Wollo University
Kombolcha Institute of Technology
School of Mechanical & Chemical Engineering

Chapter 2 part -2:

2-D Potential Flow Theory

- continued

Prepared by: **Wubishet Degife.**
E-mail: **wubesvictory@gmail.com**

2-D Potential Flow Theory

- continued

Reference:

1. **Frank M. White, Chapter 4, Sections 4.6 & 4.7;**
2. **Frank M. White, Chapter 8, Sections 8.1 to 8.4**

2-D Potential Flow Theory ….

1. Introduction, def., some basic concepts
2. General governing Eqns; reduced forms,
3. **Solving the Potential Flow**
   - ***Velocity potential function, Laplace eqn***
   - ***Stream function, Laplace eqn***
   - ***Superposition of elementary potential flow models;***

     Examples:

     » **Uniform Flow and Source Flow**

     » **Uniform Flow and Source / Sink Pair: The Rankine Oval**

     » **Uniform Flow and Doublet: *Non-lifting flow over a circular cylinder***

     » **Uniform Flow, Doublet and Free Vortex: *Lifting flow over a circular cylinder***
4. ***Conclusion : The Kutta – Joukowski Thm***

2-D Potential Flow Theory

– **Potential flow** is

- an inviscid, incompressible, irrotational, steady flow

– **Potential Flow Theory** is

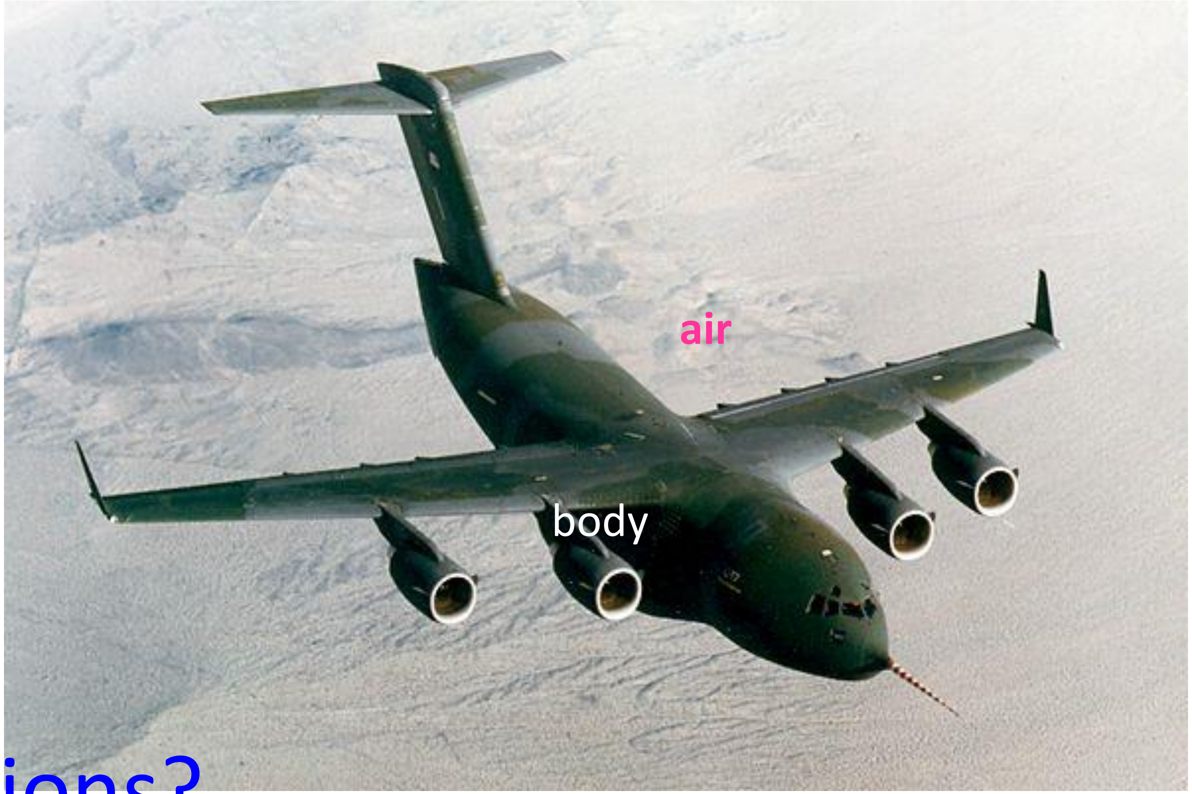
- a mathematical method developed to **solve** flow problems that can be closely approximated as a potential flow

2-D Potential Flow Theory

<u>Question</u>

What do we mean by '**solve**' ?

Consider an air craft moving in air:

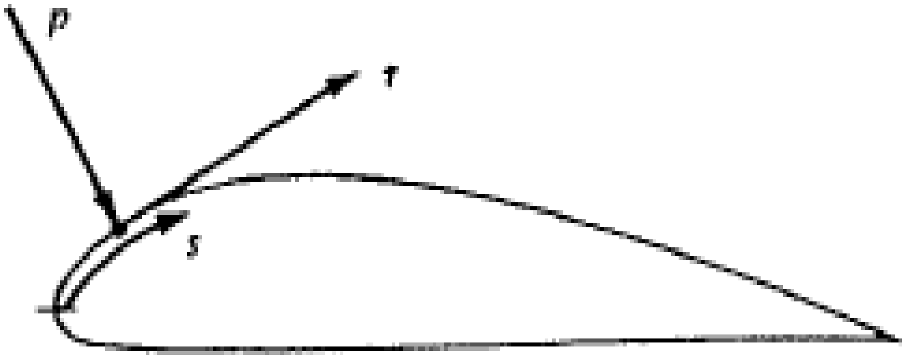


Questions?

Why doesn't it fall down ? (What keeps it floating in air?

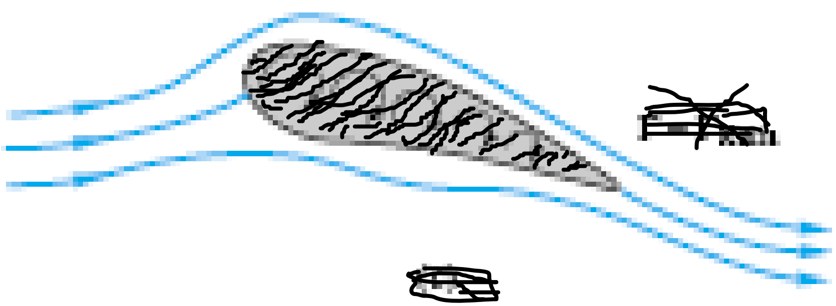
Why/How does it move forward?

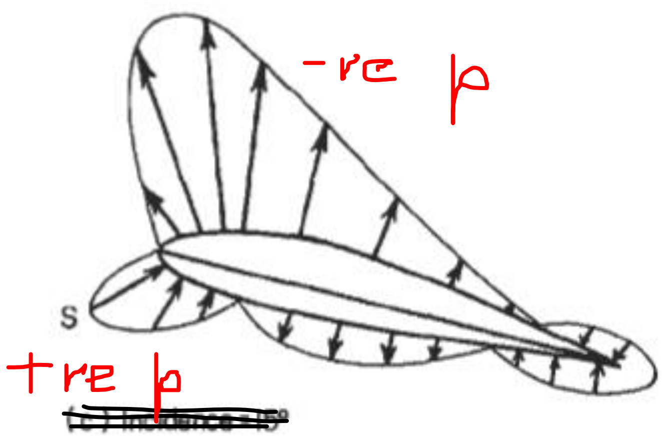
Airfoil :



$p = p(s)$ = surface pressure distribution
$\tau = \tau(s)$ = surface shear stress distribution

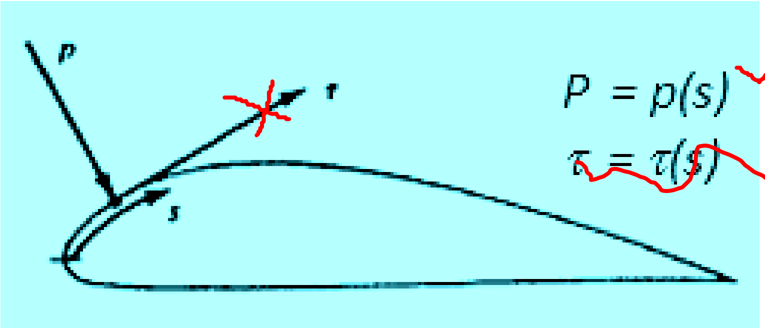
Figure 1.8 Illustration of pressure and shear stress on an aerodynamic surface.





- **Known:**
  - The ***forces*** exerted on bodies moving through a fluid are caused ***only by two reasons***:
    1. Pressure distribution *(* p *)* over the body surface,
    2. Shear-stress distribution *(* τ *)* over the body surface.

    i.e., ***pressure*** and ***Shear-stress*** are ***the only two mechanisms*** nature has for 'communication' of force & moment between the body and the moving fluid.



In general,

- the net effect of the *p*- and τ distributions integrated over the entire body surface constitutes a **resultant force**, **R**, b/n the fluid and the body; and is given by:

$$R = \int_{\text{entire surface}} p\,dA + \int_{\text{entire surface}} \tau\,dA$$

In particular,

- **For a potential flow**, **there is no** τ **; we have only** *p*

- ***A good e.g. for 'solve' is determining the velocity and pressure distribution, from which R could be determined** (i.e., **solving an eng'g problem).***

2-D Potential Flow Theory …

The starting point for solving a potential flow problem is the set of eqns governing the flow.

2-D Potential Flow Theory …

- Recall the set of ***Differential Eqns for a*** (**general**) ***fluid flow***:

Continuity: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

Momentum: $$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla p + \nabla \cdot \tau_{ij}$$

Energy: $$\rho \frac{d\hat{u}}{dt} + p(\nabla \cdot \mathbf{V}) = \nabla \cdot (k \nabla T) + \Phi$$

- The flow is steady. Therefore, the **non-steady term** of the **continuity eqn** can be **left out.** (Friction does not affect continuity eqn).

- ***The flow is inviscid & incompressible.*** Therefore, ***the momentum eqn reduces to***:

$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla p$$

- We know, the **energy eqn** is **not required for the study /analysis of incompressible flow**. So, forget the 3rd eqn.

2-D Potential Flow Theory …

- Thus, our Potential Flow problem can be fully described by the following set of eqns :

Continuity:	$\nabla \cdot \mathbf{U} = 0$
Momentum:	$\rho \frac{d\mathbf{V}}{dt} = \rho\mathbf{g} - \nabla p$
Irrotational	Curl $\mathbf{U} = 0$

integral ?

$$\frac{p}{\rho} + \frac{1}{2}V^2 = const$$

- The next **task** is: *find **U** and p from* these eqns;
  - The ***momentum eqn can be integrated***; however,
  - the **continuity eqn**, as it is, can not be integrated;
    - It can, however, be integrated through use of ***velocity potential- and stream functions***
- ***So, let us define these functions first:***

2-D Potential Flow Theory …

Velocity Potential Function *(symbol, Φ)*

1

- Known – Potential flow is irrotational flow → **Curl U = 0**;
- Known:

if Curl U = 0, then, there exists a scalar function Φ(x, y) such that:

U = $\nabla\Phi$, where: $\nabla\Phi = \frac{\partial\Phi}{\partial x} i + \frac{\partial\Phi}{\partial y} j$

- ***Comparing*** $U = ui + vj$

& $U = \frac{\partial\Phi}{\partial x} i + \frac{\partial\Phi}{\partial y} j$

we note that the following relationship exists b/n the velocity components of the flow and the function Φ:

$u = \frac{\partial\Phi}{\partial x}$ and $v = \frac{\partial\Phi}{\partial y}$

2-D Potential Flow Theory …

(Repeated),

$$u = \frac{\partial \Phi}{\partial x} \quad \text{and} \quad v = \frac{\partial \Phi}{\partial y}$$

where $\Phi = \Phi(x,y)$ represents the **scalar function**, named ***'velocity potential function'*** .

2 *Now,* these velocity components computed from Φ must of course satisfy ***the continuity eqn:***

$$\nabla \cdot \mathbf{U} = 0 \quad ,$$

i.e.,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad ,$$

$$\Rightarrow \quad \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial y}\right) = 0 \quad \Rightarrow \quad \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

2-D Potential Flow Theory …

The eqn. $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ is the well known ***Laplace eqn***.

- ***Laplace eqn*** is a ***widely studied eqn*** in mathematical physics; and its characteristics is well known.

3 Thus, (CONCLUSION)

- The pair of continuity and momentum eqns, governing a potential flow, can now been replaced by **Laplace Eqn** (*whose solution is well known*) & by **Bernoulli eqn**:

i.e.,

- Continuity: ……………………………… $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$
- Momentum: $\rightarrow$ Bernoulli eqn: $\frac{p}{\rho} + \frac{1}{2} V^2 = const$

2-D Potential Flow Theory …

- and therefore,
  - the problem of solving the potential flow is now reduced to that of ***solving simultaneously*** Laplace and Bernoulli eqns.

2-D Potential Flow Theory …

- Another Laplace eqn can be obtained in a similar way, using stream function as follows:

- Stream Function (symbol, ψ)
  - is defined based on a 2D continuity eqn and a simple (but clever) mathematical manipulation.

2-D Potential Flow Theory …

Stream Function …

1 – We start with *continuity eqn*:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

2 – *We can find a function,* $\psi = \psi(x, y)$ *such that*

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0$$

(a true statement)

i.e., letting $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$

3 and, for irrotational flow in x-y plane,

since Curl **U** = 0, $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$ (Laplace eqn)

2-D Potential Flow Theory …

- Again,
  - the problem of solving the potential flow is reduced to that of ***solving simultaneously*** *another* Laplace and Bernoulli eqns.

2-D Potential Flow Theory …

Geometric interpretation of velocity potential function and stream function:

- Const-ψ lines represent stream lines of the flow; no flow across the stream lines; flow is along these lines.
- const-Φ lines represent equipotential lines; no flow along equipotential lines; flow is across these lines.
- Const-ψ and const-Φ lines are perpendicular to each other.

2-D Potential Flow Theory …

– Const-ψ and const-Φ lines are perpendicular to each other.

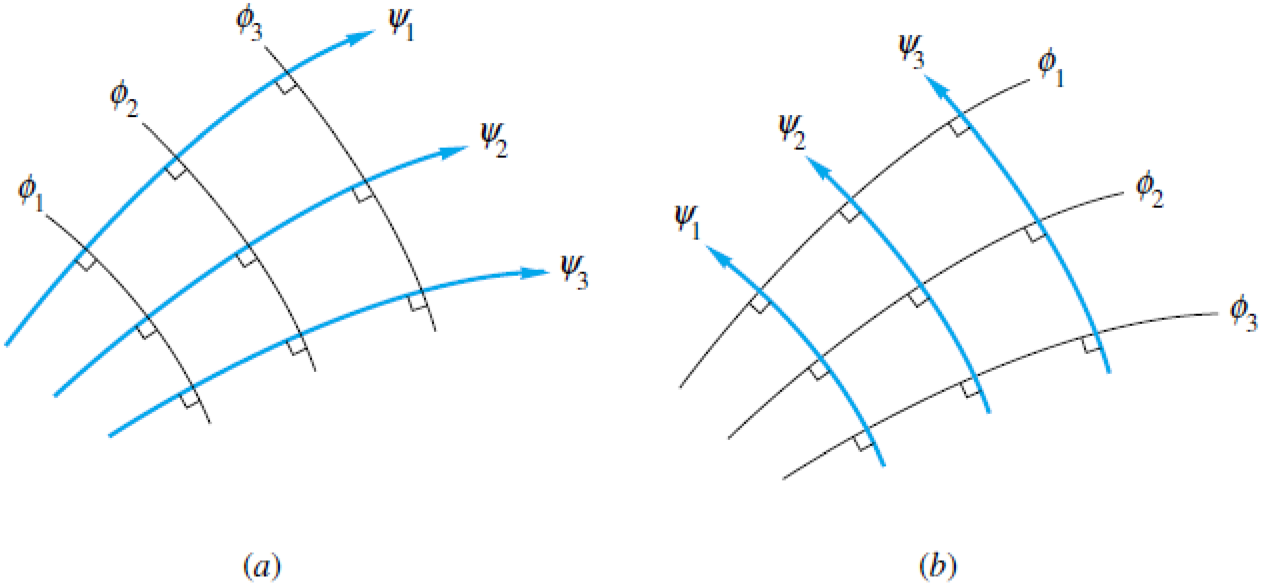


Fig. 8.2 Streamlines and potential lines are orthogonal and may reverse roles if results are useful: (*a*) typical inviscid-flow pattern; (*b*) same as (*a*) with roles reversed.

2-D Potential Flow Theory …

Note that:

- if $\psi = \psi(x, y)$ and/or $\phi(x, y)$ functions of a flow field are known, one can determine the velocity field, and then use Bernoulli eqn to find p

- If, on the other hand, the velocity field is known, the velocity potential and/or stream function of the field can be determined as illustrated next:

$$\psi = \psi(x, y) \Rightarrow d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v dx + u dy$$

$$\Rightarrow \psi = \int -v dx + \int u dy + const$$

2-D Potential Flow Theory …

Similarly,

– for velocity potential function,

$$\phi = \phi(x, y) \;\Rightarrow\; d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = udx + vdy$$

$$\Rightarrow\; \phi = \int udx + \int vdy + const$$

- **Now, returning back to the issue of <u>solving the Potential Flow problem</u>:**
  - There are different methods of finding ϕ and/or ψ of the flow, …..

2-D Potential Flow Theory …

… **solving the Potential Flow** ….

– There are **different methods of finding ϕ and/or ψ** of the flow, ***from which the velocity field could be determined***:

– These methods are:

1. **Superposition of known elementary potential flow models,**
2. *Numerical panel techniques, and*
3. *Conformal mapping*.

– ***The scope of this course is limited to no.1***

2-D Potential Flow Theory …

Concept behind the superposition principle:

- **If ϕ_1 and ϕ_2 are two separate solutions of a Laplace eqn $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$, their sum, $\phi_1+\phi_2$ is also another solution of the same eqn.**

- The same is true for the stream function; $\psi_1+\psi_2$ is also the solution of $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$ if ψ_1 and ψ_2 are known separate solutions of the Laplace eqn.

- *The superposition principle is based on this well established characteristic feature of a Laplace Eqn.*

2-D Potential Flow Theory …

Steps to apply the superposition principle:

1. Start with known elementary potential flow models like:-
   - » **Uniform flow,**
   - » **Source/sink flow,**
   - » **Doublet,**
   - » **Vortex, etc**

Note The velocity potential / stream functions of these flow models are known.

2-D Potential Flow Theory …

Steps to apply the superposition principle: …

2. Superimpose/combine linearly a number of the elementary flow models that together simulate the actual flow pattern, and then,

    a) Determine the ***stream function*** of the ***combined flow*** from the ***sum*** of the stream functions of the individual elementary flows.

    b) Determine the ***velocity components*** of the ***combined flow from the combined stream/velocity potential function***

    c) Determine the ***pressure distribution*** of the flow field from Bernoulli eqn.

    d) Determine the eng’g parameter (e.g. ***aerodynamic force /moment) by integrating the diff. pressure force over the entire surface***

2-D Potential Flow Theory …

The Elementary Flow Models

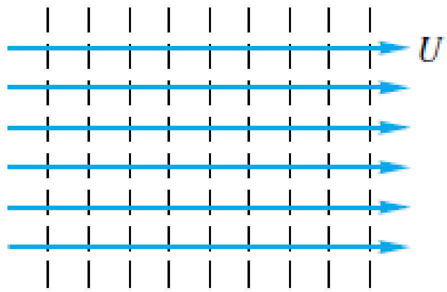
2-D Potential Flow Theory …

Elementary Flow Models

$$\phi = \int u dx + \int v dy + const$$

$$\psi = \int -v dx + \int u dy + const$$

1. Uniform Flow

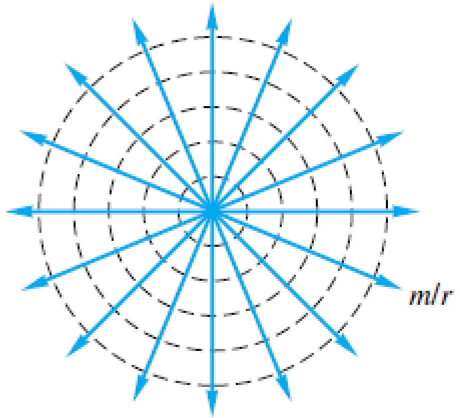


A uniform stream $\mathbf{V} = \mathbf{i}U$,

$$u = U = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad v = 0 = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$\boxed{\psi = Uy} \qquad \boxed{\phi = Ux}$$

2. Line Source/sink



$$v_r = \frac{Q}{2\pi rb} = \frac{m}{r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r}$$

$$m = \frac{Q}{2\pi b}$$

$$b = debth$$

m – source strength

$$v_\theta = 0 = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$\boxed{\psi = m\theta} \qquad \boxed{\phi = m \ln r}$$

2-D Potential Flow Theory …

Superposition of the Elementary Flows

- We can form a variety of flow patterns of practical interest by linearly combining these elementary flow models (and others).

- The streamlines of the combined flow could be obtained, e.g., graphically (i.e., by summing up the constant-ψ lines of the elementary flow models).

Examples:-

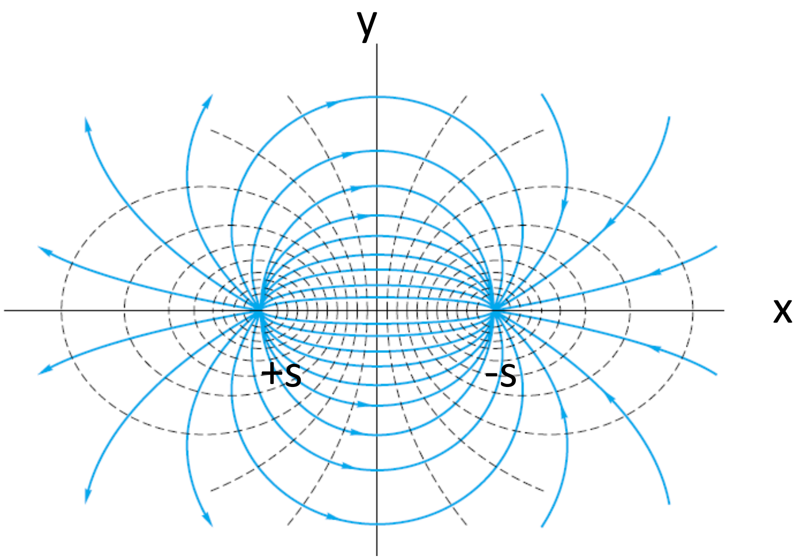
1. Source – Sink pair (PTO)

2-D Potential Flow Theory …

Superposition of the Elementary Flows …

Examples:-

1. Source – Sink pair



$$\psi = \psi_{\text{source}} + \psi_{\text{sink}} = m \tan^{-1} \frac{y}{x+a} - m \tan^{-1} \frac{y}{x-a}$$

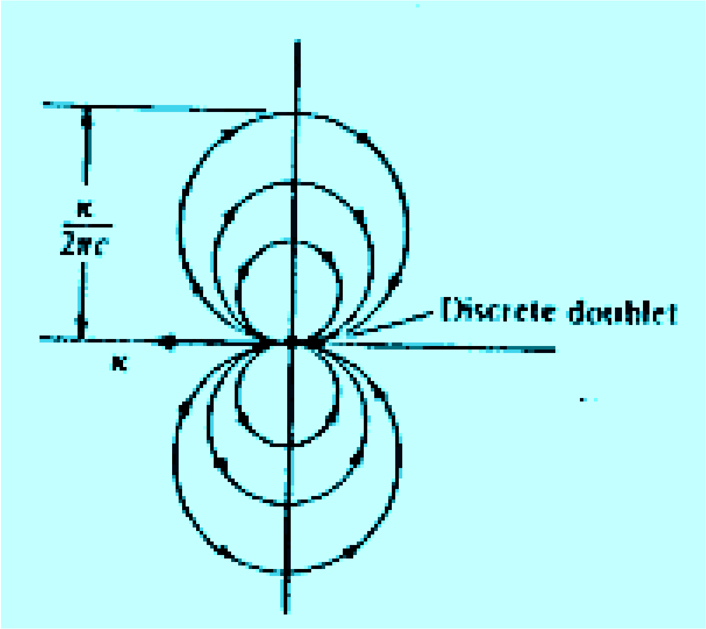
$$\phi = \phi_{\text{source}} + \phi_{\text{sink}} = \frac{1}{2} m \ln [(x+a)^2 + y^2] - \frac{1}{2} m \ln [(x-a)^2 + y^2]$$

2-D Potential Flow Theory …

Superposition of the Elementary Flows …

Examples:-

2. Doublet



$$\psi = -\frac{\kappa}{2\pi}\frac{\sin\theta}{r}$$

$$\phi = \frac{\kappa}{2\pi}\frac{\cos\theta}{r}$$

2-D Potential Flow Theory …

Examples of

superposition of elementary flows

to simulate practical flow problems

2-D Potential Flow Theory …

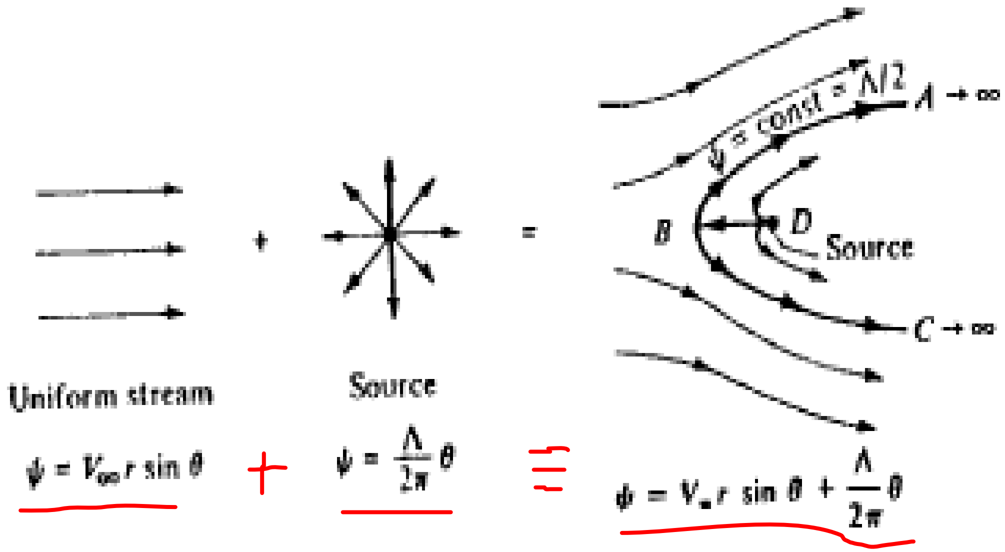
Example 1

superposition of

Uniform Flow and Source Flow

2-D Potential Flow Theory …

1. Uniform Flow and Source Flow



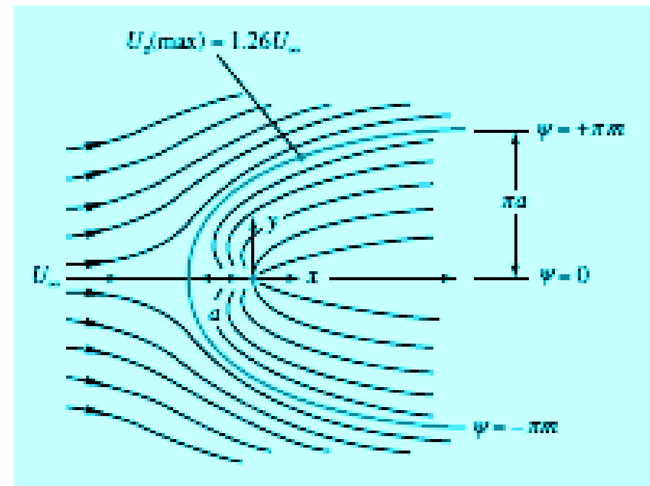


Figure 3.15 Superposition of a uniform flow and a source; flow over a semi-infinite body.

- The ***stream function of the combined flow***,

$$\psi = V_\infty r \sin\theta + \frac{\Lambda}{2\pi}\theta$$

- The ***velocity components of the comb. flow***

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = V_\infty \cos\theta + \frac{\Lambda}{2\pi r}$$

$$V_\theta = -\frac{\partial\psi}{\partial r} = -V_\infty \sin\theta$$

2-D Potential Flow Theory …

Superposition of the Elementary Flows …

1. Uniform Flow and Source Flow …

- The ***velocity components*** of the ***combined flow***:

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_\infty \cos\theta + \frac{\Lambda}{2\pi r}$$

$$V_\theta = -\frac{\partial \psi}{\partial r} = -V_\infty \sin\theta$$

- The ***stagnation pt*** of the ***combined flow***:

$$V_\infty \cos\theta + \frac{\Lambda}{2\pi r} = 0$$

$$V_\infty \sin\theta = 0$$

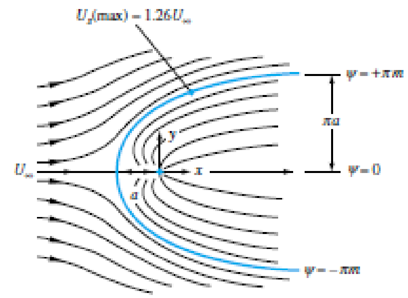
- Solving the eqns, the stagnation pt, $S(r,\theta) = S(\Lambda/2\pi V_\infty, \pi)$

Potential Flow Theory …

1. <u>Uniform Flow and Source Flow</u> …

❖ Subst. the coordinates of the <u>stagnation pt</u> in the expression of ψ of the comb flow, ***we get***

$\psi_{-cont} = \Lambda/2$ ***- an expression of the <u>dividing stream line</u>*** (see fig.);



- it ***separates*** the free stream and the source flow.
- ***The entire region inside the const*** $\psi_{-cont} = \Lambda/2$ ***can be replaced with a solid surface of the same shape;***
- and, ***<u>the flow outside can be described by</u>***:

$$\psi = V_\infty r \sin\theta + \frac{\Lambda}{2\pi}\theta$$

Potential Flow Theory …

1. <u>Uniform Flow and Source Flow</u> …

- This illustrates the practicality of adding elementary flows to obtain a more complex flow over a body of interest.

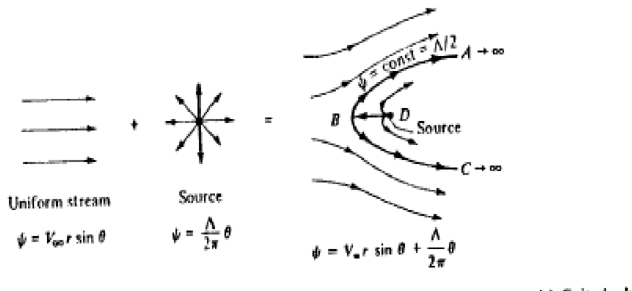
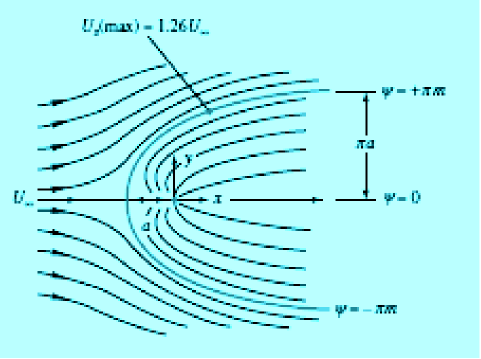


Figure 3.15 Superposition of a uniform flow and a source; flow over a semi-infinite body.

- The velocity field can then be calculated from the ψ of the combined flow.



- Then, we can calculate the pressure coeff. from:

$$C_p = 1 - \left(\frac{V}{V_\infty}\right)^2$$

- And then, the aerodynamic force coeff. can be calculated.

Potential Flow Theory …

Example 2

superposition of

Uniform Flow and Source / Sink Pair:

- The Rankine Oval

Potential Flow Theory …

2. Uniform Flow + Source / Sink Pair (Rankine Oval)

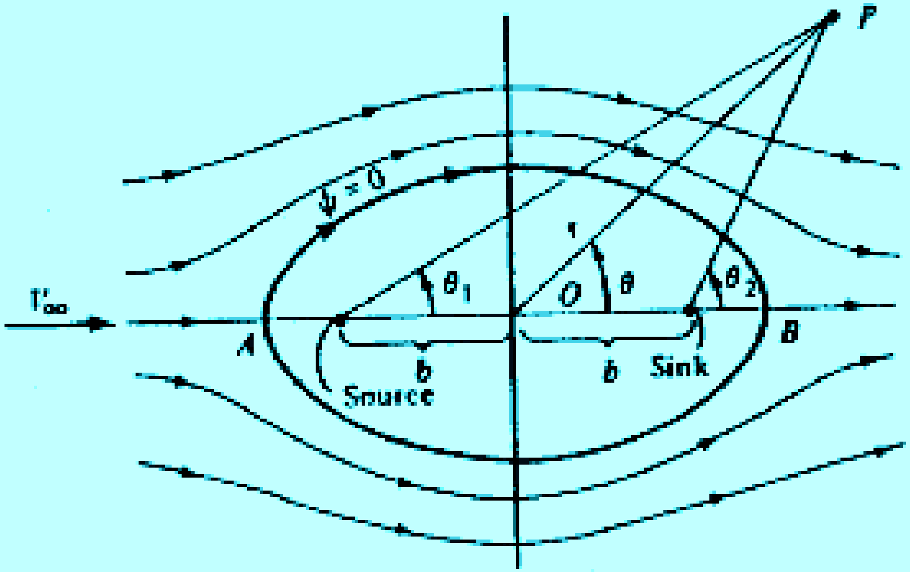


Figure 3.16 Superposition of a uniform flow and a source-sink pair; flow over a Rankine oval.

- The ***stream function*** of the ***combined*** flow at any arbitrary point P (see Fig) is:

$$\psi = V_\infty r \sin\theta + \frac{\Lambda}{2\pi}(\theta_1 - \theta_2)$$

Potential Flow Theory …

2. Uniform Flow + Source / Sink Pair (Rankine Oval)

- The ***stream function*** of the ***combined*** flow at P:

$$\psi = V_\infty r \sin\theta + \frac{\Lambda}{2\pi}(\theta_1 - \theta_2)$$

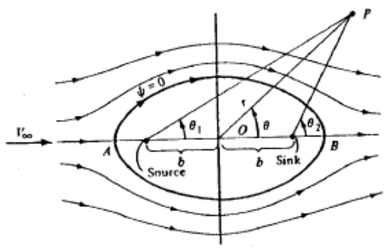


Figure 3.16 Superposition of a uniform flow and a source-sink pair; flow over a Rankine ov

- from which the ***velocity components at P***:

$u_r = \ldots..$ $\qquad u_\theta = \ldots..$

- The ***stagnation pts in the cobined flow field*** ……..

- The particular const-ψ line that passes thru the stagnation pts, ***ψ = 0, is the dividing stream line***.

- All the flow from the source is 'consumed' by the sink; and so, the source-sink pair is entirely contained inside the oval curve defined by ***the dividing stream line , ψ = 0***.

Potential Flow Theory …

2. Uniform Flow + Source / Sink Pair (Rankine Oval)

- Therefore, the region inside the oval can be replaced with the shape given by $\psi = 0$, where:

$$\psi = V_\infty r \sin\theta + \frac{\Lambda}{2\pi}(\theta_1 - \theta_2)$$

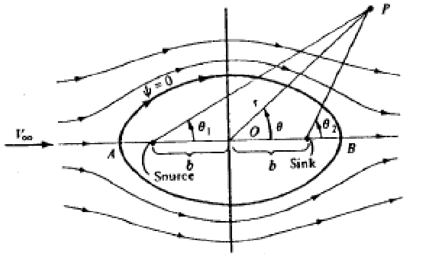


Figure 3.16 Superposition of a uniform flow and a source-sink pair; flow over a Rankine ov

- And the region outside the oval closed curve can be interpreted as an inviscid, irrot. incomp. steady flow past the solid surface.

- The ***velocity field*** can be calculated from: $\psi = V_\infty r \sin\theta + \frac{\Lambda}{2\pi}(\theta_1 - \theta_2)$

- The ***pressure coeff***. ….

- The ***aerodynamic coeff***. ……

Potential Flow Theory …

Example 3

superposition of

Uniform Flow and Doublet:

Non-lifting flow over a circular cylinder

Potential Flow Theory …

3. <u>Uniform Flow and Doublet</u>

(Non-lifting flow over a circular cylinder)

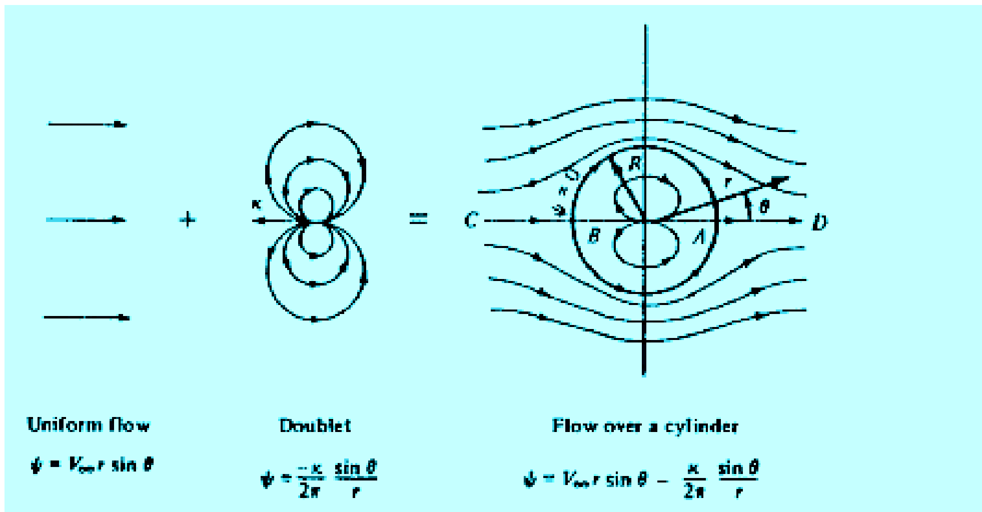


Figure 3.19 Superposition of a uniform flow and a doublet; nonlifting flow over a circular cylinder.

- The ***<u>stream function</u>*** of the ***comb flow*** is:

$$\psi = (V_\infty r \sin\theta)\left(1 - \frac{R^2}{r^2}\right)$$

$$R = \sqrt{\frac{\kappa}{2\pi V_\infty}}$$

R : radius of cylinder; k : Doublet strength

Potential Flow Theory …

3. Uniform Flow and Doublet

(Non-lifting flow)

- The ***velocity field*** can then be determined by differentiating the stream function of the ***combined flow***:

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r}(V_\infty r \cos\theta)\left(1 - \frac{R^2}{r^2}\right)$$

$$V_r = \left(1 - \frac{R^2}{r^2}\right)V_\infty \cos\theta$$

$$V_\theta = -\frac{\partial \psi}{\partial r} = -\left[(V_\infty r \sin\theta)\frac{2R^2}{r^3} + \left(1 - \frac{R^2}{r^2}\right)(V_\infty \sin\theta)\right]$$

$$V_\theta = -\left(1 + \frac{R^2}{r^2}\right)V_\infty \sin\theta$$

$$\left(1 - \frac{R^2}{r^2}\right)V_\infty \cos\theta = 0$$

$$\left(1 + \frac{R^2}{r^2}\right)V_\infty \sin\theta = 0$$

On surface,

$$V_r = 0$$

$$V_\theta = -2V_\infty \sin\theta$$

- Equating V_r and V_θ to zero, we determine ***the stagnation pts***: located at $(r, \theta) = (R, 0)$ and (R, π).

Potential Flow Theory …

3. Uniform Flow and Doublet

(Non-lifting flow)

- ***Substituting*** the co-ord. of the stagnation pts into the expression of the stream function for the comb flow, we get the eqn of the ***dividing stream line***:

$$\psi = (V_\infty r \sin\theta)\left(1 - \frac{R^2}{r^2}\right) = 0$$

satisfied by $r = R$ for all values of θ.

describes a circle with radius R,

- The ***velocity*** distribution on the surface of the cylinder is:

$$V_r = 0$$

$$V_\theta = -2V_\infty \sin\theta$$

Potential Flow Theory …

3. <u>Uniform Flow and Doublet</u>

(Non-lifting flow)

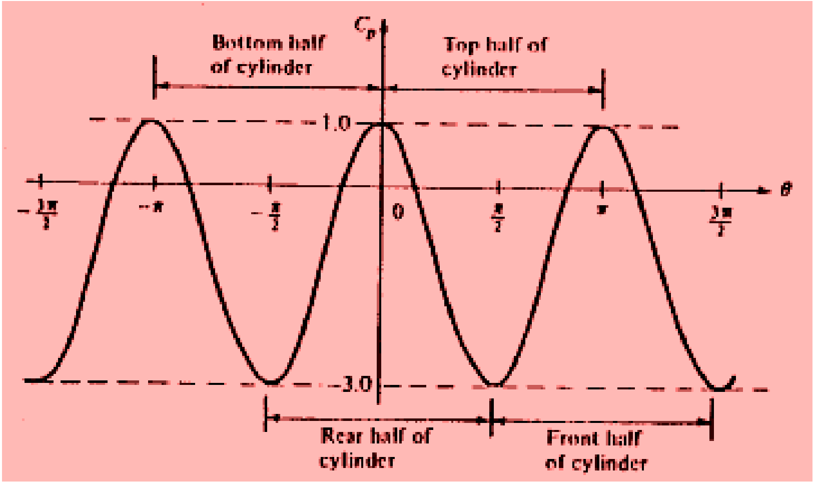
- Thus, the distribution of ***<u>pressure coeff</u>*** on the surface of the cylinder,

$$C_p = 1 - \left(\frac{V}{V_\infty}\right)^2 \quad \Rightarrow \quad C_p = 1 - 4\sin^2\theta$$

- The normal- and axial ***<u>force coeffs</u>* are each zero**.

$$c_n = \frac{1}{c}\int_0^c (C_{p,l} - C_{p,u})\,dx$$

$$c_a = \frac{1}{c}\int_{LE}^{TE} (C_{p,u} - C_{p,l})\,dy$$



Potential Flow Theory …

One More Elementary Flow Model:

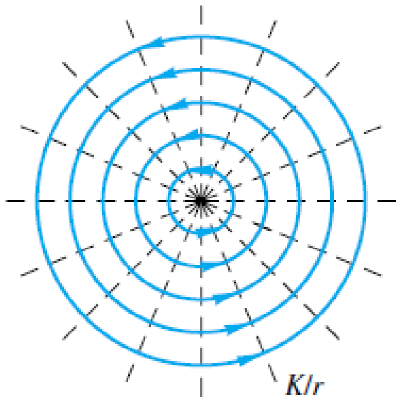
Line Vortex

Potential Flow Theory …

Line Vortex (**Ref:** See, e.g., Section 3.14, Anderson)

- is an elementary potential flow model where:
  - ***the stream lines are concentric circles,***
  - ***v_θ is const for a given streamline;***

    however, it varies from one streamline to another according to:



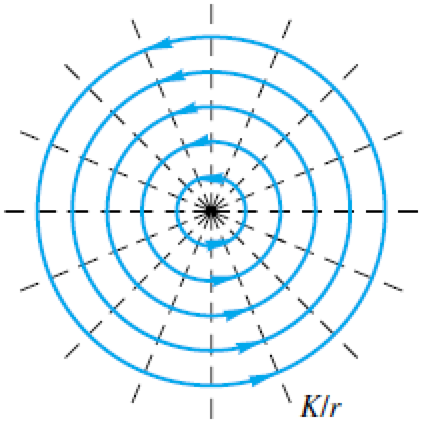
$r\, v_\theta = \text{const},$ or, $v_\theta = -\Gamma/(2\pi r)$

where Γ = **the vortex strength = - 2** π (const)

  - The ***radial velocity*** comp, $v_r = 0$

Potential Flow Theory …

Line Vortex …



- *Vortex flow* is a ***physically possible*** model and so, it ***obeys continuity eqn*** for incomp flow; ie, $\nabla \cdot \mathbf{V} = 0$.
- and, it is an ***irrotational flow***, i.e.,

  $$\text{Curl } \mathbf{V} = 0$$

  at every pt except at the origin;

  – at the origin,

  $$\text{Curl } \mathbf{V} \to \infty.$$

$$v_r = 0 = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r}$$

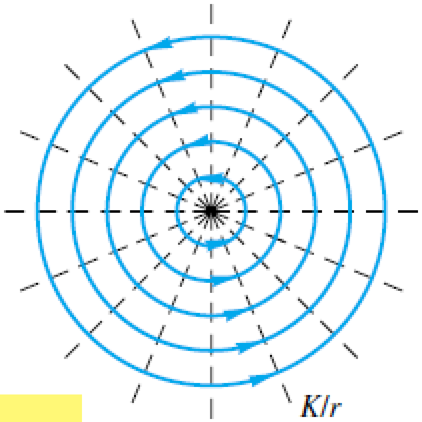
$$v_\theta = \frac{K}{r} = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$\psi = -K \ln r$$

$$\phi = K\theta$$

Potential Flow Theory …

Line Vortex …



- Since ***the velocity field is known***, ***one can determine the velocity potential and stream functions*** of the model by integration, using:

$$v_r = 0 = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r}$$

$$v_\theta = \frac{K}{r} = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

- The results: $\phi = -\dfrac{\Gamma}{2\pi}\theta$, and $\psi = -\dfrac{\Gamma}{2\pi}\ln(r)$

Potential Flow Theory …

Example 4

superposition of

Uniform Flow, Doublet and Free Vortex:

Lifting flow over a circular cylinder

Ref. See, e.g., Sec 3.15, Anderson

Potential Flow Theory …

Superposition of Uniform Flow, Doublet and Free Vortex: *<u>Lifting flow over a circular cylinder</u>*

- The Doublet and free vortex are located at the same pt.

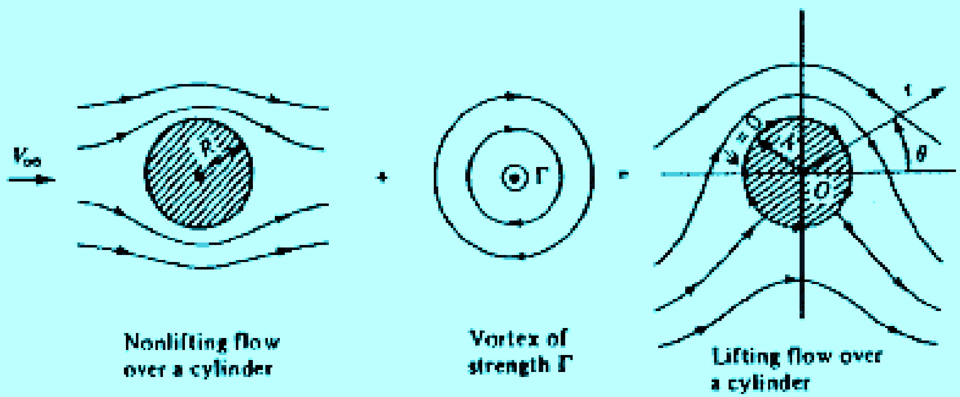


Figure 3.24 The synthesis of lifting flow over a circular cylinder.

The comb flow is:
. Symm about the y-axis; but
. Not symm about the x-axis,
(hence, lift exists)

- ***The <u>stream function</u> of the combined flow is:***

$$\psi = (V_\infty r \sin\theta)\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln\frac{r}{R}$$

Potential Flow Theory …

Lifting flow over a circular cylinder **…**

Stream function of the comb flow …

$$\psi = (V_\infty r \sin\theta)\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln\frac{r}{R}$$

- Radius of the cylinder, $R = \sqrt{\frac{\kappa}{2\pi V_\infty}}$ *k – Doublet strength*
- Since ψ = a const is the eqn of a streamline,

  *ψ = 0 represents a streamline; in fact it represents the **dividing streamline** of the comb flow.*
- At r = R, ψ = 0 for all values of θ.

Potential Flow Theory …

Lifting flow over a circular cylinder …

The stream function of the comb flow …

$$\psi = (V_\infty r \sin\theta)\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln\frac{r}{R}$$

- **The velocity field** of the comb flow can be calculated from this expression of ψ :

$$V_r = \left(1 - \frac{R^2}{r^2}\right)V_\infty \cos\theta$$

$$V_\theta = -\left(1 + \frac{R^2}{r^2}\right)V_\infty \sin\theta - \frac{\Gamma}{2\pi r}$$

- **Stagnation pts** can be obtained from:

$$V_r = \left(1 - \frac{R^2}{r^2}\right)V_\infty \cos\theta = 0$$

$$V_\theta = -\left(1 + \frac{R^2}{r^2}\right)V_\infty \sin\theta - \frac{\Gamma}{2\pi r} = 0$$

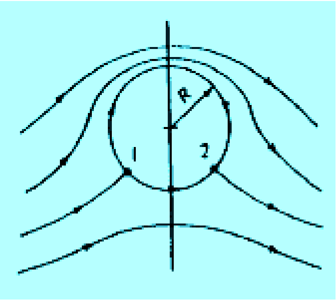
Result : at $r = R$ and,

$\theta = \arcsin(-\Gamma/4\pi \mathbf{V}_\infty R)$

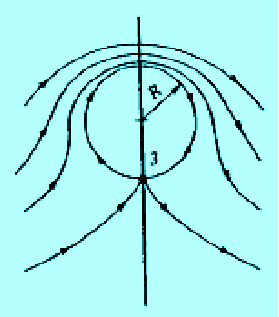
Potential Flow Theory …

Lifting flow over a circular cylinder …

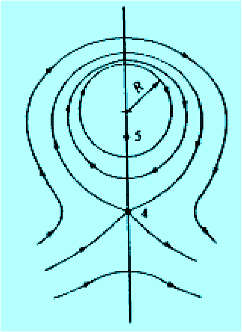
- **The flow pattern** of the ***comb flow***:



$\Gamma < 4\pi V_\infty R$



$\Gamma = 4\pi V_\infty R$



$\Gamma > 4\pi V_\infty R$

Note: In the sketch, the vortex strength, Γ is freely chosen.

- There are infinite no. of possible potential flow solutions (corresponding to the infinite choices for values of Γ).

Potential Flow Theory …

Lifting flow over a circular cylinder …

- The ***velocity components*** **on the surface** of the cylinder:

  $v_r = 0 \;;\quad v_\theta = -2V_\infty \sin\theta - \Gamma/2\pi R$

- Therefore, the ***velocity*** on the surface,

  $V = v_\theta = -2V_\infty \sin\theta - \Gamma/2\pi R$

- The ***pressure coeff*** at any pt on the surface of the cylinder:

$$C_p = 1 - \left(\frac{V}{V_\infty}\right)^2 = 1 - \left(-2\sin\theta - \frac{\Gamma}{2\pi R V_\infty}\right)^2$$

$$C_p = 1 - \left[4\sin^2\theta + \frac{2\Gamma\sin\theta}{\pi R V_\infty} + \left(\frac{\Gamma}{2\pi R V_\infty}\right)^2\right]$$

$$C_p = 1 - \left(\frac{V}{V_\infty}\right)^2$$

Potential Flow Theory …

Lifting flow over a circular cylinder …

- The drag coeff ,

$$c_d = c_a = \frac{1}{c}\int_{LE}^{TE}(C_{p,u} - C_{p,l})\,dy$$

$$c_d = \frac{1}{c}\int_{LE}^{TE} C_{p,u}\,dy - \frac{1}{c}\int_{LE}^{TE} C_{p,l}\,dy$$

$\Rightarrow$ $c_d = 0$

- The lift coeff:

$$c_l = c_n = \frac{1}{c}\int_0^c C_{p,l}\,dx - \frac{1}{c}\int_0^c C_{p,u}\,dx$$

The Lift per unit span:

$\Rightarrow$ $L' = \rho_\infty V_\infty \Gamma$

  – Very important relation in theoretical aerodynamics!!!
  – The eqn is named Kutta - Joukowski Thm

Potential Flow Theory …

Lifting flow over a circular cylinder …

Concluding Remarks on the results :

- **The drag coeff** : $c_d = 0$
- **The Lift per unit span:** $L' = \rho_\infty V_\infty \Gamma$

- ***Agreement b/n the results and the real life ???***
  - The ***prediction of zero drag is erroneous ; the error is*** due to the ***(inviscid) assumption*** underlying the Theory.
  - In reality, there is:

    skin friction --> flow separation --> finite drag.

  - The ***prediction of Lift***, however, ***is realistic !!!!*** ***The result is used extensively in the theory of aerodynamics***

This is a special case of d'Alembert's paradox, mentioned in Sec. 1.10:

> According to inviscid theory, the drag of any body of any shape immersed in a uniform stream is identically zero.

D'Alembert published this result in 1752 and pointed out himself that it did not square with the facts for real fluid flows. This unfortunate paradox caused everyone to overreact and reject all inviscid theory until 1904, when Prandtl first pointed out the profound effect of the thin viscous boundary layer on the flow pattern in the rear, as in Fig. 7.2*b*, for example.

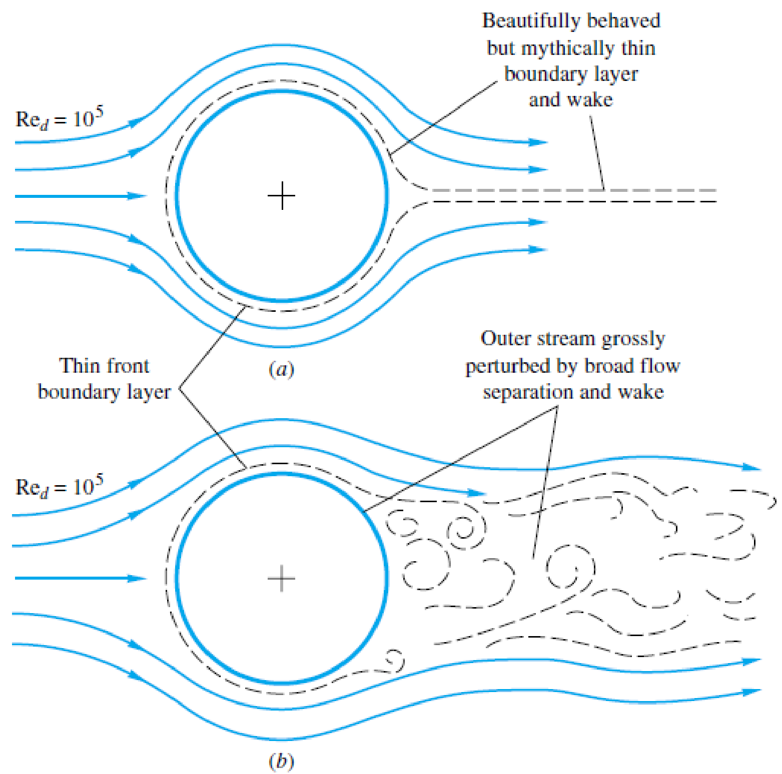
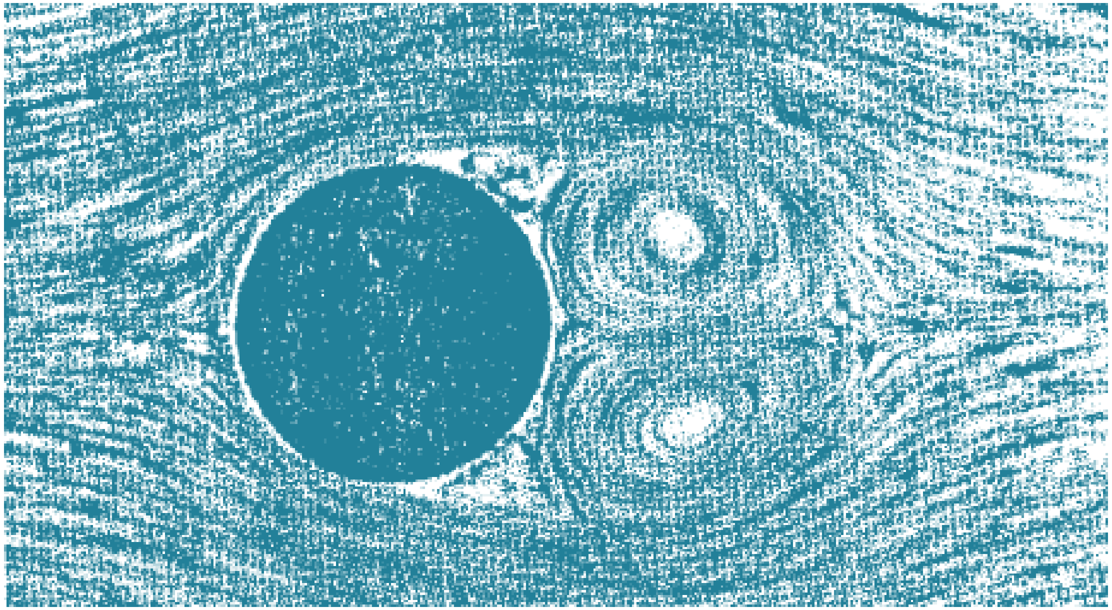


Fig. 7.2 Illustration of the strong interaction between viscous and inviscid regions in the rear of blunt-body flow: (*a*) idealized and definitely false picture of blunt-body flow; (*b*) actual picture of blunt-body flow.

Potential Flow Theory …

- **<u>Concluding Remarks on the results</u> ….**
  - The following pictures were obtained in water tunnel, where aluminum filings were scattered on the surface of the cylinder to show the directions of the streamlines.

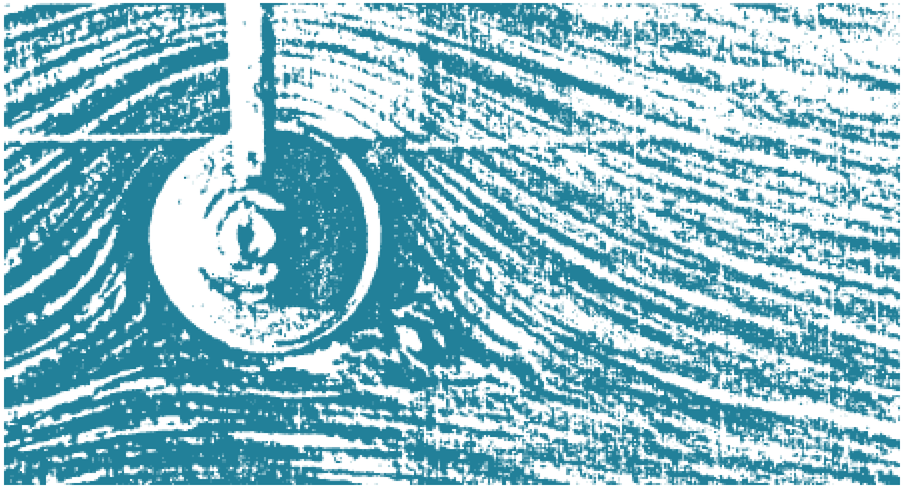
Case (a)   Non-spinning cylinder in the tunnel



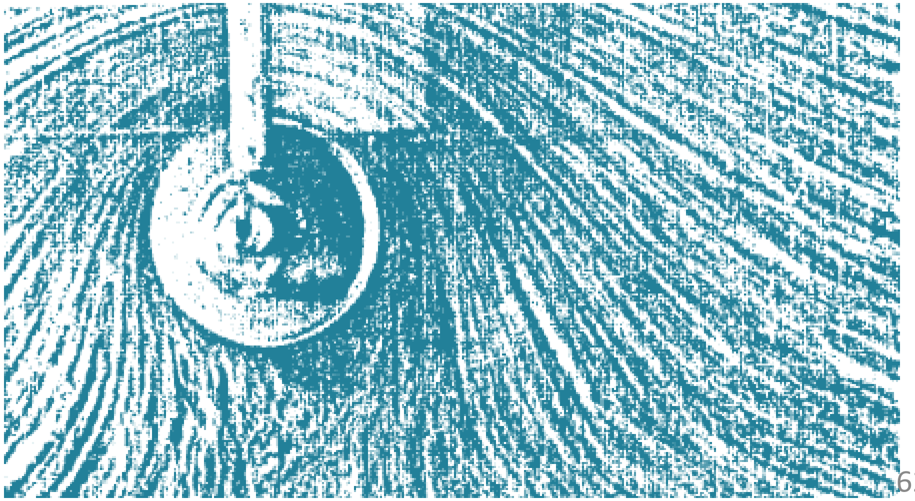
Potential Flow Theory …

- **Concluding Remarks on the results ….**

Case (b) **Spinning** cylinder – ***peripheral velocity*** of the surface = $\mathbf{3V_\infty}$

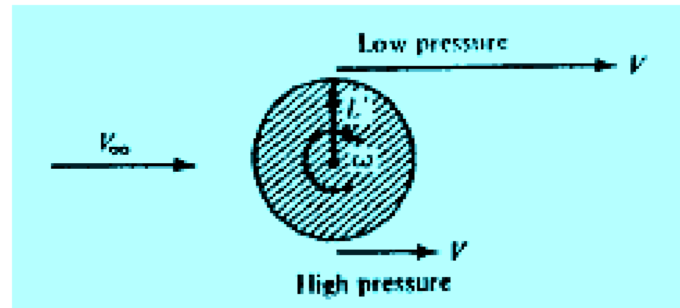


Case (c) **Spinning** cylinder – ***peripheral velocity*** of the surface = $\mathbf{6V_\infty}$



Potential Flow Theory …

- **<u>Concluding Remarks</u> <u>on the results</u> ….**
  - The physics of creation of lift on the spinning cylinder: (see the sketch)



  - ***The spinning cylinder transfers its angular momentum to the fluid due to***:
    - » (1) **the no-slip condition** and (2) **viscosity.**

hence, generating the vortex flow Γ required for the lift:

$$L' = \rho_\infty V_\infty \Gamma$$

The Kutta – Joukowski Thm

According to inviscid theory, the lift per unit depth of any cylinder of any shape immersed in a uniform stream equals $\rho u_\infty \Gamma$, where Γ is the total net circulation contained within the body shape. The direction of the lift is 90° from the stream direction, rotating opposite to the circulation.

(Courtesy of R. C. Lessmann, University of Rhode Island.)

Potential Flow Theory …

***The Kutta – Joukowski Thm*, and**

The Generation of Lift

- The eqn $L' = \rho_\infty V_\infty \Gamma$ is applicable also to cylindrical bodies of **any cross-section**, **including airfoils.**

- Consider a uniform flow past an airfoil (see sketch:
- Let A be any curve **enclosing** the airfoil.
- For the airfoil to produce Lift, the velocity field around the airfoil must be such that the **circulation**,

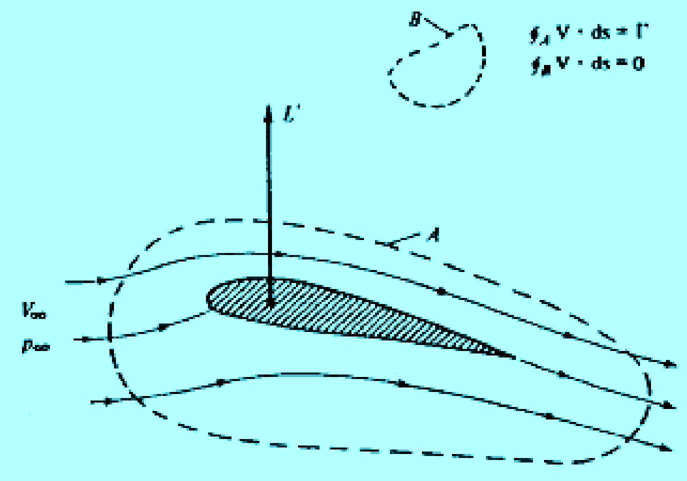


Figure 3.28 Circulation around a lifting airfoil.

$$\Gamma \equiv \oint_A V \bullet dS \neq 0$$ and, the lift, $$L' = \rho_\infty V_\infty \Gamma$$

Potential Flow Theory …

Conclusion

– The def of Γ and the use of the eqn $L' = \rho_\infty V_\infty \Gamma$ to calculate L' is the core concept of ***the circulation theory of lift in aerodynamics***.

– the development of the theory (turn of the 20th century) was a breakthrough in aerodynamics.

– *Remember, however, that*

  – *The circulation theory of lift is* ***simply an alternative way of thinking about the generation of lift****;*

  – *the* ***physical source*** *of the force is the* ***p-*** *(and τ) distribution on the body surface.*

Potential Flow Theory …

Conclusion …

- Thus, Kutta-Joukowski Thm is simply an alternative way of expressing the consequences of *the* ***p-*** *and* τ distribution;
  - So, take the Thm is a **mathematical tool** that is **consistent** with the physical situation for the inviscid incomp flow over airfoils.
- ***How can we determine the*** Γ ***of a given flow problem?***
- Possible ways are:
  1. The thin airfoil theory
  2. The Vortex Panel Method
  3. etc