C' = AC$ and from $S_k(C) = R$, $S_k(A) = A'$, we get $A'R = AC$. Similarly, $S_l(A') = A'$ and $S_l(R) = Q$ gives $A'Q = A'R$.

By combining these three equalities, we obtain $A'Q = A'C'$.

This implies that A' is equidistant from the end points of the segment determined by Q and C'. So A' is on the perpendicular bisector of the segment $C'Q$. Similarly, from $S_l \circ S_k(B) = S_l(S_k(B)) = S_l(P) = B'$ and $S_l \circ S_k(C) = S_l(S_k(C)) = S_l(R) = Q$, we have $B'Q = BC$.

Again from the hypothesis $B'C' = BC$. Thus, $B'Q = B'C'$ which implies that B' is equidistant from the end points of the segment determined by Q and C'.

This in turn results that B' is on the perpendicular bisector of the segment $C'Q$. Hence, both A' and B' are on a line which is the perpendicular bisector of the segment $C'Q$ and this line is m so then $S_m(Q) = C'$.

As a result,

$$S_m \circ S_l \circ S_k(A) = S_m \circ S_l(A') = S_m(A') = A'$$
$$S_m \circ S_l \circ S_k(B) = S_m \circ S_l(P) = S_m(B') = B'$$
$$S_m \circ S_l \circ S_k(C) = S_m \circ S_l(R) = S_m(Q) = C'$$

Therefore, there exists $f = S_m \circ S_l \circ S_k$ which maps the three non- collinear points A, B, C into three non-collinear points A', B', C'.

The second part of the proof is to show that this isometry is unique. Since, A, B, C are three non-collinear points, the uniqueness follows from Theorem 3.1. Thus, the proof is complete. Here, give attention on how to find such lines m, l, k which satisfies the given conditions.

a) Line k is the perpendicular bisector of A and A'
b) Line l is the perpendicular bisector of B' and $S_k(B)$
c) Line m is the perpendicular bisector of A' and B'

Example: Given $\Delta ABC \cong \Delta DEF$ where the vertices are

$A = (0,0),\ B = (5,0),\ C = (0,10)$ and $D = (4,2),\ E = (1,-2),\ F = (12,-4)$.

Find equations of lines such that product of reflections on these lines takes ΔABC to ΔDEF.

Solution: Follow the procedure in the proof of the above theorem.

Problem Set 3.8

1. Given the three lines $l: x = \frac{1}{2},\ m: x = 2,\ n: x = 5$. Find a line p parallel to these lines such that $S_n \circ S_m \circ S_l = S_p$. **Answer** $: p: x = \frac{7}{2}$

2. Let m be the x-axis, n the y-axis and p the line $y = x$. Find the line q such that $S_n \circ S_m \circ S_p = S_q$. **Answer** $: q: y = -x$

3. Let $l: y = 3x + 20$ and $m: y = 3x - 10$. Then, find a translation T with translator vector v such that $S_m \circ S_l = T_v$.

   Answer $: T_v(x, y) = (x + 18, y - 6)$

4. Given the lines $M: y = 0,\ n: y = 2x,\ x = 0$. Find line q such that

   a) $S_q = S_p \circ S_n \circ S_m$

   b) $S_q = S_p \circ S_m \circ S_n$

3.9 Equations of Orthogonal Transformations in Coordinates

In the previous sections, we have studied about types of isometries and their properties.

Now let's summarize and generalize their equations using coordinate geometry.

Equation of Rotations: Equation of rotation about any center $C=(h,k)$ with angle of rotation θ is given by $\rho_{C,\theta}(x,y)=(x',y')$ where

$$\begin{cases} x'=(x-h)\cos\theta-(y-k)\sin\theta+h \\ y'=(x-h)\sin\theta+(y-k)\cos\theta+k \end{cases}$$

Expanding these equations gives $\begin{cases} x'=x\cos\theta-y\sin\theta+c \\ x'=x\sin\theta-y\cos\theta+d \end{cases}$

Now, by letting $a=\cos\theta,\ b=\sin\theta,\ a^2+b^2=1$, these equations can be reduced to the form $\begin{cases} x'=ax-by+c \\ y'=bx+ay+d \end{cases},\ a^2+b^2=1$

Here, if $\theta=2\pi n,\ n\in Z$, then

$a=\cos\theta,\ b=\sin\theta \Rightarrow a=\cos(2\pi n)=1,\ b=\sin(2\pi n)=0$ and the equations become

$\begin{cases} x'=x+c \\ y'=y+d \end{cases}$ which are the general equations of a translation.

Otherwise, for $\theta\neq 2\pi n,\ a=\cos\theta,\ b=\sin\theta \Rightarrow a\neq\cos(2\pi n)\neq 1,\ b\neq\sin(2\pi n)\neq 0$, the equations are equations of a rotation.

Consequently; the equations $\begin{cases} x'=ax-by+c \\ y'=bx+ay+d \end{cases},\ a^2+b^2=1$ are the general equations of a translations and or rotations.

3.10 Equations of Even and Odd Isometries

Definition: *Even isometry* is an isometry that can be expressed as a product of even number of reflections. On the other hand, an isometry that can be expressed as a product of odd number of reflections is said to be *odd isometry*. Translation and rotations are even isometries while reflection and glide reflection are odd isometries.

3.10.1 Equations of Even Isometries

We discussed above that only translations and rotations are isometries that can be expressed as a product of even number of reflections. Thus, for any even isometry, its equations are of the form $\begin{cases} x' = ax - by + c \\ y' = bx + ay + d \end{cases}$, $a^2 + b^2 = 1$.

3.10.2 Equations of Odd Isometries

Any odd isometry is a product of odd numbers of reflections. In other words, any odd isometry is a product of even isometry followed by a reflection on any line l. Now, let β be any odd isometry. Then, for any line l, $S_l \circ \beta$ becomes even isometry where as $S_l \circ S_l \circ \beta$ is an odd isometry. For any point (x, y), taking the line l to be the $x-$axis (in particular, since it works for all), we get $S_l(x, y) = (x', y') = (x, -y)$. Hence, equation of the odd isometry α becomes,

$$\begin{aligned}(x', y') = \beta(x, y) &= S_l \circ (S_l \circ \beta)(x, y) \\ &= S_l(ax - by + c, bx + ay + d) \\ &= (ax - by + c, -[bx + ay + d])\end{aligned}$$

Thus, the general equations of any odd isometry β becomes $\beta(x, y) = (x', y')$ where $\begin{cases} x' = ax - by + c \\ y' = -[bx + ay + d] \end{cases}$, $a^2 + b^2 = 1$

Since no isometry is both even and odd, the above two equations (equations of even and odd isometries) constitute the equations for any types of isometries.

Theorem 3.17 (The generalized equation Theorem for Isometries):

Let (x, y) be any point. Then, the general equations for any isometry β is given by $\beta(x, y) = (x', y')$ where $\begin{cases} x' = ax - by + c \\ y' = \pm[bx + ay + d] \end{cases}$, $a^2 + b^2 = 1$

The plus sign is used when we assume that β is an even isometry and the minus sign is applied when we assume that β is an odd isometry. Conversely, any equations of this form are equations of isometries.

Examples:

1. Let β be an isometry whose equations are given by $\beta(x, y) = (x', y')$ where

$$\begin{cases} x' = -\tfrac{3}{5}x + \tfrac{4}{5}y + 4 \\ y' = \tfrac{4}{5}x + \tfrac{3}{5}y - 2 \end{cases}$$

Determine whether β is an even or odd isometry.

Solution:

From equations of isometries, we know that $\begin{cases} x' = ax - by + c \\ y' = \pm[bx + ay + d] \end{cases}$, $a^2 + b^2 = 1$

Equate the general equations and the given values of x' and y'. This gives,

$$\begin{cases} x' = ax - by + c = -\tfrac{3}{5}x + \tfrac{4}{5}y + 4 \\ y' = \pm[bx + ay + d] = \tfrac{4}{5}x + \tfrac{3}{5}y - 2 \end{cases}$$

So, $x' = ax - by + c = -\tfrac{3}{5}x + \tfrac{4}{5}y + 4 \Rightarrow a = -\tfrac{3}{5},\ b = -\tfrac{4}{5},\ c = 4$

Using these values, we obtain d from the equations of y' as follow.

$$y' = \pm[bx + ay + d] = \pm[-\tfrac{4}{5}x - \tfrac{3}{5}y + d] = \tfrac{4}{5}x + \tfrac{3}{5}y - 2$$

To solve these equations we have two options either to use the plus sign or the minus sign. If we use the plus sign the equation becomes

$$-\tfrac{4}{5}x - \tfrac{3}{5}y + d = \tfrac{4}{5}x + \tfrac{3}{5}y - 2 \Rightarrow -\tfrac{4}{5} = \tfrac{4}{5},\ -\tfrac{3}{5} = \tfrac{3}{5}, d = -2$$ (by equating coefficients) which is absurd (impossible!). If we use the minus sign the equation becomes $\tfrac{4}{5}x + \tfrac{3}{5}y - d = \tfrac{4}{5}x + \tfrac{3}{5}y - 2 \Rightarrow \tfrac{4}{5} = \tfrac{4}{5},\ \tfrac{3}{5} = \tfrac{3}{5}, d = 2$ (by equating coefficients)

which is the only logical option.

Now, using the values $a = -\frac{3}{5},\ b = -\frac{4}{5},\ c = 4,\ d = 2$,

we get $\begin{cases} x' = -\frac{3}{5}x + \frac{4}{5}y + 4 = ax - by + c \\ y' = \frac{4}{5}x + \frac{3}{5}y - 2 = -[bx + ay + d] \end{cases}$

But from our previous discussions, these are equations of odd isometry and hence β is odd.

2. Let β be an isometry whose equations are given by $\beta(x, y) = (x', y')$ where

$$\begin{cases} x' = -\frac{3}{5}x - \frac{4}{5}y + \frac{4}{5} \\ y' = \frac{4}{5}x - \frac{3}{5}y - \frac{2}{5} \end{cases}$$

Show that β is an even isometry and conclude that β is a rotation and find the center of the rotation.

Solution:

An isometry β is even if and only if its equations are given by

$$\begin{cases} x' = ax - by + c \\ y' = bx + ay + d \end{cases},\ a^2 + b^2 = 1$$

In our case, $x' = ax - by + c = -\frac{3}{5}x - \frac{4}{5}y + \frac{4}{5} \Rightarrow a = -\frac{3}{5},\ b = \frac{4}{5},\ c = \frac{4}{5}$ and

$$y' = bx + ay + d = \frac{4}{5}x - \frac{3}{5}y - \frac{2}{5} \Rightarrow d = -\frac{2}{5}.$$

Hence, the equations of β become $\begin{cases} x' = -\frac{3}{5}x - \frac{4}{5}y + \frac{4}{5} = ax - by + c \\ y' = \frac{4}{5}x - \frac{3}{5}y - \frac{2}{5} = bx + ay + d \end{cases}$

But, these are the forms of an even isometry and thus β is even. Since the only even isometries are rotation and translation, β must be arotation because as we see from its general equation, β cannot be a translation. Now, if β is a rotation withcenter $C = (h, k)$, it must satisfy the equations

$$\begin{cases} x' = ax - by + c = x\cos\theta - y\sin\theta + c \\ y' = bx + ay + d = x\sin\theta + y\cos\theta + d \end{cases}$$

From which we can get the equations to determine the intersection point $C = (h, k)$ which is given as $\begin{cases} h(1 - \cos\theta) + k\sin\theta = c \\ -h\sin\theta + k(1 - \cos\theta) = d \end{cases}$

Thus, h and k can be solved as follow.

$$h(1 + \tfrac{3}{5}) + k(\tfrac{4}{5}) = \tfrac{4}{5} \Rightarrow 2h + k = 1 \Rightarrow k = 1 - 2h$$

$$-h(\tfrac{4}{5}) + k(1 + \tfrac{3}{5}) = -\tfrac{2}{5} \Rightarrow 2h - 4k = 1$$

$$\Rightarrow 2h - 4(1 - 2h) = 1$$

$$\Rightarrow h = \tfrac{1}{2},\ k = 1 - 2h \Rightarrow k = 0$$

$$\Rightarrow C = (h, k) = (\tfrac{1}{2}, 0)$$

Alternatively, one can also solve the values of h and k from the equation that we derived earlier using $a=\cos\theta=-\frac{3}{5}$, $b=\sin\theta=\frac{4}{5}$, $r=\frac{4}{5}$, $t=-\frac{2}{5}$ in the equations we will get the same results.

$$h=\frac{r}{2}-t\frac{\sin\theta}{2(1-\cos\theta)}=\frac{\frac{4}{5}}{2}+\frac{2}{5}\frac{\frac{4}{5}}{2(1+\frac{3}{5})}=\frac{2}{5}+\frac{1}{10}=\frac{1}{2}$$

$$k=\frac{t}{2}+r\frac{\sin\theta}{2(1-\cos\theta)}=\frac{\frac{-2}{5}}{2}+\frac{4}{5}\frac{\frac{4}{5}}{2(1+\frac{3}{5})}=-\frac{1}{5}+\frac{1}{5}=0$$

$$\Rightarrow (h,k)=(\tfrac{1}{2},0)$$

Hence, we conclude that β is a rotation which is the product of two reflections on lines intersecting at the point $(\frac{1}{2},0)$.

3. Let α be an isometry with $\alpha(0,0)=(2,1)$, $\alpha(1,-1)=(1,0)$, $\alpha(2,3)=(5,1)$. Then find the equations of α.

Solution: Let $A=(0,0),\ B=(1,-1),\ C=(2,3),\ A'=(2,1),\ B'=(1,0),\ C'=(5,1).$

Clearly, A, B, C are non-collinear and so are A', B', C'. But we know that there is a unique isometry that takes three non-collinear points into three non collinear points. Now, let this isometry be given by $\alpha(x,y)=(x',y')$ where $\begin{cases} x'=ax-by+c \\ y'=bx+ay+d \end{cases}$ such that the constants a, b, c, d are to be determined from the given points and their images.

So, $\alpha(A)=\alpha(0,0)=(2,1)\Rightarrow a.0-b.0+c=2, b.0+a.0+d=1\Rightarrow c=2,\ d=1.$

Using these values $c=2,\ d=1$, the equation of α becomes $\alpha(x,y)=(x',y')$ where $\begin{cases} x'=ax-by+2 \\ y'=bx+ay+1 \end{cases}$

Again, $\alpha(B)=\alpha(1,-1)=(1,0)\Rightarrow\begin{cases} a+b+2=1 \\ b-a+1=0 \end{cases}\Rightarrow b=-1,\ a=0$

Hence, $\alpha(x,y)=(x',y')$ where $\begin{cases} x'=y+2 \\ y'=-x+1 \end{cases}$.

If we use the minus sign, in the general equation we get the same result.

4. Let $\beta(x,y)=(x',y')$ where $\begin{cases} x'=-3x-4y+4 \\ y'=4x-3y-2 \end{cases}$.

Verify that β is not an isometry.

Solution: If the equations of β represent equations of an isometry, it must satisfy the following equations:

$$\beta(x,y)=(x',y') \text{ where } \begin{cases} x'=ax-by+c \\ y'=\pm[bx+ay+d] \end{cases},\quad a^2+b^2=1$$

But, $x' = ax - by + c = -3x - 4y + 4 \Rightarrow a = -3,\ b = 4,\ c = 4 \Rightarrow a^2 + b^2 = 25 \neq 1$.

Hence, the equations of β does not represent equations of an isometry and thus β is not an isometry.

Theorem 3.18: The product of any two even or any two odd isometries is even and the product of an even and odd isometries is odd isometry.

Proof:

Let α and β be any two even isometries given by the following equations.

$$\alpha(x, y) = (x', y') \text{ where} \begin{cases} x' = ax - by + c \\ y' = bx + ay + d \end{cases},\ a^2 + b^2 = 1$$

$$\beta(x, y) = (x', y') \text{ where} \begin{cases} x' = mx - ny + p \\ y' = nx + my + q \end{cases},\ m^2 + n^2 = 1$$

We need to show that the product $\beta \circ \alpha$ is also a direct similarity.

For any point (x, y), $\beta \circ \alpha(x, y) = \beta(\alpha(x, y)) = (x'', y'')$ where

$$\begin{aligned} x'' &= mx' - ny' + p \\ &= m(ax - by + c) - n(bx + ay + d) + p \\ &= (ma - nb)x - (mb + na)y + mc - nd + p \\ &= tx - ly + r, \quad with\ t = ma - nb, l = mb + na, r = mc - nd + p \\ y'' &= nx' + my' + q \\ &= n(ax - by + c) + m(bx + ay + d) + q \\ &= (na + mb)x + (ma - nb)y + nc + md + q \\ &= lx + ty + s, \quad with\ s = nc + md + q \end{aligned}$$

Now, these equations to be equations of isometries, we are left to show that $t^2 + l^2 = 1$. But,

$$\begin{aligned} t^2 + l^2 &= (ma - nb)^2 + (mb + na)^2 \\ &= (ma)^2 - 2abmn + (nb)^2 + (mb)^2 + 2abmn + (na)^2 \\ &= (ma)^2 + (na)^2 + (mb)^2 + (nb)^2 \\ &= m^2(a^2 + b^2) + n^2(a^2 + b^2) \\ &= (m^2 + n^2)(a^2 + b^2) = 1,\ \text{since}\ a^2 + b^2 = 1,\ m^2 + n^2 = 1 \\ &\Rightarrow t^2 + l^2 = 1. \end{aligned}$$

Hence, $\beta \circ \alpha(x, y) = \beta(\alpha(x, y)) = (x', y')$ where $\begin{cases} x' = tx - ly + r \\ y' = lx + ty + s \end{cases},\ t^2 + l^2 = 1$

But these are equations of an isometry obtained by applying the plus sign in the general equation of isometries (Theorem 3.17).

Hence, $\beta \circ \alpha$ is an even isometry whenever α and β are even isometries. The other part follows similarly.

3.11 Test for Type of Isometries

From the observation on the type of isometries and their effects on orientation, let's develop a simple test for the type of isometry α from the image of three non-collinear points. Given three non-collinear points P, Q, R and their images $P' = \alpha(P),\ Q' = \alpha(Q),\ R' = \alpha(R)$. We need to determine whether α is a translation, rotation, glide reflection or reflection. Since an isometry maps any three non-collinear points in to non-collinear points, P', Q', R' are non-collinear points. Now, take ΔPQR and $\Delta P'Q'R'$ and determine their orientation. We know that the orientation of ΔPQR is determined from $\det\left(\overrightarrow{PQ}, \overrightarrow{PR}\right)$ and that of $\Delta P'Q'R'$ from $\det\left(\overrightarrow{P'Q'}, \overrightarrow{P'R'}\right)$.

Test-I: If ΔPQR and $\Delta P'Q'R'$ have the same orientation, then α is orientation preserving isometry. Thus, α is either a translation or a rotation (Because translation and rotation are the only types of orientation preserving isometries). Furthermore, to determine whether α is a translation or a rotation, find the vectors $\overrightarrow{PP'}, \overrightarrow{QQ'}, \overrightarrow{RR'}$. If $\overrightarrow{PP'} = \overrightarrow{QQ'} = \overrightarrow{RR'}$, then α is a translation, but if they are different, α is a rotation.

Test-II: If ΔPQR and $\Delta P'Q'R'$ have different orientation, then α is orientation reversing (changing) isometry. Thus, α is either a glide-reflection or a reflection (Because glide-reflection and reflection are the only types of orientation reversing (changing) isometries). Furthermore, to determine whether α is a reflection or glid-reflection, find the equation of the reflecting line using any of the points and its respective image.

If all the points are reflected on the same line, then α is a reflection, otherwise it is glide-rflection.

Example: Suppose α is an isometry which maps ΔPQR into $\Delta P'Q'R'$ where the vertices of the triangles are

$$P=\begin{pmatrix}4\\2\end{pmatrix},\ Q=\begin{pmatrix}1\\-2\end{pmatrix},\ R=\begin{pmatrix}12\\-4\end{pmatrix},\ P'=\begin{pmatrix}-1\\3\end{pmatrix},\ Q'=\begin{pmatrix}3\\0\end{pmatrix},\ R'=\begin{pmatrix}5\\11\end{pmatrix}.$$

Determine whether α is a translation, rotation, glide reflection or reflection and find its equation. Based on the type of α, find the translator vector, center and angle of rotation or line of reflection.

Solution: To apply the above test, first determine the orientation of ΔPQR and $\Delta P'Q'R'$ .The orientation of ΔPQR is determined from $\det\left(\overrightarrow{PQ},\overrightarrow{PR}\right)$ where $\det\left(\overrightarrow{PQ},\overrightarrow{PR}\right)=\begin{vmatrix}-3 & 8\\ -4 & -6\end{vmatrix}=50>0$ .Hence, ΔPQR has positive orientation.

Similarly, the orientation of $\Delta P'Q'R'$ is determined from $\det\left(\overrightarrow{P'Q'},\overrightarrow{P'R'}\right)$ where $\det\left(\overrightarrow{P'Q'},\overrightarrow{P'R'}\right)=\begin{vmatrix}4 & 6\\ -3 & 8\end{vmatrix}=50>0$ which shows that $\Delta P'Q'R'$ also has positive orientation. So, we have got that ΔPQR and $\Delta P'Q'R'$ have the same orientation. As a result, α is orientation preserving isometry. Therefore, α is either a translation or a rotation. Now, to determine whether α is a translation or a rotation, find the vectors $\overrightarrow{PP'}, \overrightarrow{QQ'}, \overrightarrow{RR'}$.

Here, $\overrightarrow{PP'}=\begin{pmatrix}-5\\1\end{pmatrix},\ \overrightarrow{QQ'}=\begin{pmatrix}2\\2\end{pmatrix},\ \overrightarrow{RR'}=\begin{pmatrix}-7\\15\end{pmatrix}$. But $\overrightarrow{PP'}\neq\overrightarrow{QQ'}\neq\overrightarrow{RR'}$.

So, α is not a translation. As a result, the only option for α to be is a rotation. Now, determine the equation of α. We know that rotation is an even isometry. So, we can determine the equation of α from the general equations of even isometries $\alpha\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x'\\y'\end{pmatrix}$ where $\begin{cases}x'=ax-by+c\\ y'=bx+ay+d\end{cases}$, $a^2+b^2=1$.

So, using $\alpha(P)=P'$, $\alpha(Q)=Q'$, we get $\begin{cases}4a-2b+c=-1\\ 2a+4b+d=3\end{cases}$; $\begin{cases}a+2b+c=3\\ -2a+b+d=0\end{cases}$

Collecting equations with like terms gives

$$\begin{cases} 4a - 2b + c = -1 \\ a + 2b + c = 3 \end{cases} \Rightarrow 3a - 4b = -4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

$$\begin{cases} 2a + 4b + d = 3 \\ -2a + b + d = 0 \end{cases} \Rightarrow 4a + 3b = 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

Multiplying equation (i) by 3 and equation (ii) by 4 yields

$$\begin{cases} 9a - 12b = -12 \\ 16a + 12b = 12 \end{cases} \Rightarrow 25a = 0 \Rightarrow a = 0,\ b = 1$$

Substituting these values of a and b gives $c = 1,\ d = -1$.

Therefore, the equation of α becomes $\alpha\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ where $\begin{cases} x' = -y + 1 \\ y' = x - 1 \end{cases}$.

Since α is a rotation, our last task is to find the center and angle of the rotation. But, in the general equation of isometry, we know that $a = \cos\theta,\ b = \sin\theta$ where θ is the angle of rotation. Thus, $\cos\theta = 0,\ \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$. Besides, the center of rotation $C = (h, k)$ is given by

$$h = \frac{c}{2} - \frac{d}{2\tan\frac{\theta}{2}} = \frac{1}{2} + \frac{1}{2} = 1,\ k = \frac{d}{2} + \frac{r}{2\tan\frac{\theta}{2}} = -\frac{1}{2} + \frac{1}{2} = 0.$$

As a result, α is a rotation with center $C = (1,0)$ and angle of rotation $\theta = \pi/2$.

Problem Set 3.9

1. Suppose $R_{C,\theta}$ is a counterclockwise rotation with center $C=(h,k)$ whose equations are given by $\begin{cases} x'=10-y \\ y'=x-30 \end{cases}$. Find the angle and center of this rotation.

Answer : $(h,k)=(20,-10)$

2. If $2x'=\sqrt{3}x-y+14-4\sqrt{3}$ and $2y'=x+\sqrt{3}y+8-6\sqrt{3}$ are equations of a CCW rotation, then find the center and angle of the rotation.

Answer :$C=(4,6),\ \theta=30^\circ$

3. Suppose $\alpha\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2ax+\frac{4}{5}y+7 \\ \frac{4}{5}x+\frac{3}{5}y-5 \end{pmatrix}$ is an odd isometry.Then, find the value (s) of the constant a.

Answer : $a=-3/10$

4. If $\alpha\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 3px+\frac{5}{13}y \\ \frac{5}{13}x+\frac{12}{13}y \end{pmatrix}$ is an isometry, find the value (s) of the constant p.

Answer : $p=-4/13$

5. If $x'=\dfrac{3x}{5}+\dfrac{4y}{5}$ and $y'=\dfrac{4x}{5}-\dfrac{3y}{5}$ are equations for a reflection S_m, then find equation of line m.

Answer :$m: x=2y$

6. Suppose α is an isometry that maps the three non-collinear points $A=(-1,1),\ B=(2,1),\ C=(-1,5)$ into the points $A'=(1,-1), B'=(4,-1),\ C'=(1,3)$. Determine whether α is a translation, rotation, glide reflection or reflection and find its equation.

7. Suppose $g(x,y)=(\dfrac{5}{13}x-\dfrac{12}{13}y-\dfrac{31}{13},-\dfrac{12}{13}x-\dfrac{5}{13}y+\dfrac{38}{13})$ is the equations of a glide reflection g.

Answer :$l: 2x+3y-2=0$

8. Suppose $\alpha(x,y)=(x',y')$ where $x'=ax+by+c$ and $y'=bx-ay+d$ where $a^2+b^2=1$. Show that α is an odd isometry.

Review Problems On Chapter-3

1. Given the lines $l: x+y-6=0, m: x-y-4=0$. Find the center P of a half turn H_P such that $S_m \circ S_l = H_P$. **Answer** : $P=(5,1)$

2. Given a line $m: 2x-y+3=0$. Find two possible equations for a line l such that $S_m(l)=l$ with $P=(1,5)\in l$. **Answer** : $l: 2x-y+3=0$ Or $l: x+2y-11=0$

3. Let $l: 2x-3y+1=0$ and $m: 4x-6y-5=0$. Then find the image of the point $(1,3)$ by the composite reflection $S_m \circ S_l$

4. Let $l: y=2\sqrt{3}x+1$ and $m: y=\frac{5}{3\sqrt{3}}x+\frac{2}{3}$. Show that for any point (x,y),

$S_m \circ S_l(x,y)=(x',y')$ where $\begin{cases} x'=\frac{x}{2}+\frac{\sqrt{3}}{2}y-\frac{4\sqrt{3}}{13} \\ y'=\frac{-\sqrt{3}}{2}x+\frac{1}{2}y-\frac{4\sqrt{3}}{13} \end{cases}$

5. Let $l: y=ax,\ a\neq 0$ and $m: y=bx,\ b\neq 0$. If the angle measured from l to m is θ, then $S_m \circ S_l(x,y)=(x',y')$ where $\begin{cases} x'=x\cos 2\theta+y\sin 2\theta \\ y'=x\sin 2\theta-y\cos 2\theta \end{cases}$

6. Let $l: ax-y=0$ and $m: cx+y=0$. If $ac=1$, calculate $S_m \circ S_l(10,-12)$.

Answer :$(-10,12)$

7. <u>True or False?</u> If $f=S_n \circ S_m$ and $f=S_l \circ S_t$, then $n=l$ and $m=t$.

Answer: False

8. Let $l: y=3x+20$ and $m: y=3x-10$. Then, find a translation T with translator vector v such that $S_m \circ S_l = T_v$. **Answer** : $v=(9,-3)$

9. Given two lines l and m intersecting at $(1,2)$. Suppose $S_m \circ S_l(3,5)=(-2,4)$. Then find the acute angle measured from l to m. **Answer** : $\theta=\frac{\pi}{4}$

10. Let m and n be lines intersecting at $C=(2,1)$. If $S_m \circ S_n = R_{C,\frac{\pi}{3}}$, then find the equation of line n when $m: y=x+1$.

11. Let g be a glide reflection with axis $l: y = x+1$ and glide vector $\vec{v} = (1,1)$ Then, find the image of the point $(3,2)$. **Answer** : $(2,5)$

12. Let g be a glide reflection with axis $l: x - y + 6 = 0$ and glide vector $\vec{v} = (2,2)$. Then, find the general equation of g and calculate the image of the point $(0,0)$. **Answer** : $g(x, y) = (y-4, x+8)$

13. Let g be a glide reflection with axis $l: x - y + 3 = 0$ and glide vector $\vec{v} = (4,4)$. Then, find the general equation of g and calculate the image of the point $(0,0)$. **Answer** : $g(x, y) = (y+1, x+7),\ g(0,0) = (1,7)$

14*. Let m and n be parallel lines $13cm$ a part. A point A is $4cm$ and $17cm$ from line m and n respectively. Suppose A is reflected across line m, and then its image A' is reflected across line n to create a second image A''.

Answer :$d = 26cm$

a) Draw the diagram showing the position of A, A', A'' and find the distance between A and A''. b) If the order of reflection is changed, repeat the problem of part (a).

How do the answers to part (a) and (b) differ?

15. Suppose m and n are two distinct lines. Then, show that $S_n \circ S_m(P) = P \Rightarrow P \in m \cap n$.

16. Let l and m be any two lines. Prove that $S_l \circ S_m = S_m \circ S_l \Leftrightarrow l \perp m$

17. Let m and n be lines intersecting at P. Show that if m and n are fixed lines under an isometry α, then $\alpha(P) = P$.

18. Given two lines l and m intersecting at a point $(1,2)$ and $l: y = x+1$. Suppose $R_m R_l(3,5) = (-2,4)$. Find the acute angle measured from l to m.

19. If $k // l$ and $k \perp m$, then show that $S_m \circ (S_l \circ S_k) = (S_l \circ S_k) \circ S_m$.

20*. Let α, β, θ be interior angles at the vertices A, B, C respectively of ΔABC (oriented counter clockwise). Show that $R_{A,2\alpha} \circ R_{B,2\beta} \circ R_{C,2\theta} = i.$

21. Show that non-identity rotations with different centers do not commute.

22. What must be true of two rotations $R_{A,\theta}$ and $R_{B,\beta}$ if their product $R_{A,\theta} \circ R_{B,\beta}$ is a translation?

23. Show that the composition of any two rotations is either a rotation or a translation.

24. Let R_1 and R_2 be rotations. Then show that $R_1 \circ R_2 \circ R_1^{-1} \circ R_2^{-1}$ is a translation.

25. Prove that

a) The set of all isometries forms a transformation group.

b) The set of all rotation with fixed center forms abelian group of transformations.

26. Suppose l intersects the lines m and n so that alternate interior angles are congruent. Then, using the concept of isometry, prove that m and n are parallel.

27. Show that the set of all rotations, all reflections, or all half turns does not form a transformation group. State the reasons clearly in each case.

28. Suppose the lines l and m passes through the origin in R^2 make angles α and β respectively with the positive $x-$axis. Show that $R_m R_l = R_{2(\beta-\alpha)}$. Where $R_m R_l$ is a reflection on l followed by a reflection on m and $R_{2(\beta-\alpha)}$ is a rotation by angle $2(\beta-\alpha)$ about the origin.

CHAPTER-4

SIMILARITY TRANSFORMATIONS

4.1 Introduction

Consider a right angle triangle ABC with right angle at C and hypotenuse $\overline{AB}$ as in the figure 4.1.

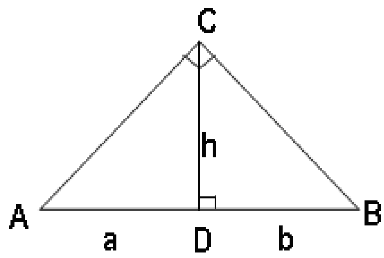


Figure 4.1

If the altitude $\overline{CD}$ is constructed, we get three similar triangles ADC, ACB, CDB. Then, using similarity of these triangles we see that there is proportion of the segments of the hypotenuse AD, DB and the altitude $\overline{CD}$ given by $\frac{AD}{CD} = \frac{DC}{DB} = \frac{AC}{CB}$.

If we assign the lengths of these segments as a, b, h in the figure, we get $\frac{a}{h} = \frac{h}{b} \Rightarrow h^2 = ab$.

In this explanation, even though the lengths, areas in general sizes of the figure are different we see that they do have the same shape and the ratio of their sides remains constant as well.

Such geometric figures which have the same shape and their sides proportional are said to be similar figures and a transformation which maps one figure in to a similar figure is known as *similarity transformation*.

Definition: A transformation α of the plane is said to be a similarity transformation if there exists a positive number k such that for all points A and B of the plane and their images A' and B', we have $A'B' = kAB$. That is $\alpha(A)\alpha(B) = A'B' = kAB$. The constant k is called the coefficient or factor or ratio of the similarity.

Examples: Determine whether the following transformations are similarity transformation or not.

a) $\alpha : R^2 \to R^2$ given by $\alpha(x, y) = (2x + 5, 2y - 6)$

b) $\beta : R^2 \to R^2$ given by $\beta(x, y) = (x - 1, y - 3)$

c) $\delta : R^2 \to R^2$ given by $\delta(x, y) = (2x + 3, 3y - 5)$

Solution:

a) Clearly, α is a transformation. Now, let $A = (a, b)$ and $B = (c, d)$ be any two points in the plane R^2. Then,

$$\alpha(A) = A' = (2a + 5, 2b - 6), \alpha(B) = B' = (2c + 5, 2d - 6)$$

$$\Rightarrow \overrightarrow{A'B'} = B' - A' = (2c - 2a, 2d - 2b)$$

$$\Rightarrow \overrightarrow{A'B'} = 2\overrightarrow{AB} \Rightarrow A'B' = kAB, \text{ for } k = 2$$

Hence, α is a similarity transformation with ratio $k = 2$.

b) Similar calculation here also yields $A'B' = AB \Rightarrow A'B' = kAB,\ for\ k = 1$.

So, β is a similarity transformation with ratio $k = 1$.

c) In this case, $A'B' = (3c - 3a, 2d - 2b),\ AB = (c - a, d - b) \Rightarrow A'B' \neq kAB,\ \forall k \in R$.

It means there is no positive constant k for which $A'B' \neq kAB$. So, δ is not a similarity transformation.

4.2 Properties of Similarity Transformations

Proposition 4.1: The product of any two similarities is again a similarity.

Proof: Let α and β be any two similarities with ratios k and t respectively. For any two points, A and B, let $\alpha(A) = A'$, $\alpha(B) = B'$.

Then, $\beta \circ \alpha(A) = \beta(\alpha(A)) = \beta(A') = A''$ and $\beta \circ \alpha(B) = \beta(\alpha(B)) = \beta(B') = B''$

Since β is a similarity with ratio t, it follows that $A''B'' = tA'B'$.

Again, as α is a similarity with ratio k, it follows that $A'B' = kAB$.

So, $A''B'' = tA'B' = t(kAB) = tkAB$. Thus, $\beta \circ \alpha$ is a similarity with ratio tk.

Proposition 4.2: The inverse of a similarity is again a similarity.

Proof: Let α be a similarity with ratio k. Then, for any two points, A and B, $A'B' = kAB$ where $\alpha(A) = A'$, $\alpha(B) = B'$. But, α is a transformation (Definition of similarity) implies $\alpha(A) = A'$, $\alpha(B) = B' \Leftrightarrow \alpha^{-1}(A') = A$, $\alpha^{-1}(B') = B$. Thus,

$$A'B' = kAB \Leftrightarrow A'B' = k\alpha^{-1}(A')\ \alpha^{-1}(B') \Leftrightarrow \alpha^{-1}(A')\alpha^{-1}(B') = \frac{1}{k}A'B' \Leftrightarrow AB = \frac{1}{k}A'B'.$$

But for any positive constant k, $\frac{1}{k}$ is also defined and positive. Hence, α^{-1} is a similarity transformation with ratio $\frac{1}{k}$ whenever α is a similarity with ratio k.

Therefore, for any similarity α with ratio k, $A'B' = kAB$ if and only if $AB = \frac{1}{k}A'B'$ if and only if α^{-1} is a similarity transformation with ratio $\frac{1}{k}$.

Proposition 4.3: A similarity maps triangles in to similar triangles.

Proof: Let α be a similarity with ratio k and let $\Delta A'B'C'$ be the image of ΔABC under α. We need to show that $\Delta A'B'C' \sim \Delta ABC$, where $\alpha(A) = A'$, $\alpha(B) = B'$, $\alpha(C) = C'$.

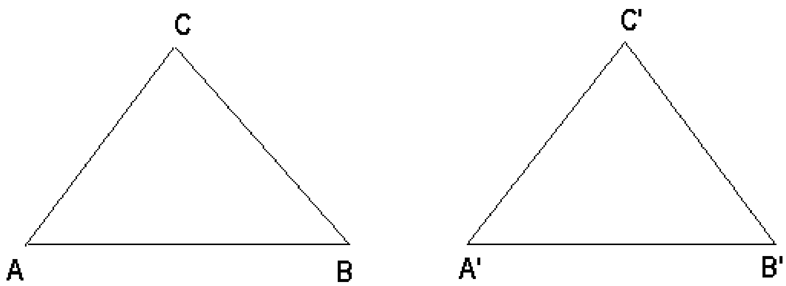


Figure 4.2

Since α is a similarity with ratio k, it follows that

$$\left.\begin{array}{l} A'B' = kAB \Rightarrow \dfrac{A'B'}{AB} = k \\ A'C' = kAC \Rightarrow \dfrac{A'C'}{AC} = k \\ B'C' = kBC \Rightarrow \dfrac{B'C'}{BC} = k \end{array}\right\} \Rightarrow \frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC} = k.$$

This shows that the three sides of the two triangles are proportional. Hence, by Side-Side-Side similarity theorem, we get $\Delta A'B'C' \sim \Delta ABC$.

Here, $\Delta A'B'C' \sim \Delta ABC$ in turn enables us to infer that the three interior angles of ΔABC are congruent to the three interior angles of $\Delta A'B'C'$ (Because corresponding angles of similar triangles are congruent).

So, the three interior angles of ΔABC are preserved and this will enable us to state the following immediate corollary.

Proposition 4.4: Any similarity preserves angles.

Proof: Immediate from proposition 4.3.

Proposition 4.5: Any similarity preserves co-linearity, between ness and mid point.

Proof: Let α be a similarity with ratio k and let A, B, C be three collinear points where B is between A and C. Then, $AB + BC = AC$ since B is between A and C. Now, whenever $\alpha(A) = A', \alpha(B) = B', \alpha(C) = C'$, we have that

$A'B' = kAB,\ B'C' = kBC,\ A'C' = kAC.$ As a result,

$A'B' + B'C' = kAB + kBC = k(AB + BC) = kAC = A'C' \Rightarrow A'B' + B'C' = A'C'$

But, this is true if and only if the image points A', B', C' are collinear and B' is between A' and C'. Besides, if B is the mid point of A and C, then

$AB = BC \Rightarrow A'B' = kAB = kBC = B'C'$ which implies B' is the mid point of A' and C'. This result tells us that the images of any three or more collinear points are again collinear. If we consider, the proposition in detail, it further implies that any similarity is a collineation.

Because if A', B' are the images of the points A, B on a line l, then for any other point P on l, P' is on a line determined by the image points A', B' (As similarity preserves co linearity and between ness). Thus, one can use those ideas together in order to conclude that the image of a line under a similarity is again a line.

Theorem 4.1: A similarity with at least two fixed points is an isometry.

Proof: Let α be a similarity with ratio k such that $\alpha(A) = A, \alpha(B) = B$. By definition of similarity, for any two distinct points A and B, $A'B' = kAB$. Then, it follows that $A'B' = kAB \Rightarrow AB = kAB \Leftrightarrow k = 1$.This shows that for any other points, P and Q, $\alpha(P) = P', \alpha(Q) = Q' \Leftrightarrow P'Q' = PQ$. (Any given similarity has exactly one ratio). Hence, α is an isometry.

Corollary 4.1: A similarity with ratio of $k = 1$ is an isometry.

4.3 Common Types of Similarity Transformations

4.3.1 Isometries

Since for any isometry f, $f(A)f(B) = AB$ for every pair of points A and B, it shows that every isometry is a similarity transformation.

a) Translations
b) Reflections
c) Rotations
d) Glide reflections

All these isometries transformations are similarity transformations.

4.3.2 Homothety (Homothetic Transformations)

Definition: A transformation is said to be homothety transformation if and only if for a scalar $k \neq 0,1$ and a fixed point C it maps every point X to X' such that $\overrightarrow{X'C} = k\overrightarrow{XC}$.

Here, the fixed point C is called center of the homothety and the scalar $k \neq 0,1$ is called ratio or factor of the homothety.

Such a transformation is denoted by $H_{C,k}$. Thus,

$$\overrightarrow{H_{C,k}(X)C} = k\overrightarrow{XC} \Rightarrow C - H_{C,k}(X) = kC - kX$$
$$\Rightarrow H_{C,k}(X) = kX + (1-k)C$$

Therefore, for any point X, $H_{C,k}(X) = kX + (1-k)C$.

If $H_{C,k} : R^2 \to R^2$ and $C = (a,b) \in R^2$, then $H_{C,k}$ is given by $H_{C,k}(x,y) = (x',y')$

where $\begin{cases} x' = kx + (1-k)a \\ y' = ky + (1-k)b \end{cases}$

Hence, all such types of transformations are named as homothety and they are similarity transformations. Such type of similarity transformations are usually called *homothetic* similarities.

Examples:

1. Find the equation of a homothety with center $(2,4)$ and factor $k = 2$ and calculate the image of the point $(3,-5)$.

Solution:

For any point (x,y), $H_{C,k}(x,y) = (x',y')$ where $\begin{cases} x' = kx + (1-k)a \\ y' = ky + (1-k)b \end{cases}$

$$H_{C,2}(x,y) = (x',y') \text{ where } \begin{cases} x' = 2x + (1-2)2 = 2x - 2 \\ y' = 2y + (1-2)4 = 2y - 4 \end{cases}$$

$H_{C,2}(x,y) = (2x-2, 2y-4)$. In particular, $H_{C,2}(3,-5) = (4,-14)$

2. A homothety takes the point $(1,1)$ to $(-1,5)$ and the point $(0,2)$ to $(-4,8)$. Find the equation of this homothety.

Solution: Let $H_{C,k}$ be a homothety with center $C=(a,b)$ and factor k. Then,

$$H_{C,k}(X)=kX+(1-k)C \Rightarrow H_{C,k}(x,y)=k(x,y)+(1-k)(a,b)$$

$$H_{C,k}(1,1)=k(1,1)+(1-k)(a,b)=(-1,5) \Rightarrow (k+a-ka,k+b-kb)=(-1,5)$$

$$H_{C,k}(0,2)=k(0,2)+(1-k)(a,b)=(-4,8) \Rightarrow (a-ak,2k+b-bk)=(-4,8)$$

$$\Rightarrow \begin{cases} k+a-ka=-1 \\ a-ka \quad =-4 \end{cases}, \qquad \begin{cases} k+b-kb=5 \\ 2k+b-kb=8 \end{cases}$$

$$\Rightarrow k=3,\ a=2,\ b=-1 \Rightarrow C=(a,b)=(2,-1)$$

Hence, $H_{C,3}(x,y)=(3x-4,3y+2)$.

Proposition 4.6: Every homothety is a similarity transformation.

Proof: Let $H_{C,k}$ be a homothety. Then, for any two points A and B,

$$H_{C,k}(A)=A'=kA+(1-k)C,\ H_{C,k}(B)=B'=kB+(1-k)C$$

$$\Rightarrow A'B'=B'-A'=kB-kA=kAB$$

$$\Rightarrow A'B'=kAB$$

So, $A'B'=kAB$ for any two points A and B where $H_{C,k}(A)=A',\ H_{C,k}(B)=B'$. Hence, any homothety is a similarity.

4.4 Representation of Similarity Transformations

Theorem 4.2: Every similarity is the product of an isometry and a homothety.

Proof: Let α be a similarity with ratio k and consider a homothety $H_{C,\frac{1}{k}}$ with center C and factor $\frac{1}{k}$. From proposition 4.1 and 4.2, $H_{C,\frac{1}{k}}\circ\alpha$ is a similarity with ratio $\frac{1}{k}.k=1$. But, a similarity with ratio of 1 is an isometry. Thus, let this isometry be $H_{C,\frac{1}{k}}\circ\alpha=\beta$. So,

$$H_{C,\frac{1}{k}}\circ\alpha=\beta \Rightarrow H_{C,k}\circ H_{C,\frac{1}{k}}\circ\alpha=H_{C,k}\circ\beta$$

$$\Rightarrow i\circ\alpha=H_{C,k}\circ\beta,\quad H_{C,\frac{1}{k}}\circ H_{C,k}=i$$

$$\Rightarrow \alpha=H_{C,k}\circ\beta$$

This means that any similarity with ratio k is the product of a homothety with ratio k and any isometry.

Definition: A transformation that maps each vector in to a vector parallel to itself is known as dilation. That means δ is a dilation if and only if for all A and B, $\overrightarrow{A'B'} = k\overrightarrow{AB}$ where $A' = \delta(A),\ B' = \delta(B).$

Example: Show that $\alpha : R^2 \to R^2$ given by $\alpha(x, y) = (3x - 4, 3y + 2)$ is dilation.

Solution: Let $A = (a,b),\ B = (c,d)$ be arbitrary points. Then,

$$A' = \alpha(A) = (3a - 4, 3b + 2), \quad B' = \alpha(B) = (3c - 4, 3d + 2)$$
$$\Rightarrow \overrightarrow{A'B'} = (3c - 3a, 3d - 3b) = 3(c - a, d - b) = 3\overrightarrow{AB} \Rightarrow \overrightarrow{A'B'} // \overrightarrow{AB}$$

Hence, the given transformation is dilation.

Proposition 4.7: Every dilation is a similarity.

Proof: Let δ be dilation with factor $k \neq 0$.

Then, $\overrightarrow{A'B'} = k\overrightarrow{AB}$, for all points A, B. But from vector analysis, for any two vectors v and u, $v = ku \Rightarrow |v| = |k||u|$ for any scalar k.

Thus, $\overrightarrow{A'B'} = k\overrightarrow{AB} \Rightarrow \left|\overrightarrow{A'B'}\right| = \left|k\overrightarrow{AB}\right| \Rightarrow \left|\overrightarrow{A'B'}\right| = |k|\left|\overrightarrow{AB}\right| \Rightarrow \overline{A'B'} = |k|\overline{AB}.$

But, $\overline{A'B'} = |k|\overline{AB}$ is a condition for similarity if $k \neq 0.$ Hence, δ is a similarity.

Be careful! The converse of this theorem need not be true.

Let $\delta : R^2 \to R^2$ be a transformation given by $\delta(x, y) = (3 - y, x - 1).$ Take any two points. $A = (a,b)$ and $B = (c,d)$. Then,

$$\delta(A) = A' = (3 - b, a - 1),\ \delta(B) = B' = (3 - d, c - 1)$$
$$\Rightarrow \overline{A'B'} = \left\|\overrightarrow{A'B'}\right\| = \|(b - d, c - a)\| = \|(d - b, c - a)\| = \left\|\overrightarrow{AB}\right\| = \overline{AB} \Rightarrow \overline{A'B'} = k\overline{AB}, \quad with\ k = 1.$$

Thus, δ is a similarity. On the other hand,

$$\overrightarrow{A'B'} = (b - d, c - a),\ \overrightarrow{AB} = (d - b, c - a) \Rightarrow \overrightarrow{A'B'} \neq k\overrightarrow{AB},\ \forall k \in R.$$

This means $\overrightarrow{A'B'}$ and $\overrightarrow{AB}$ are not parallel. Thus, δ is not a dilation.

Hence, from this observation δ is a similarity but not a dilation which shows that the converse of the theorem is not true.

Theorem 4.4 (The Classification Theorem):

This theorem is useful in classifying whether a given dilation is a translation or a homothety. That is why it is named as *classification theorem.*

Let $\alpha : W \to W$ be a dilation. Then, $\frac{\overrightarrow{A'B'}}{\overrightarrow{AB}} = k$ (constant) for all A, B in W.

Furthermore, if $k \neq 1,$ then α is a homothety with ratio of k itself and if $k = 1,$ then α is a translation.

Proof: Assume that W is not a line. Let α be a dilation such that $\alpha(A) = C,\ \alpha(B) = D$ as shown in figure 4.3.

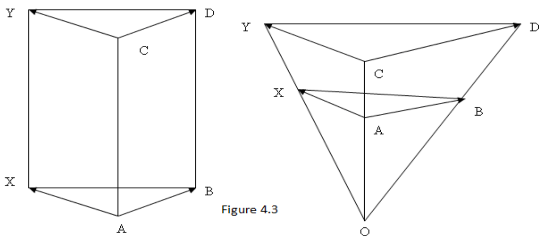


Figure 4.3

Now, suppose the points A, B, C, D are not collinear. Take any point X not on $\langle A, B\rangle$ with $\alpha(X) = Y$. As α is a dilation, $\overrightarrow{AB} \| \overrightarrow{CD}, \overrightarrow{AX} \| \overrightarrow{CY}$ and $\overrightarrow{BX} \| \overrightarrow{DY}$.

Thus, by Desagrues Theorem, $\frac{\overrightarrow{CD}}{\overrightarrow{AB}} = \frac{\overrightarrow{CY}}{\overrightarrow{AX}} = \frac{\overrightarrow{DY}}{\overrightarrow{BX}} = k.$ Since X is arbitrary, $\frac{\overrightarrow{A'B'}}{\overrightarrow{AB}} = k$ for all points A, B in W. Furthermore, if $k \neq 1$(refer figure 4.3), then $\overrightarrow{CD} = k\overrightarrow{AB}, \overrightarrow{CY} = k\overrightarrow{AX}, \overrightarrow{DY} = k\overrightarrow{BX}$ still by Desagrues Theorem, the lines $< A, C >, < B, D >$ and $< X, Y >$ are concurrent at a point O with $\frac{\overrightarrow{OC}}{\overrightarrow{OA}} = \frac{\overrightarrow{CY}}{\overrightarrow{AX}} = \frac{\overrightarrow{OY}}{\overrightarrow{OX}} = k.$ As a result,

$$\frac{\overrightarrow{OY}}{\overrightarrow{OX}} = k \Rightarrow \overrightarrow{OY} = k\overrightarrow{OX}$$

$$\Rightarrow \overrightarrow{O\alpha(X)} = k\overrightarrow{OX}$$

$$\Rightarrow \alpha(X) - O = kX - kO$$

$$\Rightarrow \alpha(X) = kX + (1-k)O$$

But, $\alpha(X) = kX + (1-k)O$ is an equation of a homothety with center O and factor k. On the other hand if $k = 1$ (refer figure 4.3), then $\overrightarrow{CD} = \overrightarrow{AB}, \overrightarrow{CY} = \overrightarrow{AX}, \overrightarrow{DY} = \overrightarrow{BX}$. But,

$$\begin{aligned}
\overrightarrow{CY} = \overrightarrow{AX} &\Rightarrow \overrightarrow{XY} = \overrightarrow{AC} \\
&\Rightarrow \overrightarrow{X\alpha(X)} = \overrightarrow{AC} \\
&\Rightarrow \alpha(X) - X = C - A \\
&\Rightarrow \alpha(X) = X + \overrightarrow{AC}
\end{aligned}$$

Thus, $\alpha(X) = X + \vec{b}$ for any point X where $\vec{b} = \overrightarrow{AC}$ which shows that α is a translation.

Corollary 4.2: Every dilation is either a translation or a homothety.

Proof : (*Use the classification theorem)*

Example: A dilation takes the point $(0,20)$ to $(1,-12)$ and the point $(10,5)$ to $(3,-15)$. Determine whether this dilation is a translation or a homothety and write its equation.

Solution: Let α be a dilation. Then, by the classification theorem, $\frac{\overrightarrow{A'B'}}{\overrightarrow{AB}} = k,$ for all A, B. In particular, it is true for $A = (0,20)$ and $B = (10,5)$.

Thus, $k = \frac{\overrightarrow{A'B'}}{\overrightarrow{AB}} = \frac{B'-A'}{B-A} = \frac{(3,-15)-(1,-12)}{(10,5)-(0,20)} = \frac{(2,-3)}{(10,-15)} = \frac{1}{5}\frac{(2,-3)}{(2,-3)} = \frac{1}{5} \Rightarrow k = \frac{1}{5} \neq 1.$

Hence, α is a homothety with factor $k = \frac{1}{5}$ by the classification theorem.

So, $\alpha(x, y) = \frac{1}{5}(x, y) + (1-\frac{1}{5})C = \frac{1}{5}(x, y) + \frac{4}{5}C,\ C = (a,b).$ This is true for all (x, y).

In particular, it is true for $(0,20)$, so

$$\begin{aligned}
\alpha(0,20) = (1,-12) &\Rightarrow \tfrac{1}{5}(0,20) + \tfrac{4}{5}C = (1,-12) \\
&\Rightarrow \tfrac{4}{5}C = (1,-12) - (0,4) = (1,-16) \\
&\Rightarrow C = \tfrac{5}{4}(1,-16) = (\tfrac{5}{4},-20)
\end{aligned}$$

Therefore, the equation of α is given by $\alpha(x, y) = (\frac{1}{5}x + 1, \frac{1}{5}y - 16).$

Corollary 4.3: Every dilation is a product of a homothety and an isometry.

Proof: The proof of this theorem follows from proposition 4.7 (every dilation is a similarity)and from theorem 4.2, every similarity is a product of a homothety and an isometry. Hence, every dilation is a product of a homothety and an isometry. This theorem explains that for any dilation δ, there exist a homothety $H_{C,k}$, and an isometry β such that $\delta = H_{C,k} \circ \beta$.

Theorem 4.5: If $\overleftrightarrow{A'B'} // \overleftrightarrow{AB}$, then there is a unique dilation δ for which $\delta(A) = A',\ \delta(B) = B'$.

Proof: Left as an exercise.

From this theorem, we are assured that the dilation δ that takes A, B to A', B' respectively exists and is unique whenever $\overleftrightarrow{A'B'} // \overleftrightarrow{AB}$. But, the main point is not only to assure its existence but also how to find its equation. Now, the main question is for a given similarity δ, how can we find the homothety $H_{C,k}$, and the isometry β such that $\delta = H_{C,k} \circ \beta$. To find $H_{C,k}$ and β explicitly, from $\delta = H_{C,k} \circ \beta$, let's proceed case by case as follow:

Case I: When the similarity δ is dilation:

Suppose δ is a dilation where $\delta(A) = A',\ \delta(B) = B'$ such that $\overleftrightarrow{A'B'} // \overleftrightarrow{AB}$ (The line through A', B' is parallel to the line through A, B). Thus, by theorem 4.5, there is a unique dilation that takes A to A' and B to B' .To find such dilation follow the following procedures:

Step-1: Find a translation T_v that takes A to A' and calculate the image of B under T_v . say $T_v(B) = P$

Step-2: Write the formula for a homothety with center A' and factor k, say $H_{A',k}$.

Step-3: Solve the equation $H_{A',k}(P) = B'$ for k.

Step-4: Write the equations of $H_{A',k}$ and T_v using any object point (x, y).

Hence, these are the required homothety and isometry for which $\delta = H_{A',k} \circ \beta$.

Example: Let δ be a dilation where $\delta(1,2)=(2,5)$ and $\delta(2,4)=(5,11)$. Find a homothety $H_{C,k}$ and an isometry β such that $\delta = H_{C,k} \circ \beta$.

Solution: From the given, $A=(1,2),\ B=(2,4),\ A'=(2,5),\ B'=(5,11)$

Clearly, $\overleftrightarrow{A'B'} // \overleftrightarrow{AB},$ because the line m through A', B' and the line n through A, B have the same slope 2. So, to find the decompositions of δ as a homothety $H_{C,k}$ and an isometry β such that $\delta = H_{C,k} \circ \beta$, proceed as follow using the above steps.

Step-1: Let's find a translation T_v that takes A to A' and calculate the image of B under T_v . If T_v is a translation that takes A to A' , then $v = A' - A = (1,3)$.

Hence, for any point (x, y), $T_v(x, y) = (x+1, y+3)$ and $P = T_v(B) = T_v(2,4) = (3,7)$.

Step-2: A homothety with center A' and factor k is given by

$H_{A',k}(x, y) = (kx - 2k + 2, ky - 5k + 5)$.

Step-3: Equating $H_{A',k}(P) = B'$, yields $H_{A',k}(3,7) = (k+2, 2k+5) = (5,11) \Rightarrow k = 3.$

Step-4: For any object point (x, y),

$$H_{A',3}(x, y) = (3x - 4, 3y - 10)$$
$$\beta(x, y) = T_v(x, y) = (x+1, y+3)$$

Hence, these are the required homothety and isometry for which $\delta = H_{C,k} \circ \beta.$

Case II: When the similarity δ is not dilation:

Note that from the example above, we could not generalize that the isometry part β is always a translation rather we must investigate the case when $\overrightarrow{A'B'}$ and $\overrightarrow{AB}$ are not parallel so as to make generalization. This follows with similar procedures as above with a little modification and it is left to the readers to develop the procedures.

4. 5 Equations of Similarity Transformations in Coordinates

So far we discussed about the classes of similarities. Now, we are going to see the equation of similarities in details using coordinates. First let's consider special cases which help us to drive the general equations of similarities.

I) Equations of isometric similarities

We have already discussed that the equations of any isometry β is given by

$$\beta(x,y)=(x',y') \text{ where } \begin{cases} x'=(x-h)\cos\theta-(y-k)\sin\theta+h \\ y'=\pm[(x-h)\sin\theta+(y-k)\cos\theta+k] \end{cases}$$

Expanding the equations gives $\begin{cases} x'=x\cos\theta-y\sin\theta+r \\ y'=\pm[x\sin\theta+y\cos\theta+t] \end{cases}$ where

$$\begin{cases} r=h(1-\cos\theta)+k\sin\theta \\ t=-h\sin\theta+k(1-\cos\theta) \end{cases}$$

This can be generalized as $\beta\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x' \\ y' \end{pmatrix}$ where $\begin{cases} x'=ax-by+c \\ y'=-(bx+ay+d) \end{cases}$, $a^2+b^2=1$.

Since any isometry is a similarity, this is one form of equations of similarities.

II) Equations of Homothetic Similarities

The equation of a any homothety with factor r and center $C=(h,k)$ is

$$H_{C,r}(x,y)=(x',y') \text{ where } \begin{cases} x'=rx+(1-r)h \\ y'=ry+(1-r)k \end{cases}$$

III) General Equations of Similarities

The general equations of similarities are derived from the above results using the fact that any similarity is a homothety followed by an isometry. That is for any similarity
α, $\alpha=\beta\circ H_{C,r}$ where $H_{C,r}$ is a homothety with ratio r, center C and β is an isometry. Thus, using the results in I and II above, we get

$$\alpha(x,y)=\beta\circ H_{C,r}(x,y)=(x',y') \text{ where } \begin{cases} x'=xr\cos\theta-yr\sin\theta+c \\ y'=\pm[xr\sin\theta+yr\cos\theta+d] \end{cases}$$

Here c and d are real numbers.

Now, by letting $a=r\cos\theta,\ b=r\sin\theta,\ a^2+b^2=r^2\neq 0$, for $r\neq 0$, we state the general equations of similarities as follows.

Theorem 4.6[The Generalized Equation Theorem for Similarities]:

Any similarity α in a plane has equations of the form $\alpha(x,y)=(x',y')$ where

$$\begin{cases} x' = ax - by + c \\ y' = \pm[bx + ay + d] \end{cases}, \quad a^2 + b^2 \neq 0$$

Conversely, any equations of this form are equations of a similarity.

4.6 Direct and Opposite Similarities

i) Direct Similarities

Definition: A similarity α is said to be direct similarity if and only if α is the product of a homothety about any point followed by an even isometry. Thus, from the equation of even isometry, the equation of a direct similarity α becomes $\begin{cases} x' = ax - by + c \\ y' = bx + ay + d \end{cases}, \quad a^2 + b^2 \neq 0$

ii) Opposite Similarities

Definition: A similarity α is said to be opposite similarity if and only if α is the product of a homothety about any point followed by an odd isometry.
Thus, from the equation of odd isometry, the equation of an odd similarity α becomes $\begin{cases} x' = ax - by + c \\ y' = -bx - ay - d \end{cases}, \quad a^2 + b^2 \neq 0$

Therefore, in the general equation of similarity we discussed above, the plus sign applies to direct similarities where as the minus sign is taken for opposite similarities.
Note that a similarity is either direct or opposite but can not be both.
Theorem 4.7: The product of any two direct or any two opposite similarities is direct similarity while the product of a direct and opposite similarities is opposite similarity.

Proof: Let α and δ be any two direct similarities given by the following equations. $\alpha(x,y)=(x',y')$ where $\begin{cases} x'=ax-by+c \\ y'=bx+ay+d \end{cases}$, $a^2+b^2 \neq 0$

$\delta(x,y)=(x',y')$ where $\begin{cases} x'=mx-ny+p \\ y'=nx+my+q \end{cases}$, $m^2+n^2 \neq 0$

We need to show that the product $\delta \circ \alpha$ is also a direct similarity.

For any point (x,y), $\delta \circ \alpha(x,y)=\delta(\alpha(x,y))=(x'',y'')$ where

$$\begin{aligned}
x'' &= mx'-ny'+p \\
&= m(ax-by+c)-n(bx+ay+d)+p \\
&= (ma-nb)x-(mb+na)y+mc-nd+p \\
&= tx-ly+r, \quad with\ t=ma-nb, l=mb+na, r=mc-nd+p \\
y'' &= nx'+my'+q \\
&= n(ax-by+c)+m(bx+ay+d)+q \\
&= (na+mb)x+(ma-nb)y+nc+md+q \\
&= lx+ty+s, \quad with\ s=nc+md+q
\end{aligned}$$

Now, these equations to be a similarity, we are left to show that $t^2+l^2 \neq 0$.

But,

$$\begin{aligned}
t^2+l^2 &= (ma-nb)^2+(mb+na)^2 \\
&= (ma)^2-2abmn+(nb)^2+(mb)^2+2abmn+(na)^2 \\
&= (ma)^2+(na)^2+(mb)^2+(nb)^2 \\
&= m^2(a^2+b^2)+n^2(a^2+b^2) \\
&= (m^2+n^2)(a^2+b^2) \neq 0, \ \sin ce\ a^2+b^2 \neq 0,\ m^2+n^2 \neq 0 \\
&\Rightarrow t^2+l^2 \neq 0.
\end{aligned}$$

Hence, $\delta \circ \alpha(x,y)=\delta(\alpha(x,y))=(x',y')$ where $\begin{cases} x'=tx-ly+r \\ y'=lx+ty+s \end{cases}$, $t^2+l^2 \neq 0$

But by, these are equations of a similarity obtained by applying the plus sign in the general equation of similarities. Hence, $\delta \circ \alpha$ is a direct similarity whenever α and δ are direct. Similarly, show that the product of any two opposite similarities is a direct similarity. Now, let's proof the last part.

The product of a direct and opposite similarities is opposite similarity.

Let α be a direct similarity given by $\alpha(x, y) = (x', y')$ where

$$\begin{cases} x' = ax - by + c \\ y' = bx + ay + d \end{cases}, \quad a^2 + b^2 \neq 0$$

Let δ be opposite similarity given by $\delta(x, y) = (x', y')$ where

$$\begin{cases} x' = mx - ny + p \\ y' = -nx - my - q \end{cases}, \quad m^2 + n^2 \neq 0$$

We need to show that the product $\delta \circ \alpha$ is opposite similarity.

For any point (x, y),we have $\delta \circ \alpha(x, y) = \delta(\alpha(x, y)) = (x'', y'')$ where

$$\begin{aligned} x'' &= mx' - ny' + p \\ &= m(ax - by + c) - n(bx + ay + d) + p \\ &= (ma - nb)x - (mb + na)y + mc - nd + p \\ &= tx - ly + r, \quad with\ t = ma - nb, l = mb + na, r = mc - nd + p \\ y'' &= -nx' - my' - q \\ &= -n(ax - by + c) - m(bx + ay + d) - q \\ &= -(na + mb)x - (ma - nb)y - nc - md - q \\ &= -lx - ty - s \end{aligned}$$

Where the constants l, t, s are as given in the first part. Besides, we showed above that $t^2 + l^2 \neq 0$.

Thus, we got $\delta \circ \alpha(x, y) = \delta(\alpha(x, y)) = (x', y')$ where $\begin{cases} x' = tx - ly + r \\ y' = -lx - ty - s \end{cases}, \quad t^2 + l^2 \neq 0$

So, these are equations of a similarity obtained by applying the minus sign in the general equations of similarities. Hence, $\delta \circ \alpha$ is an opposite similarity whenever α and δ are direct and opposite similarities.

Example: Let α be a transformation given by $\alpha(x, y) = (x', y')$ where

$$\begin{cases} x' = -3x + 4y + 2 \\ y' = 4x + 3y - 2 \end{cases}.$$

Show that α is a similarity and find the ratio k of this similarity. Finally, determine whether it is direct or opposite similarity.

Solution: To show that α is a similarity, it suffices to show $\overline{A'B'} = k\overline{AB}$ for any two arbitrary points A, B and for some positive constant k.

Let $A=(x,y),\ B=(z,w)$. Then, $A'=\alpha(A)=(x',y')$ where $\begin{cases} x'=-3x+4y+2 \\ y'=4x+3y-2 \end{cases}$ and

$B'=\alpha(B)=(z',w')$ where $\begin{cases} z'=-3z+4w+2 \\ w'=4z+3w-2 \end{cases}$.

Thus,

$$\begin{aligned}
\overline{A'B'} &= \sqrt{(z'-x')^2+(w'-y')^2} \\
&= \sqrt{[-3(z-x)+4(w-y)]^2+[4(z-x)+3(w-y)]^2} \\
&= \sqrt{9(z-x)^2-24(z-x)(w-y)+16(w-y)^2+16(z-x)^2+24(z-x)(w-y)+9(w-y)^2} \\
&= \sqrt{25(z-x)^2+25(w-y)^2} \\
&= 5\sqrt{(z-x)^2+(w-y)^2} \\
&= 5\overline{AB},\ \ \overline{AB}=\sqrt{(z-x)^2+(w-y)^2}
\end{aligned}$$

This implies that α is a similarity with ratio $k=5$.

Now, we are left to determine whether it is direct or opposite similarity . From the general similarity equation, we have $\alpha(x,y)=(x',y')$ where

$$\begin{cases} x'=ax-by+c \\ y'=\pm[bx+ay+d] \end{cases},\ \ a^2+b^2\neq 0.$$

Equating the equation of α with this gives us $\begin{cases} x'=ax-by+c=-3x+4y+2 \\ y'=\pm[bx+ay+d]=4x+3y-2 \end{cases}$

Here, from the firs equation we get,

$ax-by+c=-3x+4y+2 \Rightarrow a=-3,\ b=-4, c=2.$ From the second we have $\pm[bx+ay+d]=4x+3y-2 \Rightarrow \pm[-4x-3y+d]=4x+3y-2$. If we apply the plus sign and equate corresponding coefficients, we get $+[-4x-3y+d]=4x+3y-2 \Rightarrow -4=4, -3=3, d=-2$ which is not possible. Again by applying the minus sign, $-[-4x-3y+d]=4x+3y-2 \Rightarrow d=2$.

Thus, with the values $a=-3, b=-4, c=2,\ d=2$, $\alpha(x,y)=(x',y')$ where

$$\begin{cases} x'=ax-by+c \\ y'=-[bx+ay+d] \end{cases},\ \ a^2+b^2\neq 0.$$

This shows that α is an opposite similarity.

Review Problems on Chapter-4

1. If $x' = 3x + 5y + 2$ and $y' = tx - 3y$ are the equations of a similarity, then find the value of t. **Answer** $: t = 5$

2. Let H be a *homothety* with factor k and center c. For what value of k, H will be a half turn about c? **Answer** $: k = -1$

3. If $\delta(0,0) = (1,0),\ \delta(1,0) = (2,2),\ \delta(2,2) = (-1,6)$ for a similarity δ, then find $\delta(-1,6)$

4. Let δ be a dilation with center $C = (2,3)$ and factor $k = 3$. If $d(C, P') = 15cm$, where $\delta(P) = P'$, find $d(C, P)$. **Answer** $: 5cm$

5. A dilation maps the point $(-1,4)$ to $(6,-9)$ and the point $(0,1)$ to $(-1,12)$. Is it a homothety or a translation? Write the formula for this dilation.

Answer : A homothety given by $H(x, y) = (-7x - 1, -7y + 19)$

6. *Consider the line $L : y = 2x + 17$ such that $f : L \to R$ is given by $f(x, y) = \sqrt{5}x, \forall (x, y) \in L$. Show that f is a transformation and conclude that it is a similarity. Finally, find a point P if $f(P) = -7\sqrt{5}$. **Answer** $: P = (-7,3)$

7. Given a homothety $H_{P,k}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x - \frac{8}{3} \\ \frac{1}{3}y + \frac{10}{3} \end{pmatrix}$. Then give the ratio k and the center P of this homothety. **Answer** : Ratio $k = 1/3$, Center $C = (-4,5)$

8. Let $\alpha\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} px + 4y + q \\ 4x + 3y + r \end{pmatrix}$ be a similarity with ratio $k = 5$.

If $\alpha\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ 2 \end{pmatrix}$, then find the value of the constants p, q and r.

Answer $: p = -3, q = -6, r = -9$

9. Show that a mapping $f(X) = mX + B$ is

   a) A translation if $m = 1$

   b) A homothety with factor m and center $C = \dfrac{B}{1 - m}$ if $m \neq 1$.

10. Given a line $\ell:(x,y)=(1,3)+r(2,-5)$ and a homothety $f(x,y)=5(x,y)+(-6,1)$. Find the image $f(l)$,the factor k and center C of the homothety.

Answer :By problem 7, the factor is $k=5$ and center $C=\frac{(-6,1)}{1-5}=-\frac{1}{4}(-6,1)=(\frac{3}{2},-\frac{1}{4})$

11. Let the mapping $f(x,y)=(h\circ g)(x,y)$ be defined by $f(x,y)=(x,y)+(-6,-2)$. Then find

a) The formula for $g(x,y)$ if h is a homothety with factor 2 and center $(3,2)$

b) The formula for $h(x,y)$ if g is a half turn at $(-1,-1)$.

Answer : $a)\ h(x,y)=\frac{1}{2}(x-9,y-2)$

12. Give an example of similarity with one fixed point but not isometry.

13. Let $H_{C,k}$ be a homothety and l be any line. Show that $H_{C,k}(l)=l \Leftrightarrow C\in l$.

14. Let $\delta(x,y)=(-2x+1,-2y-3)$ be a similarity. Find a homothety $H_{C,k}$ and an isometry β such that $\delta=H_{C,k}\circ\beta$.

15. Consider a dilatation $\delta_{C,r}$ with $r=3$, $C=(1,1)$. Find a vector v and a linear map β such that $\delta_{C,r}=T_v\circ\beta$. **Answer** : $v=(-2,-2),\ \beta(x,y)=(3x,3y)$

16. If the transformation α with equations $\begin{cases} x'=5x-7y+2 \\ y'=tx-5y+4 \end{cases}$ is an opposite similarity, then find the value of the constant t. **Answer** : $t=-7$

17. If a homothety $H_{C,k}(x,y)=(3x-6,3y+2t)$ fixes the line $l:4x-3y+9=0$, find the value of the constant t.(Hint: Use problem 13). **Answer** : $t=-7$

18. If a half turn $H_P(x,y)=(6-x,k-y)$ fixes the line $l:4x-3y+9=0$, find the value of the constant k. **Answer** : $k=14$

19. Let $H_{o,k}$ be homothety about the origin and let δ be any similarity with factor $\frac{1}{k}$. Show that $U=H_{o,k}\circ\delta$ is an isometry.

20. If $H_{P,\frac{1}{k}}$ and $H_{Q,k}$ are any two homotheties, then show that $H_{Q,k} \circ H_{P,\frac{1}{k}} = T_{\overrightarrow{PR}}$ for some point R.

21. Given $A = (2,1)$, $B = (-2,4), A' = (-2,4)$ and $B' = (-14,13)$. Find a dilation α such that $\alpha(A) = A'$ and $\alpha(B) = B'$.

22. Let α be a transformation given by $\alpha(x, y) = (x', y')$ where $\begin{cases} x' = -3x + 4y + 2 \\ y' = 4x + 3y - 2 \end{cases}$.

Show that α is a similarity and find the ratio k of this similarity. Finally, determine whether it is direct or opposite similarity.

23. A dilation δ maps the point $(-1,4)$ to $(6,-9)$ and the point $(0,1)$ to $(-1,12)$.

   a) Is it a homothety or a translation?
   b) Write the formula for the dilation.
   c) Find a homothety $H_{C,k}$ and an isometry β such that $\delta = H_{C,k} \circ \beta$.

24. Show that

   a) The ratio of every similarity is unique.
   b) The image of a vector under a *homothety* is a parallel vector.
   c) Any similarity is a collineation.
   d) The product of two homotheties with common center is commutative
   e) If a dilatation fixes two points, then it is an identity transformation.

25. Let $\Delta A'B'C'$ be the medial triangle of ΔABC. Prove that there exists a dilation δ which maps ΔABC to $\Delta A'B'C'$. Find the center and scaling factor.

CHAPTER-5

AFFINE TRANSFORMATIONS

5.1 Introduction

In affine space, any three points $A=(x,y),\ B=(z,w),\ C=(u,v)$ are said to be collinear if and only if $\begin{vmatrix} A & 1 \\ B & 1 \\ C & 1 \end{vmatrix} = \begin{vmatrix} x & y & 1 \\ z & w & 1 \\ u & v & 1 \end{vmatrix} = 0$

Consider two mappings in a plane $f(x,y)=(x^3, y+1), g(x,y)=(y-1, x+2)$.

Let $A=(0,0),\ B=(1,2),\ C=(-2,-4)$. Are these points collinear?

$\begin{vmatrix} A & 1 \\ B & 1 \\ C & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ -2 & -4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -2 & -4 \end{vmatrix} = 0$. Hence A, B, C are collinear.

On the other hand, consider the two given mappings above.

Under f, $A' = f(A) = (0,0),\ B' = f(B) = (1,3),\ C' = f(C) = (-8,-3)$ and

$\begin{vmatrix} A' & 1 \\ B' & 1 \\ C' & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ -8 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -8 & -3 \end{vmatrix} = 21 \neq 0$. Thus, A', B', C' are not collinear.

Again, using g instead of f, we have

$A' = g(A) = (-1,2),\ B' = g(B) = (0,4),\ C' = g(C) = (-3,-2)$ and

$\begin{vmatrix} A' & 1 \\ B' & 1 \\ C' & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 1 \\ 0 & 4 & 1 \\ -3 & -2 & 1 \end{vmatrix} = -\begin{vmatrix} 4 & 1 \\ -2 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ -3 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 4 \\ -3 & -2 \end{vmatrix} = -6-6+12=0$.

Thus, A', B', C' are collinear. Such transformations like g which maps collinear points into collinear points are said to be affine transformations, but mappings like f are not considered as affine transformation.

Definition: Let $g : W \to W$ be a transformation. Then, g is said to be affine transformation if and only if it preserves co-linearity. (It maps any collinear points into collinear points).

Example: Let $g : R^2 \to R^2$ be given by $g(x, y) = (x - y, x + y)$. Show that g is affine transformation.

Solution: Clearly g is a transformation.

Let $A = (x, y),\ B = (z, w),\ C = (u, v)$ be any collinear points.

Then, $\begin{vmatrix} A & 1 \\ B & 1 \\ C & 1 \end{vmatrix} = \begin{vmatrix} x & y & 1 \\ z & w & 1 \\ u & v & 1 \end{vmatrix} = 0$

Now, consider their images under g,

$A' = g(A) = (x - y, x + y),\ B' = g(B) = (z - w, z + w),\ C' = g(C) = (u - v, u + v)$.

Thus, $\begin{vmatrix} A' & 1 \\ B' & 1 \\ C' & 1 \end{vmatrix} = \begin{vmatrix} x-y & x+y & 1 \\ z-w & z+w & 1 \\ u-v & u+v & 1 \end{vmatrix} = \begin{vmatrix} x & x+y & 1 \\ z & z+w & 1 \\ u & u+v & 1 \end{vmatrix} = \begin{vmatrix} x & y & 1 \\ z & w & 1 \\ u & v & 1 \end{vmatrix} = 0$.

(Here we used properties of determinants)

Thus, A', B', C' are collinear. This means the three collinear points A, B, C are mapped into collinear points under g. So g is affine transformation.

Coordinate definition of affine transformation: A mapping $g : R^2 \to R^2$ is said to be affine transformation if and only if it is of the form

$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by + e \\ cx + dy + f \end{pmatrix}$, where $ad - bc \neq 0$. This definition is equivalent to the definition we stated first. In this book, we mainly follow this definition.

Consider the two mappings in a plane that we saw above

$f(x, y) = (x^3, y + 1), g(x, y) = (y - 1, x + 2)$. Here, g is of the form given in this definition and thus it is affine transformation but f is not of the form given in the definition because the power of x is 3 and thus it is not affine transformation.

5.2 Basic Properties of Affine Transformations

1.Any affine transformation maps lines into lines.

Proof: Let l be any line and g be any affine transformation. Suppose A, B, C are collinear points on l.

Since g preserves co-linearity, $A' = g(A),\ B' = g(B),\ C' = g(C)$ are also collinear. Hence, $A',\ B',\ C'$ are all on the same line say l'. Now for any point P on l collinear with A, B, $P' = g(P)$ is collinear with $A',\ B'$ and found on l'. Otherwise, if for at least one point Q on l, $Q' = g(Q)$ is not on l', Q' will be non-collinear with $A',\ B'$ which means g maps three collinear points A, B, Q into non-collinear points A', B', Q'. But this is a contradiction with the definition of affine transformation (as g is supposed to be affine transformation)

2. An affine transformation maps *parallel* lines into parallel lines.

Proof: Let m and n be any two distinct parallel lines and g be an affine transformation such that $m' = g(m),\ n' = g(n)$. We need to show that $m' // n'$.

Suppose they are not parallel. Then there exist at least one common point P'.

But, $P' \in m' \cap n' \Rightarrow \exists P \in m,\ Q \in n,\ \ni g(P) = P',\ g(Q) = P' \Rightarrow g(P) = g(Q) \Rightarrow P = Q.$

Because g is one to one. This means, $P' \in m' \cap n' \Rightarrow \exists P \in m \cap n,\ \ni g(P) = P'.$ But this is not possible as the lines m and n are parallel and distinct. Consequently, $m' // n'$ whenever $m // n$.

3. The image of a *plane* under affine transformation is again a plane.

Proof: Let π be any plane. If m and n are any two intersecting or parallel lines on π, then from above properties their images are intersecting or parallel under a given affine transformation. But any two intersecting or parallel lines determine a plane.

4. An affine transformation preserves *'betweeness'*: If P is any point between A and B, then $g(P)$ is a point between $g(A)$ and $g(B)$.

5. An affine transformation preserves *mid-point*: If M is the mid-point of A and B, then $g(M)$ is the mid-point of $g(A)$ and $g(B)$.
6. An affine transformation preserves *length ratio* of line segments lying along a line. If A, B, C are collinear points, then it is always true that

$$\frac{g(A)-g(B)}{g(B)-g(C)} = \frac{A-B}{B-C}.$$

7. The image of a triangle under affine transformation is another triangle.
8. The image of a parallelogram under affine transformation is another parallelogram.
9. An affine transformation preserves *cenroid* of a triangle.
10. An affine transformation preserves *bay centric* coordinates or center of mass.
11. The image of an *ellipse* under affine transformations is another ellipse.

Geometric Properties that are not preserved under AT

As there are preserved properties of geometric figures under affine transformations, there are also properties which are not preserved under affine transformations.

1. Lengths of line segments (Distances) are not preserved under affine transformations.

Example: Let $g: R^2 \to R^2$ be affine transformation given by $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$.

Take any two points $A = \begin{pmatrix} x \\ y \end{pmatrix}$, $B = \begin{pmatrix} z \\ w \end{pmatrix}$. Then, $\overline{AB} = \sqrt{(z-x)^2 + (w-y)^2}$.

But,

$$A' = \begin{pmatrix} 2x \\ 2y \end{pmatrix},\ B' = \begin{pmatrix} 2z \\ 2w \end{pmatrix} \Rightarrow \overline{A'B'} = \sqrt{4(z-x)^2 + 4(w-y)^2}$$

$$= 2\sqrt{(z-x)^2 + (w-y)^2}$$

$$= 2\overline{AB} \Rightarrow \overline{A'B'} \neq \overline{AB}$$

2. Circles are not preserved under affine transformations. Under affine transformations, generally *circles* are mapped into *ellipses*.

Example: Let $g: R^2 \to R^2$ be affine transformation given by $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x \\ \frac{1}{4}y \end{pmatrix}$.

Find the image of the circle $C: x^2 + y^2 = 144$.

Solution: Let $\begin{pmatrix} x \\ y \end{pmatrix}$ be any point on the given circle. Then, $\begin{pmatrix} x' \\ y' \end{pmatrix} = g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x \\ \frac{1}{4}y \end{pmatrix}$.

Solving this equation for x and y in terms of the images x' and y' yields,

$x = 3x'$, $y = 4y'$.

Now put these values in the equation of the circle to find the required image.

$$C : x^2 + y^2 = 144$$
$$\Rightarrow C' : (3x')^2 + (4y')^2 = 144$$
$$\Rightarrow C' : 9x'^2 + 16y'^2 = 144$$
$$\Rightarrow C' : \frac{x'^2}{16} + \frac{y'^2}{9} = 1$$

Thus, the image is an ellipse.

3. *The norm of a vector is* not preserved under affine transformations.
4. *Angle measures* are not preserved under affine transformations.
5. *Area* and *volume* are not preserved under affine transformations.
6. Affine transformations do not preserve *orientation* of plane figures.
7. *Right*, *Isosceles* and *equilateral* triangles are not preserved under affine transformations.
8. *Rectangles* are not preserved under affine transformations.
9. Generally, affine transformations do not preserve *shape and size.*

Proposition 5.1: Let g be an affine transformation that fixes two distinct points A, B. Then g fixes the whole line through A, B point wise.

Proof: Given $g(A) = A, g(B) = B$. We need to show that $g(P) = P$ for any point on a line through A, B. Let P be any point on a line through A, B. Then, A, B, P are collinear so that $\overrightarrow{AP} = r\overrightarrow{AB}$. On the other hand as g preserves co linearity, A', B', P' are also collinear. So, $\overrightarrow{A'P'} = r\overrightarrow{A'B'}$. But $A' = A, B' = B$ implies $\overrightarrow{AP'} = r\overrightarrow{AB}$. Thus combining $\overrightarrow{AP} = r\overrightarrow{AB}$ and $\overrightarrow{AP'} = r\overrightarrow{AB}$, gives

$\overrightarrow{AP'} = r\overrightarrow{AB} = \overrightarrow{AP} \Rightarrow P' - A = P - A \Rightarrow P' = P$

Hence, g fixes the whole line through A, B point wise.

5.3 Types of Affine Transformations

5.3.1 Line (Skew)-Reflections

Definition: Let l and k be two intersecting lines in a plane. Then, *line refection* is a *reflection* on line k in the *direction* of line l that maps every point P to the point P' such that the line through P and P' is parallel to the line l and the midpoint of $P \text{ and } P'$ lies on the line of reflection k.

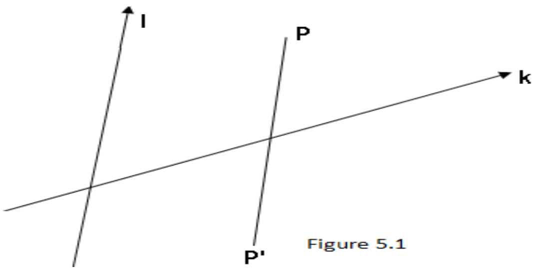


Figure 5.1

Notation: Let S be line reflection on k in the *direction* of l.

Then, $$S(P) = \begin{cases} P, \text{ if } P \in k \\ P', \ni \overleftrightarrow{PP'} // l, \ \dfrac{P+P'}{2} \in k, \text{ if } P' \notin k \end{cases}$$

Here, the line k is called axis of reflection and the line l is called direction of reflection.

Example: Find the image of the point $(2,-1)$ by a reflection on a line $k: 2x - y + 1 = 0$ in the direction of the line $l; 3x + y - 6 = 0$.

Solution: Let the image of the point $(2,-1)$ be (x', y'). Then, the line through $(2,-1)$ and (x', y') is parallel to the direction line l. As any two parallel lines have the same slope, we get

$$\frac{y'+1}{x'-2} = -3 \Rightarrow 3x'+y'-5 = 0 \text{...}(i)$$

On the other hand the mid point of $(2,-1)$ and (x', y') lies on the axis of reflection k, hence it satisfies the equation of line k.

$$(\tfrac{x'+2}{2}, \tfrac{y'-1}{2}) \in k \Rightarrow 2(\tfrac{x'+2}{2}) - (\tfrac{y'-1}{2}) + 1 = 0 \Rightarrow 2x'-y'+7 = 0 \text{..................................}(ii)$$

Combining (i) and (ii), we get the following system of linear equations

$$\begin{cases} 2x' - y' + 7 = 0 \\ 3x' + y' - 5 = 0 \end{cases} \Rightarrow 5x' = -2 \Rightarrow x' = -\tfrac{2}{5},\ y' = \tfrac{31}{5} \Rightarrow P' = (-\tfrac{2}{5}, \tfrac{31}{5})$$

Therefore, the image of the point $(2,-1)$ by a reflection on the line $k: 2x - y + 1 = 0$ in the direction of the line $l; 3x + y - 6 = 0$ is $P' = (-\tfrac{2}{5}, \tfrac{31}{5})$.

Remarks:

1. In reflection problems, the line through a point P and its image P' is perpendicular to the line of reflection, but in skew or line reflection, this is not necessarily true. For instance, in the above example the line through the point $P = (2,-1)$ and its image $P' = (-\tfrac{2}{5}, \tfrac{31}{5})$ is not perpendicular to the line

$k: 2x - y + 1 = 0$ .This is because their slopes are $m = 2$ and $m' = -3$ such that $m.m' = -6 \neq -1$, but the product of the slope of
two non- vertical and perpendicular lines must be -1.

2. If the direction line of reflection is perpendicular to the line of reflection, then the line through P and its image P' is also perpendicular to the line of reflection or axis of reflection. In this case, it becomes the normal or the usual reflection that we know it before. It is simply called orthogonal reflection. That means, every reflection is line reflection but every line reflection is not orthogonal reflection.

5.3.2 Compressions

Definition: Let l and k be two intersecting lines in a plane. Then, a *compression* on line k in the *direction* of line l with factor t is a transformation that maps every point P to the point P' such that the line through Q and P is parallel to the line l where Q is a point on line k.

Notation: Let C be a compression on k in the *direction* of l. Then,

$$C(P) = \begin{cases} P, \text{ if } P \in k \\ P', \ni \overrightarrow{QP'} = t\overrightarrow{QP}, Q \in k, \text{ if } P \notin k \end{cases}$$

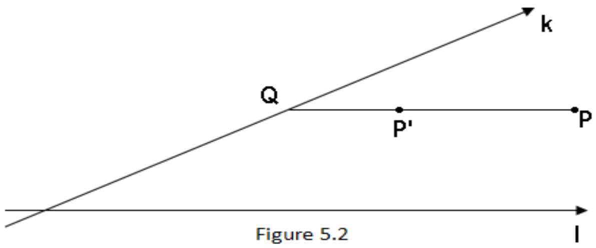


Figure 5.2

Here, the line k is called axis of compression, the line l is called direction of compression, the scalar t is called factor of the compression and the point Q is called the critical point.

Example: Find the image of the point $(1,4)$ by a compression on the line $k: 2x - y - 3 = 0$ in the direction of the line $l: 3x + y - 6 = 0$ with factor $t = \frac{2}{3}$

Solution: Let $Q = (a,b)$ be the critical point on line k such that the line $\overleftrightarrow{QP}$ is parallel to l. Since parallel lines have the same slope, we have $\frac{b-4}{a-1} = -3 \Rightarrow 3a + b - 7 = 0$. On the other hand Q is on line k means it satisfies the equation $k: 2x - y - 3 = 0$. Thus, $2a - b - 3 = 0$. Collecting these two equations yields $\begin{cases} 3a + b - 7 = 0 \\ 2a - b - 3 = 0 \end{cases} \Rightarrow 5a = 10 \Rightarrow a = 2, b = 1 \Rightarrow Q = (a,b) = (2,1)$

But from the definition of compression, we have

$$C(P) = P', \ni \overrightarrow{QP'} = t\overrightarrow{QP} \Rightarrow P' - Q = \tfrac{2}{3}(P - Q) = (x', y') - (2,1) = \tfrac{2}{3}[(1,4) - (2,1)]$$

$$\Rightarrow (x', y') = (2,1) + (-\tfrac{2}{3}, 2) = (\tfrac{4}{3}, 3)$$

Note that to solve compression problems, the first task is to identify the axis of compression and the direction line, then to find the critical point Q and finally to find the image P' using $\overrightarrow{QP'} = t\overrightarrow{QP}$ where t is factor of the compression.

5.3.3 Shears

Definitions:

a) Horizontal Shears:

Shears in the $x-$direction with factor k that sends each point (x,y) parallel to the $x-$axis by an amount of ky to the point $S_x(x,y)=(x+ky,y)$ are known as horizontal shears.

Under a horizontal shear points on the $x-$axis are invariant or unmoved because on the $x-$axis $y=0$ so that $x+ky=x+0=x$.

But when we move away from the $x-$axis, the magnitude of y increases, so that points farther from the $x-$axis moves a greater distance than which are closer to or on the $x-$axis.

b) Vertical Shears:

Shears in the $y-$direction with factor t that sends each point (x,y) parallel to the $y-$axis by an amount of tx to the point $S_y(x,y)=(x,y+ty)$ are known as vertical shears.

Under such types of shears points on the $y-$axis are invariant or fixed and points farther from the $y-$ axis moves a greater distance than closer points to the $y-$axis.

c) Total Shears (Simply called *shears*):

A total shear or simply a shear with factors k and t is a transformation that maps any point (x,y) to the point $S(x,y)=(x+ky,y+tx)$.

Examples: Find the equation of a shear that maps

$a)\ (2,3)$ to $(8,3)$ $\qquad b)\ (-3,1)$ to $(-3,0)$ $\qquad c)\ (2,-6)$ to $(8,2)$

Solution: To find the equations, first identify the type of the shear from the images.

a) Here, the x-coordinate is changed and the y-coordinate is fixed. Hence, the shear is horizontal.

So, $S_x(x,y)=(x+ky,y)\Rightarrow S_x(2,3)=(2+3k,3)$

$$\Rightarrow (2+3k,3)=(8,3)$$
$$\Rightarrow k=2\Rightarrow S_x(x,y)=(x+2y,y)$$

b) Here, the y-coordinate is changed and the x-coordinate is fixed.

Hence, the shear is vertical.

So,

$$S_y(x,y)=(x,y+tx)\Rightarrow S_y(-3,1)=(-3,1-3t)$$
$$\Rightarrow (-3,1-3t)=(-3,0)\Rightarrow t=\frac{1}{3}\Rightarrow S_y(x,y)=(x,y+\frac{1}{3}x)$$

c) Here, both coordinates are changed and thus it is a total shear.

So,

$$S(x,y)=(x+ky,y+tx)\Rightarrow S(2,-6)=(2-6k,2t-6)$$
$$\Rightarrow (2-6k,2t-6)=(8,2)\Rightarrow k=-1,t=4$$
$$\Rightarrow S(x,y)=(x-y,y+4x)$$

5.3.4 Sililarities

All **isometries** are affine transformations.Generally, all **similarities** (Homothety and or Dilations) are affine transformations. All isometries are similarities. But there are similarities that are not isometries. For instance, homotheties are similarities but not isometries. All similarities are affine transformations but there are affine transformations which are not similarities. For instance, shears, line reflections and compressions are affine transformations but not similarities.

Theorem 5.1(Characterization): Let $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ be affine transformation where $\begin{cases} x' = ax + by + h \\ y' = cx + dy + k \end{cases}$, $ad - bc \neq 0$. Then,

I. g is a similarity if and only if $a^2 + c^2 = b^2 + d^2,\ ab + cd = 0$.

II. g is an isometry if and only if $a^2 + c^2 = b^2 + d^2 = 1,\ ab + cd = 0.$

Proof: Let $A = \begin{pmatrix} x \\ y \end{pmatrix},\ B = \begin{pmatrix} z \\ w \end{pmatrix},\ C = \begin{pmatrix} u \\ v \end{pmatrix}$ be vertices of ΔABC .Then, $\overrightarrow{AB} = \begin{pmatrix} z - x \\ w - y \end{pmatrix}$

For the first part, g is a similarity if and only if $\overline{A'B'} = r\overline{AB}$ where,

$$\overline{AB} = \sqrt{(z-x)^2 + (w-y)^2},$$

$$\overline{A'B'} = \sqrt{(a^2+c^2)(z-x)^2 + (b^2+d^2)(w-y)^2 + (2ab+2cd)(z-x)(w-y)}$$

From these relations,

$$\overline{A'B'} = r\overline{AB}$$

$$\Leftrightarrow \sqrt{(a^2+c^2)(z-x)^2 + (b^2+d^2)(w-y)^2 + (2ab+2cd)(z-x)(w-y)}$$

$$= r\sqrt{(z-x)^2 + (w-y)^2}$$

$$\Leftrightarrow a^2 + c^2 = r^2,\ b^2 + d^2 = r^2,\ 2ab + 2cd = 0$$

$$\Leftrightarrow a^2 + c^2 = b^2 + d^2,\ ab + cd = 0$$

Thus, g is a similarity if and only if $a^2 + c^2 = b^2 + d^2,\ ab + cd = 0$.

Finally, g is an isometry if and only if $\overline{A'B'} = \overline{AB}$.

$$\overline{A'B'} = \overline{AB} \Leftrightarrow \sqrt{(a^2+c^2)(z-x)^2 + (b^2+d^2)(w-y)^2 + (2ab+2cd)(z-x)(w-y)}$$

$$= \sqrt{(z-x)^2 + (w-y)^2}$$

$$\Leftrightarrow a^2 + c^2 = 1,\ b^2 + d^2 = 1,\ 2ab + 2cd = 0$$

$$\Leftrightarrow a^2 + c^2 = b^2 + d^2 = 1,\ ab + cd = 0$$

Problem Set 5.1

1. If $\beta : R^2 \to R^2$ is an affine transformation given by $\beta\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x+4y-5 \\ px-2y+1 \end{pmatrix}$, find all values of p.

2. Show that the mapping given by $S(x, y) = (x', y')$ where $\begin{cases} x' = x + y \\ y' = y \end{cases}$ is a shear.

3. Determine whether the set of shears forms a group of transformation with composition or not.

4. If the image of $(1,3)$ under a compression C on a line $k : y = 2x + 3$ in the direction of the line $l : 2x + y + 2 = 0$ is the point $P' = (\frac{3}{4}, \frac{7}{2})$, find the image of the line $m : x - y + 1 = 0$.

5. Prove that any affine transformation maps two intersecting lines into intersecting lines. (Affine transformation preserves point of concurrency). Preserves ratio of the areas of triangles.

6. Show that the inverse of a horizontal (Vertical) shear is again a horizontal (vertical) shear with opposite factor.

7. Let u and w be perpendicular vectors in R^2. Define a mapping $T : R^2 \to R^2$ by $T(v) = v + (u.v)w$ for all v in R^2. Show that a) T is a linear transformation.
   b) T is a shear parallel to a line L spanned by w.
   c) If $u = (2,-1)$ and $w = (3,6)$, find $T(v)$ where $v = (3,4)$.

8. Let S be a shear parallel to a line L in R^2. Show that
   a) $S(v) = v$ for all v on L b) $S(v) - v // L$ for all v in R^2.

9. Find the image of the point (2,3) and the line $l : 4x - 2y + 6 = 0$
   a) By a reflection on the line $l_1 : 6x - 3y + 12 = 0$ in the direction of the line $l_2 : 3x + y - 2 = 0$
   b) By a compression on the line $l_1 : 6x - 3y + 12 = 0$ in the direction of the line $l_2 : 3x + y - 2 = 0$ with factor $\frac{1}{3}$. c) By a shear with factor 2.

10. Let g be a reflection on the line $l_1 : 15x - 2y + 10 = 0$ in the direction of the line $l_2 : 3x + 7y - 10 = 0$. Give at least three fixed points and three fixed line of g.

5.4 Affine Transformations and Linear Mappings

Definition: Linear transformations are any transformations from a plane onto a plane given by $\beta(x,y)=(x',y')$ where $\begin{cases} x' = ax+by \\ y' = cx+dy \end{cases}$, $a,b,c,d \in R,\ ad-bc \neq 0$

Notation: From now onwards we use the notation $\begin{pmatrix} x \\ y \end{pmatrix}$ instead of (x,y) for simplicity.

Using this notation, the above linear transformation becomes

$$\beta\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

Examples:

a) Reflection on the $x-$axis and $y-$axis are linear.

Let S_x be a reflection on $x-$axis. Then, $S_x\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$.

Hence, S_x is linear.

b) Rotations about the origin are linear.

Let ρ_θ be a rotation by an angle θ about the origin.

Then, for any point $\begin{pmatrix} x \\ y \end{pmatrix}$,

$$\rho_\theta\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}.$$

c) $\beta: R^2 \to R^2$ given by $\beta\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x-3y \\ 3x+2y \end{pmatrix}$ is linear.

d) $\beta: R^2 \to R^2$ given by $\beta\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 3x-1 \end{pmatrix}$ is not linear.

5.5 Matrix Representation of Affine Transformations

So far we have seen that any linear transformation f is given by the formula $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$, $ad-bc \neq 0$. This formula can be equivalently expressed as $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$. Now, denote the first matrix by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the second by $X = \begin{pmatrix} x \\ y \end{pmatrix}$.Thus any linear transformation f is given by $f(X) = MX$. This is known as *matrix Representation (formula)* of linear transformation and M is called the *standard matrix* of f.

From now onwards, we use the notation $f(X) = MX$, but to use this formula we have to see the method how to find the matrix M.

The formula $f(X) = MX = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$ is true for a unique M and for any point $X = \begin{pmatrix} x \\ y \end{pmatrix}$. In particular, this formula works for $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

So, $f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$, $\quad f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$.

This means the first column of M is obtained by calculating $f\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the second by calculating $f\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so that $M = \left(f\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad f\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Examples:

a) Let $f : R^2 \to R^2$ be given by $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x-y \\ 3x+7y \end{pmatrix}$. Find the standard matrix M of f and write its matrix representation.

Solution: Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the standard matrix of f. Then, from the above relation, $\begin{pmatrix} a \\ c \end{pmatrix} = f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\quad \begin{pmatrix} b \\ d \end{pmatrix} = f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \end{pmatrix} \Rightarrow M = \begin{pmatrix} 2 & -1 \\ 3 & 7 \end{pmatrix}$.

b) Let ρ_θ be a rotation by an angle θ about the origin. Find the matrix M of ρ_θ

Solution: For any point $X = \begin{pmatrix} x \\ y \end{pmatrix}$, a rotation ρ_θ by an angle θ about the origin is given by $\rho_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$ so that for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\begin{pmatrix} a \\ c \end{pmatrix} = f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \begin{pmatrix} b \\ d \end{pmatrix} = f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \Rightarrow M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

In conclusion, for any linear transformation f, there is a unique matrix M whose first and second columns are the images of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

Conversely, to each $2x2$ invertible matrix M there corresponds a unique linear transformation f whose formula is obtained by multiplying any point $\begin{pmatrix} x \\ y \end{pmatrix}$ by the matrix M as $f\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix}$.

For instance, for $M = \begin{pmatrix} 2 & -5 \\ 4 & -9 \end{pmatrix}$, the formula of the corresponding linear transformation f is $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - 5y \\ 4x - 9y \end{pmatrix}$.

Rules of Thumb!

i. To obtain the standard matrix of a linear transformation whose formula is given, simply take the coefficients of x and y. Then, form the matrix by putting the coefficients of x in the first column and the coefficients of y as second column. For instance, let $f: R^2 \to R^2$ be a linear transformation given by $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x - 2y \\ 3x + 4y \end{pmatrix}$. Here, the coefficients of x are $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and that of y are $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$.

So, putting the coefficients of x in the first column and the coefficients of y as second column yields a $2x2$ invertible matrix $M = \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$.

ii. To find the formula of a linear transformation, whose standard matrix M is given, simply multiply M by the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

For instance, let $f: R^2 \to R^2$ be a linear transformation whose standard matrix is $M = \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix}$. Then, the formula of f will be

$$f\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3y \\ 2x - y \end{pmatrix}.$$

At this point, we are equipped with enough concepts that are useful to find the matrix representation of any affine transformations.

Theorem 5.2 (Representation Theorem): Let g be any affine transformation. Then, g can be represented uniquely as a product of linear transformation f and a translation with translation vector $\vec{b}$ as $g(X) = f(X) + \vec{b}$.

Proof: Consider the general coordinate definitions of affine transformation that we discussed at the beginning of this Chapter,

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by + e \\ cx + dy + f \end{pmatrix}, \text{ where } ad - bc \neq 0.$$

Here, $g(X) = g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by + e \\ cx + dy + f \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$.

Now, observe that $g\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$. So, by taking

$f(X) = f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$, $\vec{b} = g\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$, we get that $g(X) = f(X) + \vec{b}$.

Besides, $f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which shows that f is linear transformation.

The uniqueness follows from the uniqueness of $\vec{b}$ which is $\vec{b} = g\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and from the uniqueness of the coefficients in the formula of the image of $f\begin{pmatrix} x \\ y \end{pmatrix}$. (Note that if either the image of $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or the coefficients are not unique, the map g will not be bijective. That is why we used these facts).

Example: Represent the transformation $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y-3 \\ 2x-5y+2 \end{pmatrix}$ as $g(X) = f(X) + \vec{b}$.

Solution: Using $\vec{b} = g\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-5y \end{pmatrix}$, we get $g(X) = f(X) + \vec{b}$.

Theorem 5.3 (Matrix Representation Theorem (MRT) of Affine Transformations): Let $g: R^2 \to R^2$ be any affine transformation. Then, there exist a unique matrix M and a unique vector $\vec{b}$ such that $g(X) = MX + \vec{b}$. This representation is said to be matrix representation of g.

Proof: From theorem 5.2, for any affine transformation g, there is a unique linear transformation f and a unique vector $\vec{b}$ such that $g(X) = f(X) + \vec{b}$. Again from the above discussions, to each linear transformation f, there is a unique matrix M such that $f(X) = MX$.

Consequently, we get that $g(X) = f(X) + \vec{b} = MX + \vec{b}$. Our next task will be determining the matrix associated with a given affine transformation g.

How to determine M from $g(X) = MX + \vec{b}$?

Here we need to investigate how to find the standard matrix M associated with a given affine transformation.

From $g\begin{pmatrix} x \\ y \end{pmatrix} = f\begin{pmatrix} x \\ y \end{pmatrix} + \vec{b}$, since any linear transformation preserves the origin

$g\begin{pmatrix} 0 \\ 0 \end{pmatrix} = f\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \vec{b} = \vec{b}$ and $f\begin{pmatrix} x \\ y \end{pmatrix} = g\begin{pmatrix} x \\ y \end{pmatrix} - \vec{b}$.

But from the previous observations we have seen that the standard matrix M of any linear transformation is completely determined from the images of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as $\begin{pmatrix} a \\ c \end{pmatrix} = f\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} b \\ d \end{pmatrix} = f\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so that $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Thus, from $f\begin{pmatrix} x \\ y \end{pmatrix} = g\begin{pmatrix} x \\ y \end{pmatrix} - \vec{b}$,

$\begin{pmatrix} a \\ c \end{pmatrix} = f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = g\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \vec{b}$, $\begin{pmatrix} b \\ d \end{pmatrix} = f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = g\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \vec{b}$ so that $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Example: Let $g: R^2 \to R^2$ be affine given by $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y + 3 \\ 2x + 3y - 7 \end{pmatrix}$. Find the standard matrix of g and give its matrix representation.

Solution: Let the matrix representation be $g(X) = MX + \vec{b} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \vec{b}$.

Now, find $g\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ from the formula of g ,that is $\vec{b} = g\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$.

Besides, from the above relations,

$\begin{pmatrix} a \\ c \end{pmatrix} = g\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \vec{b} = \begin{pmatrix} 4 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} b \\ d \end{pmatrix} = f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = g\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \vec{b} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ -7 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

Consequently, the standard matrix is found to be $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$.

Therefore, the matrix representation is given by

$$g\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix} + \vec{b} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ -7 \end{pmatrix}.$$

How can we determine affine transformations from their images?

In our discussion of isometries, we have seen that any isometry is uniquely determined from its effect on three non-collinear points. Likewise, any affine transformation is uniquely determined from the images of three non-collinear points. That means there is a unique affine transformation that sends three non-collinear points into three non-collinear points. In other words, for any three non- collinear points A, B, C, and any two affine transformations f and g if $g(A) = f(A),\ g(B) = f(B),\ g(C) = f(C)$, then $g = f$.

Therefore, to determine the formula of any affine transformation g, it suffices to know the images of any three non- collinear points A, B, C.

This basic and important concept is illustrated by the following example.

Example: Let $g: R^2 \to R^2$ be affine transformation with

$g\begin{pmatrix}2\\0\end{pmatrix}=\begin{pmatrix}6\\0\end{pmatrix}$, $g\begin{pmatrix}0\\5\end{pmatrix}=\begin{pmatrix}-10\\-7\end{pmatrix}$ and $g\begin{pmatrix}4\\10\end{pmatrix}=\begin{pmatrix}-23\\10\end{pmatrix}$. Then, determine the formula of g and calculate $g\begin{pmatrix}0\\0\end{pmatrix}$.

Solution: Since the points $A=\begin{pmatrix}2\\0\end{pmatrix}$, $B=\begin{pmatrix}0\\5\end{pmatrix}$, $C=\begin{pmatrix}4\\10\end{pmatrix}$ are non-collinear, there is a unique affine transformation g given by $g(X)=MX+\vec{b}$, where $M=\begin{pmatrix}a & b\\c & d\end{pmatrix}$ and $\vec{b}=\begin{pmatrix}e\\f\end{pmatrix}$ so that $g(A)=MA+\vec{b},\ g(B)=MB+\vec{b},\ g(C)=MC+\vec{b}$. Thus,

$$i.\ M(B-A)=g(B)-g(A)\Rightarrow\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}-2\\5\end{pmatrix}=\begin{pmatrix}-2a+5b\\-2c+5d\end{pmatrix}=\begin{pmatrix}-16\\-7\end{pmatrix}$$

$$ii.\ M(C-B)=g(C)-g(B)\Rightarrow\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix}=\begin{pmatrix}4a+5b\\4c+5d\end{pmatrix}=\begin{pmatrix}-13\\17\end{pmatrix}$$

Equating components and collecting like terms from (i) and (ii), we get

$$\begin{cases}-2a+5b=-16\\4a+5b=-13\end{cases}\Rightarrow 6a=3\Rightarrow a=\tfrac{1}{2},\ b=-3$$

$$\begin{cases}-2c+5d=-7\\4c+5d=17\end{cases}\Rightarrow 6c=24\Rightarrow c=4,\ d=\tfrac{1}{5}$$

Hence, $M=\begin{pmatrix}a & b\\c & d\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & -3\\4 & \frac{1}{5}\end{pmatrix}$.

Now using one of the given points A, B or C, we obtain the vector $\vec{b}$ as follow:

$$g(A)=MA+\vec{b}\Rightarrow\vec{b}=g(A)-MA=g\begin{pmatrix}2\\0\end{pmatrix}-\begin{pmatrix}\frac{1}{2} & -3\\4 & \frac{1}{5}\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}=\begin{pmatrix}6\\0\end{pmatrix}-\begin{pmatrix}1\\8\end{pmatrix}=\begin{pmatrix}5\\-8\end{pmatrix}$$

Therefore, for any $X=\begin{pmatrix}x\\y\end{pmatrix}$, $g(X)=MX+\vec{b}$ gives

$$g\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & -3\\4 & \frac{1}{5}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}5\\-8\end{pmatrix}=\begin{pmatrix}\frac{1}{2}x-3y+5\\4x+\frac{1}{5}y-8\end{pmatrix},\text{and }\ g\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}5\\-8\end{pmatrix}.$$

5.6 Orientation and Affine Transformations (Revisited)

Definitions: Let g be any affine transformation. Then, ee say that g preserves orientation if and only if for any positively oriented vectors X and Y, their images $X' = g(X)$, $Y' = g(Y)$ are again positively oriented vectors. In this case, g is said to be orientation preserving affine transformation. In general, if the pair (X, Y) and the pair $(g(X), g(Y))$ have the same orientation, then g preserves orientation. But, if they have opposite orientation, then g reverses (changes) orientation. In this case, g is said to be orientation reversing (changing) affine transformation.

Theorem 5.4 (Orientation Characterization Theorem):

Let $g : W \to W$ be any affine transformation given by $g(X) = MX + \vec{b}$. Then,

a) g preserves orientation if and only if $\det M > 0$.

b) g reverses (changes) orientation if and only if $\det M < 0$.

Proof: Whether a given transformation g preserves or reverses orientation is determined from its effect on the orientation of a triangle. This means if g preserves the orientation of any triangle ABC, then it is orientateon preserving and if g reverses the orientation of ΔABC, then it is orientation reversing affine transformation.

Now, having this fact as basis, let's prove our theorem. Let ΔABC be arbitrary triangle. Then its orientation is determined from the orientation of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Suppose ΔABC has positive orientation. That means $\det(\overrightarrow{AB}, \overrightarrow{AC}) > 0$

On the other hand, let $\Delta A'B'C'$ be the image of ΔABC under g. Then, the orientation of $\Delta A'B'C'$ is determined from the orientation of the pair $(\overrightarrow{A'B'}, \overrightarrow{A'C'})$.

But from $g(X) = MX + \vec{b}$,

$$\begin{aligned} &A' = g(A) = MA + \vec{b},\ B' = g(B) = MB + \vec{b},\ C' = g(C) = MC + \vec{b} \\ &\Rightarrow \overrightarrow{A'B'} = M(B - A),\ \overrightarrow{A'C'} = M(C - A) \end{aligned}.$$

Thus,

$$\det(\overrightarrow{A'B'},\overrightarrow{A'C'}) = \det[M(B-A), M(C-A)] = \det[M(B-A, C-A)]$$
$$= \det[M(\overrightarrow{AB},\overrightarrow{AC})] = \det M . \det(\overrightarrow{AB},\overrightarrow{AC})$$

Hence, g preserves orientation if and only if

$\det(\overrightarrow{A'B'},\overrightarrow{A'C'}) > 0 \Leftrightarrow \det M . \det(\overrightarrow{AB},\overrightarrow{AC}) > 0 \Leftrightarrow \det M > 0$ because $\det(\overrightarrow{AB},\overrightarrow{AC}) > 0$ from our assumption. Similarly, g reverses orientation if and only if

$\det(\overrightarrow{A'B'},\overrightarrow{A'C'}) < 0 \Leftrightarrow \det M . \det(\overrightarrow{AB},\overrightarrow{AC}) < 0 \Leftrightarrow \det M < 0$.

Note that since any affine transformation g is bijective, it is invertible and so is its standard matrix M and hence $\det M$ can not be zero.

So, any affine transformation g is either orientation preserving or orientation reversing but cannot be both.

Examples: Determine whether the following affine transformations are orientation preserving or orientation reversing.

$a)$ $g : R^2 \to R^2$ given by $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x-2y+7 \\ x+5y-11 \end{pmatrix}$

$b)$ $\alpha : R^2 \to R^2$ given by $\alpha\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+1 \\ y-11 \end{pmatrix}$

Solution:

a) Here, $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x-2y+7 \\ x+5y-11 \end{pmatrix} = \begin{pmatrix} 3x-2y \\ x+5y \end{pmatrix} + \begin{pmatrix} 7 \\ -11 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 7 \\ -11 \end{pmatrix}$

$$\Rightarrow M = \begin{pmatrix} 3 & -2 \\ 1 & 5 \end{pmatrix} \Rightarrow \det M = 17 > 0$$

Hence, by the above theorem g preserves orientation.

b) Similarly, $\alpha\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+1 \\ 2-y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\Rightarrow M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \det M = -1 < 0$$

Hence, by the above theorem α reverses orientation.

Theorem 5.5: The inverse of orientation-preserving affine transformation is again orientation-preserving. The inverse of orientation-reversing affine transformation is again orientation-reversing.

Proof: Let $g(X) = MX + \vec{b}$ be orientation-preserving affine transformation.

Then, we need to show that g^{-1} is also orientation-preserving. Since g is orientation-preserving, $\det M > 0$. Besides, from $g(X) = MX + \vec{b}$, by Inverse Characterization Theorem, $g^{-1}(X) = M^{-1}X - M^{-1}\vec{b}$ which gives us the standard matrix of g^{-1} is M^{-1}.

Thus, $\det(M^{-1}) = \dfrac{1}{\det M} > 0,\ \det M > 0$.

Hence, $g^{-1}(X) = M^{-1}X - M^{-1}\vec{b}$ is also orientation-preserving whenever $g(X) = MX + \vec{b}$ is orientation -preserving. Similarly, if $g(X) = MX + \vec{b}$ is orientation –reversing, then $\det M < 0$. So, $\det(M^{-1}) = \dfrac{1}{\det M} < 0,\ \det M < 0$.

Theorem 5.6: The composition of two orientation-preserving affine transformations is orientation-preserving. The composition of two orientation-reversing affine transformations is orientation-preserving. The composition of an orientation-preserving and an orientation-reversing affine transformation is orientation-reversing.

Proof: Let $g(X) = MX + \vec{b}$ and $h(X) = NX + \vec{c}$ be any affine transformations. In section 6.6, we have seen that the matrix of the composition $g \circ h$ is the product of the matrices of g and h. Thus, the standard matrix of $g \circ h$ is MN.

Now, suppose g and h are orientation-preserving.

That means, $\det M > 0,\ \det N > 0$. So, $\det(MN) = \det M . \det N > 0$. On the other hand, if g and h are orientation-reversing, $\det M < 0,\ \det N < 0$ gives $\det(MN) = \det M . \det N > 0$.

Thus, in either cases, $g \circ h$ is orientation-preserving.

Finally, if g is orientation preserving and h is orientation reversing, we will have $\det M > 0,\ \det N < 0 \Rightarrow \det MN = \det M . \det N < 0$.

Hence, if g and h have opposite effect on orientation, then their composition $g \circ h$ will be orientation reversing.

5.7 Area and Affine Transformations

Let π be a parallelogram whose adjacent sides are given by the vectors $X=\begin{pmatrix} x \\ y \end{pmatrix}$ and $Y=\begin{pmatrix} z \\ w \end{pmatrix}$ as shown in the figure 5.3 below. For simplicity of calculation, assume one of the vertices to be the origin and $X=\begin{pmatrix} x \\ y \end{pmatrix}$, $Y=\begin{pmatrix} z \\ w \end{pmatrix}$ to be positively oriented vectors. That means $\det(X,Y)=\begin{vmatrix} x & z \\ y & w \end{vmatrix}>0$.

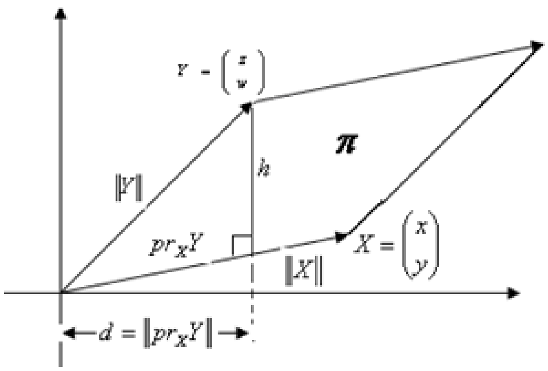


Figure 5.3

Now let's calculate the area of the parallelogram π denoted by $a(\pi)$.

That is $a(\pi)=b.h$ where b is the base of the parallelogram π given by $b=\|X\|$ and h is the altitude from the vertex Y to the base X. Here, how to find h will need a little algebraic calculation as follow. From Pythagoras Theorem,

$$h^2+d^2=\|Y\|^2 \Rightarrow h^2=\|Y\|^2-d^2=\|Y\|^2-\|\mathrm{Pr}_X\, Y\|^2=\|Y\|^2-\left\|\left(\frac{X.Y}{\|X\|^2}\right)X\right\|^2=\|Y\|^2-\frac{(X.Y)^2}{\|X\|^2}$$

$$=\frac{\|X\|\|Y\|-(X.Y)^2}{\|X\|^2}=\frac{(x^2+y^2)(z^2+w^2)-(xz+yw)^2}{x^2+y^2},\quad X.Y=xz+yw$$

$$=\frac{(xz)^2+(xw)^2+(yz)^2+(yw)^2-[(xz)^2+2xzyw+(yw)^2]}{x^2+y^2}$$

$$=\frac{(xw)^2-2(xw)(yz)+(yz)^2}{x^2+y^2}=\frac{(xw-yz)^2}{x^2+y^2}$$

$$\therefore h=\frac{|xw-yz|}{\sqrt{x^2+y^2}}$$

Hence, the area of the parallelogram π is computed by

$$a(\pi) = b.h = (\sqrt{x^2+y^2})(\frac{|xw-yz|}{\sqrt{x^2+y^2}}) = |xw-yz| = \left\| \begin{matrix} x & z \\ y & w \end{matrix} \right\|$$

Notice: The first bar is to indicate absolute value and the second is to indicate determinant in the above area formula. From this area of a parallelogram, if we are interested on the area of the triangle with adjacent sides the vectors $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $Y = \begin{pmatrix} z \\ w \end{pmatrix}$, we simply multiply by half. $a(\Delta OXY) = \frac{1}{2}\left\| \begin{matrix} x & z \\ y & w \end{matrix} \right\|$.

In general, for arbitrary parallelogram π with vertices

$A = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} z \\ w \end{pmatrix}, C = \begin{pmatrix} u \\ v \end{pmatrix}, D = \begin{pmatrix} r \\ t \end{pmatrix}$, the area of π is given by

$a(\pi) = \left\| \begin{matrix} z-x & u-x \\ w-y & v-y \end{matrix} \right\|$. For arbitrary triangle ABC with vertices

$A = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} z \\ w \end{pmatrix}, C = \begin{pmatrix} u \\ v \end{pmatrix}$, we have $a(\Delta ABC) = \frac{1}{2}\left\| \begin{matrix} z-x & u-x \\ w-y & v-y \end{matrix} \right\|$

(The derivations of these formulas are similar as we did above). The plus sign of the determinant is obtained when the triangle has positive orientation where as the minus sign is taken when the triangle has negative orientation. Notation from now onwards, we use the following notations.

For vertices $A = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} z \\ w \end{pmatrix}, C = \begin{pmatrix} u \\ v \end{pmatrix}$, the pair $(\overrightarrow{AB}, \overrightarrow{AC})$ is a matrix formed by putting the coordinates of $\overrightarrow{AB}$ as first column and the coordinates of $\overrightarrow{AC}$ as the second column as follow: $(\overrightarrow{AB}, \overrightarrow{AC}) = \begin{pmatrix} z-x & u-x \\ w-y & v-y \end{pmatrix}$.

Let $\det(\overrightarrow{AB}, \overrightarrow{AC})$ denotes determinant of $(\overrightarrow{AB}, \overrightarrow{AC})$ so that we can write

$\det(\overrightarrow{AB}, \overrightarrow{AC}) = \left| \begin{pmatrix} z-x & u-x \\ w-y & v-y \end{pmatrix} \right|$. Hence,

$$a(\Delta ABC) = \frac{1}{2}\left|\det(\overrightarrow{AB}, \overrightarrow{AC})\right| = \frac{1}{2}\left\| \begin{matrix} z-x & u-x \\ w-y & v-y \end{matrix} \right\|.$$

Now let g be any affine transformation. We need to investigate the effect of g on the area of ΔABC. Let $\Delta A'B'C'$ be the image of ΔABC under g where $A' = g(A),\ B' = g(B),\ C' = g(C)$.

Since g is affine transformation, there is a unique matrix M and a vector $\vec{b}$ such that $g(X) = MX + \vec{b}$ for all points X in the plane. Thus,

$$A' = g(A) = MA + \vec{b},\ B' = g(B) = MB + \vec{b},\ C' = g(C) = MC + \vec{b}$$
$$\Rightarrow \overrightarrow{A'B'} = MB + \vec{b} - (MA + \vec{b}) = MB - MA,$$
$$\overrightarrow{A'C'} = MC + \vec{b} - (MA + \vec{b}) = MC - MA$$

Hence, from the above area formula,

$$a(\Delta A'B'C') = \tfrac{1}{2}\left|\det(\overrightarrow{A'B'}, \overrightarrow{A'C'})\right| = \tfrac{1}{2}\left|\det(MB - MA, MC - MA)\right|$$
$$= \tfrac{1}{2}\left|\det[M(B - A, C - A)]\right| = \tfrac{1}{2}\left|\det M \det(B - A, C - A)\right|, \qquad \det(XY) = \det X \det Y$$
$$= \tfrac{1}{2}\left|\det(B - A, C - A)\right|\left|\det M\right|, \qquad |xy| = |x||y|$$
$$= \tfrac{1}{2}\left|\det(\overrightarrow{AB}, \overrightarrow{AC})\right|\left|\det M\right| = a(\Delta ABC)\left|\det M\right|, \qquad \tfrac{1}{2}\left|\det(\overrightarrow{AB}, \overrightarrow{AC})\right| = a(\Delta ABC)$$

This relation in general is given by the following theorem.

Theorem 5.7 (Area Relation Theorem, ART):

Let g be any affine transformation given by $g(X) = MX + \vec{b}$. Then, $a(\Delta A'B'C') = a(\Delta ABC)\left|\det M\right|$ where $\Delta A'B'C'$ is the image of ΔABC.

Examples:

1. Given ΔABC with vertices $A = (4,1),\ B = (-2,0),\ C = (2,-2)$. Let g be affine transformation given by $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 4y - 3 \\ 3x + 7y + 2 \end{pmatrix}$. If $\Delta A'B'C'$ is the image of ΔABC under g, compute $a(\Delta A'B'C')$.

Solution: First find the area of ΔABC so that to apply the area relation theorem. In ΔABC, $\overrightarrow{AB} = (-6,-1),\ \overrightarrow{AC} = (-2,-3).$ Thus,

$$a(\Delta ABC) = \tfrac{1}{2}\left|\det(\overrightarrow{AB}, \overrightarrow{AC})\right| = \tfrac{1}{2}\left|\det\begin{pmatrix} -6 & -2 \\ -1 & -3 \end{pmatrix}\right| = \tfrac{1}{2}|18 - 2| = 8\, sq.units$$

Now from the formula of g, its standard matrix is given by $M = \begin{pmatrix} 1 & 4 \\ 3 & 7 \end{pmatrix}$ such that $\det M = -5$. Therefore, from area relation theorem,

$$a(\Delta A'B'C') = a(\Delta ABC)\left|\det M\right| = 8\left|-5\right| = 40\,sq.units.$$

2. Suppose a certain affine transformation g takes ΔABC into $\Delta A'B'C'$ such that $a(\Delta A'B'C') = \frac{1}{3}a(\Delta ABC)$ .If g takes ΔDEF into $\Delta D'E'F'$ where $a(\Delta D'E'F') = 7cm^2$, calculate $a(\Delta DEF)$.

Solution: For an affine transformation g, if $\Delta A'B'C'$ is the image of ΔABC under g, then $a(\Delta A'B'C') = a(\Delta ABC)\left|\det M\right|$. But we are given that,

$$a(\Delta A'B'C') = \tfrac{1}{3}a(\Delta ABC) \Rightarrow a(\Delta ABC)\left|\det M\right| = \tfrac{1}{3}a(\Delta ABC) \Rightarrow \left|\det M\right| = \tfrac{1}{3} \Rightarrow \det M = \pm\tfrac{1}{3}$$

Thus,

$$a(\Delta D'E'F') = a(\Delta DEF)\left|\det M\right| \Rightarrow 7cm^2 = a(\Delta DEF)\left|\pm\tfrac{1}{3}\right| \Rightarrow a(\Delta DEF) = 21cm^2$$

Definition: *Equi-affine* transformation is an affine transformation that preserves area.

An affine transformation preserves area of ΔABC if $a(\Delta A'B'C') = a(\Delta ABC)$.

Corollary:

An affine transformation $g(X) = MX + \vec{b}$ is equi affine if and only if $\det M = \pm 1$

Proof: Suppose g is equi affine transformation. Thus, for any ΔABC, $a(\Delta A'B'C') = a(\Delta ABC)$. But from area relation theorem,

$$a(\Delta A'B'C') = a(\Delta ABC)\left|\det M\right| \Rightarrow a(\Delta ABC)\left|\det M\right| = a(\Delta ABC) \Rightarrow \left|\det M\right| = 1 \Rightarrow \det M = \pm 1.$$

Conversely, suppose $\det M = \pm 1$. Then,

$$a(\Delta A'B'C') = a(\Delta ABC)\left|\det M\right| = a(\Delta ABC)\left|\pm 1\right| = a(\Delta ABC).$$

Hence, g preserves area.

Theorem 5.8: Let $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ be any affine transformation where

$\begin{cases} x' = ax + by + h \\ y' = cx + dy + k \end{cases}$, $ad - bc \neq 0$. Then, g is equi affine if and only if $|ad - bc| = 1$.

Proof: Let $A = \begin{pmatrix} x \\ y \end{pmatrix}$, $B = \begin{pmatrix} z \\ w \end{pmatrix}$, $C = \begin{pmatrix} u \\ v \end{pmatrix}$ be vertices of ΔABC.

Then, $\overrightarrow{AB} = \begin{pmatrix} z - x \\ w - y \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} u - x \\ v - x \end{pmatrix}$ so that

$$a(\Delta ABC) = \tfrac{1}{2}\left|\det(\overrightarrow{AB}, \overrightarrow{AC})\right| = \tfrac{1}{2}\left|\det\begin{pmatrix} z - x & u - x \\ w - y & v - y \end{pmatrix}\right|.$$

On the other hand, let $\Delta A'B'C'$ be the image of ΔABC under g, then

$$A' = g(A) = \begin{pmatrix} ax + by + h \\ cx + dy + k \end{pmatrix},\ B' = g(B) = \begin{pmatrix} az + bw + h \\ cz + dw + k \end{pmatrix},\ C' = g(C) = \begin{pmatrix} au + bv + h \\ cu + dv + k \end{pmatrix}$$

$$\Rightarrow \overrightarrow{A'B'} = \begin{pmatrix} a(z - x) + b(w - y) \\ c(z - x) + d(w - y) \end{pmatrix},\ \overrightarrow{A'C'} = \begin{pmatrix} a(u - x) + b(v - y) \\ c(u - x) + d(v - y) \end{pmatrix}$$

Hence, using the general area formula for the image $\Delta A'B'C'$, we have

$$\begin{aligned}
a(\Delta A'B'C') &= \tfrac{1}{2}\left|\det(\overrightarrow{A'B'}, \overrightarrow{A'C'})\right| \\
&= \tfrac{1}{2}\left|\det\begin{pmatrix} a(z - x) + b(w - y) & a(u - x) + b(v - y) \\ c(z - x) + d(w - y) & c(u - x) + d(v - y) \end{pmatrix}\right| \\
&= \tfrac{1}{2}\left|\det\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z - x & u - x \\ w - y & v - y \end{pmatrix}\right]\right| \\
&= \tfrac{1}{2}\left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\det\begin{pmatrix} z - x & u - x \\ w - y & v - y \end{pmatrix}\right| \\
&= \tfrac{1}{2}\left|\det\begin{pmatrix} z - x & u - x \\ w - y & v - y \end{pmatrix}\right|\left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| \\
&= a(\Delta ABC)|ad - bc| \Rightarrow a(\Delta A'B'C') = a(\Delta ABC)|ad - bc)|
\end{aligned}$$

Hence, g is equi affine if and only if

$$a(\Delta A'B'C') = a(\Delta ABC) \Leftrightarrow a(\Delta ABC)|ad - bc| = a(\Delta ABC) \Leftrightarrow |ad - bc| = 1$$

5.8 Inverse of Affine Transformations

Let $g(X) = MX + \vec{b}$ and $h(X) = NX + \vec{c}$ be any two affine transformations. Then, their composition is given by

$$i.\quad g \circ h(X) = g(h(X)) = g(NX + \vec{c}) = MNX + M\vec{c} + \vec{b}$$

$$\Rightarrow g \circ h(X) = CX + \vec{d},\quad C = MN,\ \vec{d} = M\vec{c} + \vec{b}$$

$$ii.\quad h \circ g(X) = h(g(X)) = h(MX + \vec{b}) = NMX + N\vec{b} + \vec{c}$$

$$\Rightarrow h \circ g(X) = DX + \vec{e},\ D = NM,\ \vec{e} = N\vec{b} + \vec{c}$$

In either case, the matrix of the composition is the product of the matrices of the individual affine transformations.

Definition: Any two affine transformations g and f are said to be inverse of each other if and only if $f \circ g(X) = g \circ f(X) = i(X) = X$ for all X in the domain. For instance in R, $f(x) = x+1$ and $g(x) = x-1$ are inverses of each other because $f \circ g(x) = g \circ f(x) = i(x) = x, \forall x \in R.$ Now our aim is to find the inverse of an affine transformation whose formula is given. Consider any two affine transformations $g(X) = MX + \vec{b}$ and $f(X) = M^{-1}X - M^{-1}\vec{b}$. Then,

$$i.\ f \circ g(X) = f(MX + \vec{b}) = M^{-1}(MX + \vec{b}) - M^{-1}\vec{b} = X + M^{-1}\vec{b} - M^{-}\vec{b} = X$$

$$ii.\ g \circ f(X) = g(M^{-1}X - M^{-1}\vec{b}) = M(M^{-1}X - M^{-1}\vec{b}) + \vec{b} = X - \vec{b} + \vec{b} = X$$

Thus, from these two cases, $f \circ g(X) = g \circ f(X) = i(X) = X$ for all X in the given space. Hence, the given affine transformations g and f are inverse of each other. This observation is generalized by the following theorem.

Theorem 5.9 (Inverse Characterization Theorem, ICT):

Let $g(X) = MX + \vec{b}$ be affine transformation. Then, the inverse of g is given by $g^{-1}(X) = M^{-1}X - M^{-1}\vec{b}.$

Notice: To find the inverse of any given affine transformation $g : R^2 \to R^2$ given by $g(X) = MX + \vec{b}$, it is advisable to remember the following steps:

Step-1: Find carefully the standard matrix M and the vector $\vec{b}$ of g using:

i) $\vec{b} = g\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

ii) $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $\begin{pmatrix} a \\ c \end{pmatrix} = g\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \vec{b},\qquad \begin{pmatrix} b \\ d \end{pmatrix} = g\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \vec{b}$

Step-2: Calculate M^{-1} using $M^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Step-3: Compute g^{-1} using inverse characterization theorem.

$$g^{-1}(X) = M^{-1}X - M^{-1}\vec{b}$$

Example: Find the inverse of an affine transformation g given by

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9x-4y+2 \\ 2x-y-5 \end{pmatrix}.$$

Solution: To find the inverse of g, the first task is to identify its standard matrix M so that to apply theorem 5.9.

Let $g(X) = MX + \vec{b}$ where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Hence, $\vec{b} = g\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$ and

$$\begin{pmatrix} a \\ c \end{pmatrix} = g\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \vec{b} = \begin{pmatrix} 11 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ -5 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} b \\ d \end{pmatrix} = g\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \vec{b} = \begin{pmatrix} -2 \\ -6 \end{pmatrix} - \begin{pmatrix} 2 \\ -5 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$$

Thus, $M = \begin{pmatrix} 9 & -4 \\ 2 & -1 \end{pmatrix}$. Now calculate M^{-1}.

For any $2x2$ invertible matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its inverse is given by

$$M^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

So, for $M = \begin{pmatrix} 9 & -4 \\ 2 & -1 \end{pmatrix}$, $M^{-1} = \frac{1}{-1}\begin{pmatrix} -1 & 4 \\ -2 & 9 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 2 & -9 \end{pmatrix}$.

Therefore,

$$g^{-1}(X) = M^{-1}X - M^{-1}\vec{b} \Rightarrow g^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 2 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 & -4 \\ 2 & -9 \end{pmatrix}\begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

$$= \begin{pmatrix} x-4y \\ 2x-9y \end{pmatrix} - \begin{pmatrix} 22 \\ 49 \end{pmatrix}$$

$$= \begin{pmatrix} x-4y-22 \\ 2x-9y-49 \end{pmatrix}$$

Review Problems on Chapter-5

1. Suppose T is an affine transformation given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-2y \\ x+2y \end{pmatrix}$.Then, find

   a) the image of the y-axis under T

   b) the image of the line $x+y=1$ under T

   c) the image of the circle $x^2+y^2=1$ under T

Answer : $a)\ y=-x \qquad b)\ x+3y-4=0 \qquad c)\ 4(x+y)^2+(y-x)^2=16$

2. Find the image of the circle $x^2+y^2=16$ under affine transformation

$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$. **Answer** : $x^2+y^2=64$

3. Find the image of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ under affine transformation

$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{6} \\ \frac{y}{9} \end{pmatrix}$

Answer : $x^2+y^2=1/9$

4. Suppose β is a linear transformation with standard matrix $M=\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$.

Find the image of the line $l: y=2x+1$ under β. **Answer** : $4x-5y+1=0$

5*. Suppose g is an affine transformation with standard matrix $M=\begin{pmatrix} 2 & 2 \\ -1 & 0 \end{pmatrix}$.

If $g\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, find the image of $l: y=3x$ under g. **Answer** : $x+8y-20=0$

6. Let S_l and S_m be a reflections on lines a long the vectors $u=(1,1)$ and $v=(1,2)$. Then, find the equations of $S_l \circ S_m$ and give its standard matrix.

Answer : $S_l \circ S_m(x,y) = (\frac{4}{5}x+\frac{3}{5}y, -\frac{3}{5}x+\frac{4}{5}y)$, $M=\begin{pmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{pmatrix}$

7. Let $T : R^2 \to R^2$ be defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x-2y \\ 4x-3y \end{pmatrix}$.

a) Find the matrix representation of T.

b) Determine whether T preserves or changes orientation.

c) Find the formula for T^{-1}.

d) If $A'B'C'D'$ is the image of the parallelogram $ABCD$ under T where a $A(1,1),\ B(4,1),\ C(4,3),\ D(1,3)$, calculate the area of $A'B'C'D'$.

8. Let g be a reflection on the line $l_1 : 2x - y = 0$ in the direction of the line $l_2 : 12x + 3y - 2 = 0$. If $l : tx - 3y + 5 = 0$ is the fixed line of g, then find t

9. Let S be a shear which takes the point $(2,6)$ to $(5,6)$. Then find $S^{-1}(3,-2)$

10. Let $T : R^2 \to R^2$ be a *one to one* linear transformation. Then,

a) If $T\begin{pmatrix} 2x-6 \\ 3y+9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then find the point $\begin{pmatrix} x \\ y \end{pmatrix}$

b) If $T\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$, then find the image $T\begin{pmatrix} 16 \\ -24 \end{pmatrix}$

11. Let $g : R^2 \to R^2$ be affine transformation. If g is a reflection on the line $x - y - 1 = 0$ in the direction of the line $y = 2x$. Then,

a) Find the formula for the linear mapping $f(x)$ associated with $g(x)$

b) Find the matrix representation of $g(x)$

c) Give at least three fixed points of g

d) Is the line $l : 6x - 3y + 12 = 0$ a fixed line of g?

12. If the image of the $(1,3)$ under a compression C on a line $k : y = 2x + 3$ in the direction of the line $l : 2x + y + 2 = 0$ is the point $P' = (\frac{3}{4}, \frac{7}{2})$, find the image of the line $m : x - y + 1 = 0$ under this compression.

13. Let $\psi : R^2 \to R^2$ be linear transformation such that $\psi\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\psi\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find the formula for ψ and the image of $y = 5x - 2$ under ψ.

14. Let α be a shear such that $\alpha(-1,4)=(7,11)$. Then, find the equations of α .

Answer $:\alpha(x,y)=(x+2y,y-7x)$

15. Suppose f is an orientation-reversing similarity. Prove that $f\circ f$ is either dilation or a translation.

16. Suppose $g\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}ax-by\\bx+ay\end{pmatrix}$ is an equi-affine transformation. Then, prove that $a^2+b^2=1$.

17. Given the affine mapping $g(x)=\begin{bmatrix}3 & 0\\-3 & -1\end{bmatrix}x+\begin{bmatrix}6\\-2\end{bmatrix}$.

a) Is g orientation *preserving* or orientation *changing*?

b) Is g *isometric* mapping?

18. Given the affine mapping $g\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}5x\\6y\end{bmatrix}+\begin{bmatrix}3\\4\end{bmatrix}$ and triangle ABC with vertices A(0,1), B(2,1), C(2,5). Find the area of the image triangle $A'B'C'$ under g .

19. Let $g(x)=MX+b$ be affine tansformation. If $g(x)$ preserves area , then prove that $\det(M)=\pm1$.

20. Prove that any homothety preserves orientation.

21. Determine the effects of shears on area and orientation.

22. Let S_l be a reflection where l is a line through the origin that makes an angle of θ with the positive $x-$axis.

a) Show that the standard matrix of S_l is given by $M=\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}$.

b) Using part (a), conclude that S_l preserves area but changes orientation.

23. Every affine transformation given by $f\begin{pmatrix}x\\y\end{pmatrix}=M\begin{pmatrix}x\\y\end{pmatrix}+\vec{b}$ is a similarity if and only if there exists $k>0$ such that $\left|M\begin{pmatrix}x\\y\end{pmatrix}\right|=k\left|\begin{pmatrix}x\\y\end{pmatrix}\right|$ for all vectors $\begin{pmatrix}x\\y\end{pmatrix}$ in R^2.

24. In the cell of the following table, put " ✓ " if the geometric concept will be **preserved** or "✗" if it will be changed under the given type of transformations in general.

Geometric concepts	Types of Transformations		
	Similarity	Affine	Isometry
1. Size			
2. Orientation			
3. Mid-point			
4. Angles			

25. Produce an example of similarity but not an isometry

a) That preserves area but not orientation.

b) That preserves orientation but not area.

c) That preserves both area and orientation.

26. Let α_k be a mapping for $k \neq 0$ given by $\alpha_k(x, y) = (x', y')$ where $\begin{cases} x' = kx \\ y' = y \end{cases}$

a) Show that α_k is affine transformation. Can α_k be a similarity?

b) Determine whether α_k is equi-affine or not. Is it orientation preserving?

27. Let $T : R^2 \to R^2$ be defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + 2a \\ by + 3b \end{pmatrix}$, where a and b are non zero constants. Show that T is affine transformation and find the formula for T^{-1}. Let Π be a plane with area $A = \dfrac{1}{16}$. Give the area of $T(\Pi)$? Is T equi-affine? Is T orientation preserving or reversing?

28. Determine the orientation of ΔABC whose vertices are given as follow

$$A = \begin{pmatrix} -4 \\ 1 \end{pmatrix}, B = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, C = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

29. Given $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 4y - 2 \\ -4x + 3y + 1 \end{pmatrix}$

a) Show that f is an affine transformation

b) Show that f is a similarity

c) Find a dilation δ, a rotation R_θ and a translation T_v such that $f = \delta \circ T_v \circ R_\theta$

d) Is f orientation preserving?

30. Classify the following affine transformations as orientation preserving and orientation changing.

a. Dilation
b. Line reflection
c. Shears
d. Compression

31. Show that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the image of the circle $x^2 + y^2 = 1$ under some affine transformation.

32. If $2x' = -\sqrt{3}x - y + 2$ and $2y' = x - \sqrt{3}y - 1$ are equations of a rotation, then find the center and angle of the rotation.

33. Let α be an isometry with $\alpha(0,0) = (2,1)$, $\alpha(1,-1) = (1,0)$, $\alpha(2,3) = (5,1)$. Then find the equations of α. Determine whether α even or odd isometry?

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