Chapter Two

Time Response Analysis of control systems

Topics:

- *Classification of Time Responses*
- *Standard Test signals*
- *Steady State Analysis*
- *Analysis of First Order System*
- *Analysis of Second Order System*
- *Transient Response Specification*

Time Response Analysis of control systems

2.1 Introduction

- Time response of a control system means, how output behaves with respect to time. So it can be defined as below.
- **Definition:** the response given by the system which is the function of time, to the applied excitation is *called time response of a control system.*

2.2 Classification of Time Responses

- Total response of any control system is made up of two parts of response

i. **Steady- state response** – When the final state achieved by the o/p

ii. **Transient response** – while the o/p variations within the time takes to achieve the steady state is called transient response of the system.

2.2.1 Transient Response

- Transient response is that part of the response which goes to ***zero*** after some interval of time to achieve ***the final value.***
- The time required to achieve the final value is called *transient period.*
- The transient response may be *exponential or oscillatory* in nature. Symbolically it is denoted as $C_t(t)$.
- Transient response must vanish after some time to get the final value closer to the desired value. Such system in which transient response dies out after some time are called ***Stable systems***.

Mathematically for stable operating systems,

$$\lim_{t \to \infty} c_t(t) = 0$$

2.2.2 Steady- state Response

- It is that part of time response which remains after all transient responses have died down.

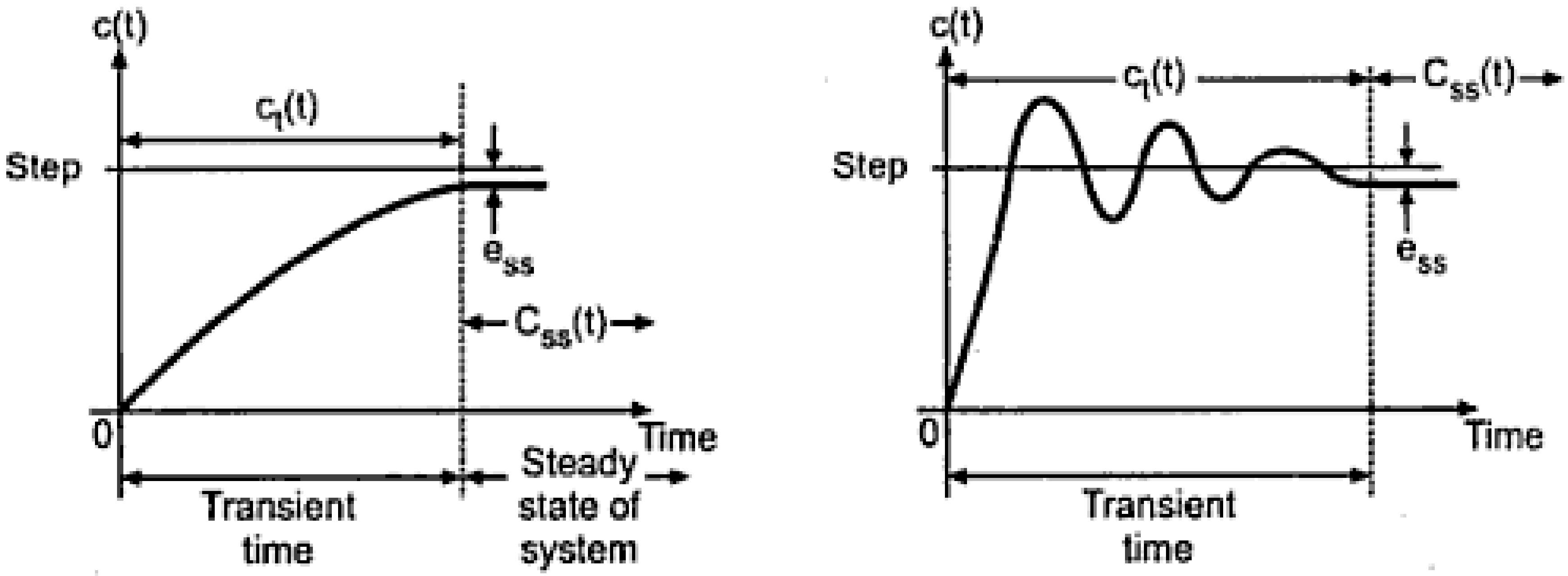
- Steady-state is the equilibrium state attained such that there is *no change with respect to time of any of the system variables.*

- *The steady state response indicates the accuracy of the system. The symbol for steady state output is* C_{SS}.

  - Hence, total time response C(t) we can write as, $C(t) = C_{SS}(t) + C_t(t)$

- The difference b/n the desired output and the actual output of the system is called **steady state error** which is denoted as e_{ss}. This error indicate the accuracy and play an important role in designing the system.

- The above definitions can be shown in the waveform as in the Fig 2.1 (a), (b) where input applied to the system is step type of input.



(a) Ct(t) is exponential

(b) Ct(t) is oscillatory

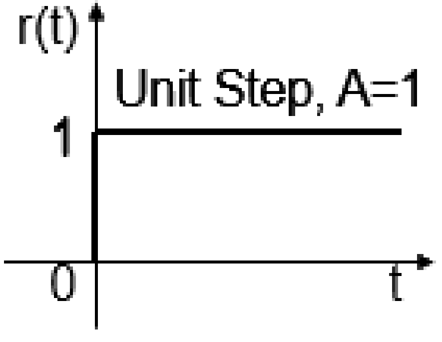
Fig: 2.1

2.3 Standard Test signals

- In practice, many signals are available which are the functions of time but, from the analysis point of view, those signals which are commonly used as reference inputs are defined as *Standard Test Inputs*.
- These standard test signals are,

1) Step Input (Position function):

- ➢ It is the sudden application of the input at a specified time as shown in the Fig.
- ➢ Let the step input of magnitude A be applied. The Laplace transform of step input r(t) having magnitude A is given by:



$r(t) = A\,u(t)$

Where, $u(t) = 1$ for $t > 0$

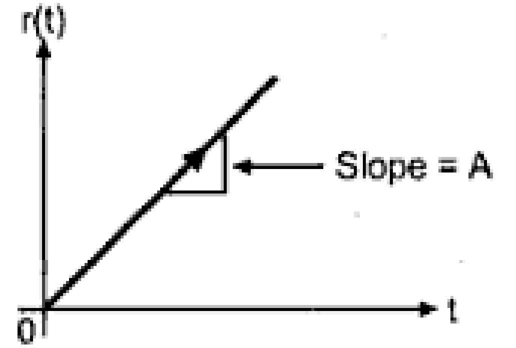
$u(t) = 0$ for $t < 0$

$R(s) = A/S$

2) Ramp Input (Velocity function): it is constant rate of change in input i.e. gradual application of input as shown in the Fig.

- Magnitude of ramp input is nothing but it's slope.
- Mathematically it is defined as,

r(t) = At	for $t \geq 0$
= 0	for $t < 0$

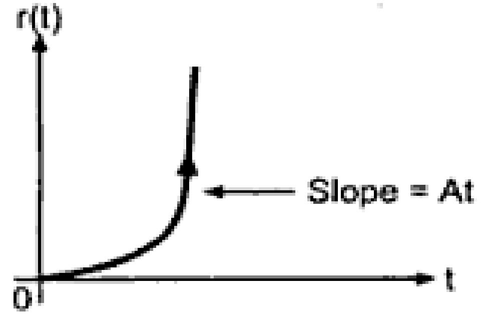


- If A = 1, it is called *Unit Ramp input.* It is denoted as r(t). Its LT is R(s) = $\frac{A}{S^2}$

3) Parabolic Input (Acceleration function): this is the input signal which is one degree faster than a ramp type of input as shown.

- Mathematically it is defined as,

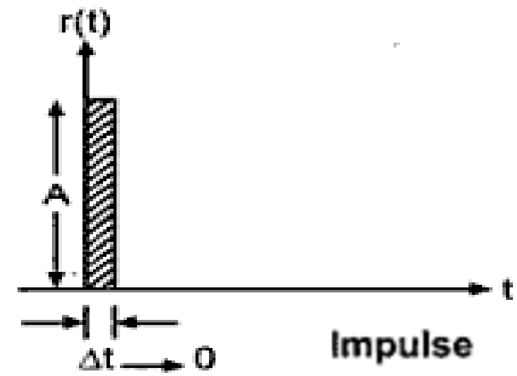
r(t) = $\frac{A}{2}t^2$	for $t \geq 0$
= 0	for $t < 0$



- If A = 1, i.e. r(t) = $\frac{t^2}{2}$ it is called *unit parabolic input*. Its LT is R(s) = $\frac{A}{S^3}$

4) Impulse input: it is input applied instantaneously (*for short duration of time*) of very high amplitude as shown in the Fig.

- It is the pulse whose magnitude is infinite while its width tends to zero i.e. t → 0, applied momentarily.



- Area of the impulse is nothing but its magnitude. If its area is unity it is called Unit Impulse Input, denoted as $\delta(t)$.
- Mathematically it is defined as,

$$r(t) = A, \quad \text{for } t = 0$$
$$= 0, \quad \text{for } t \neq 0$$

- Its *Laplace transform* ***is always 1.*** If A = 1. i.e. for unit impulse response.

r (t)	Symbol	R(s)
Unit step	u(t)	1/s
Unit ramp	r(t)	$1/s^2$
Unit parabolic	–	$1/s^3$
Unit impulse	δ (t)	1

2.4 Steady State Analysis

- As discussed earlier, steady state is that part of output which remains after transients completely vanish from the output.
- Mainly the steady state response has following two specifications.

  i) How much time system takes to reach its steady state which is called ***setting time***.

  ii) How far away actual output is reached from its desired value which is called ***steady state error (e_{ss}).***

Definition: **Steady State Error**: it is the difference b/n the actual output and the desired output.

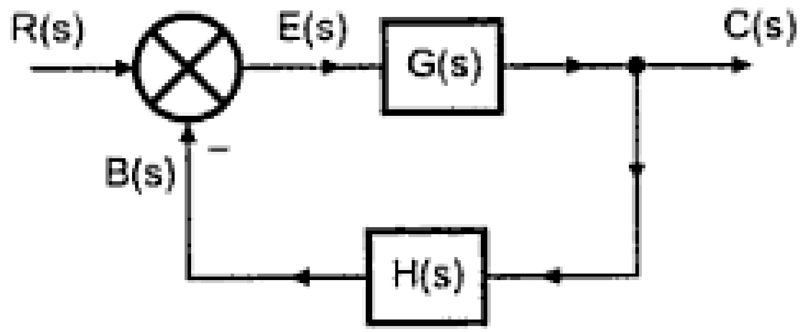
- Mathematically it is defined in Laplace domain as,

L[e(t)] = E(s) = R(s) – C(s)H(s), for non unity feedback system

L[e(t)] = E(s) = R(s) – C(s), for unity feedback system

2.4.1 Derivation of Steady State Error

- Consider a simple closed loop system using negative feedback as shown in fig below,



where $E(s)$ = Error signal, and $B(s)$ = Feedback signal

Now, $E(s) = R(s) - B(s)$

But $B(s) = C(s)H(s)$

$\therefore$ $E(s) = R(s) - C(s)H(s)$

and $C(s) = E(s)G(s)$

$\therefore$ $E(s) = R(s) - E(s)\,G(s)H(s)$

$E(s) + E(s)G(s)H(s) = R(s)$

$\therefore$

$$E(s) = \frac{R(s)}{1 + G(s)H(s)} \quad \text{for nonunity feedback}$$

$$E(s) = \frac{R(s)}{1 + G(s)} \quad \text{for unity feedback}$$

- E(s) – is the error in Laplace domain and is expressed in ‘**s**’. But, to calculate the error value in time domain, corresponding error will be e(t). Now steady state of the system is that state which remains as **t→ ∞**.

Therefore, Steady state error, $e_{ss} = \lim_{t \to \infty} e(t)$

- Now we can relate this in Laplace domain by using **final value theorem** which states that,

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sf(s) \qquad \text{where } F(s) = L\{ f(t) \}$$

Therefore,

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) \quad \text{where } E(s) \text{ is } L\{ e(t) \}.$$

Substituting E(s) from the expression derived, we can write

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)}$$

Equation (1)

For negative feedback systems use positive sign in denominator while use negative sign in denominator if system uses positive feedback.

Types of Input and Steady State Error

Error for step input

- Reference input for step input of magnitude 'A' is, $R(s) = \frac{A}{S}$
- Using Equation (1) and substituting R(s) we get,

$$e_{ss} = \underset{s \to 0}{\mathrm{Lt}} \frac{sR(s)}{1+G(s)H(s)} = \underset{s \to 0}{\mathrm{Lt}} \frac{s\left(\frac{A}{s}\right)}{1+G(s)H(s)}$$

$$\Rightarrow \quad e_{ss} = \frac{A}{1+\underset{s \to 0}{\mathrm{Lt}}\, G(s)\, H(s)}$$

- For a system $\underset{s \to 0}{\mathrm{Lt}}\, G(s)H(s)$ is constant called **positional error constant *(K_p)***

Then, $e_{ss} = \frac{A}{1+K_P}$

❑ Error for Ramp input

- For magnitude 'A', R(s) = $\frac{A}{S^2}$

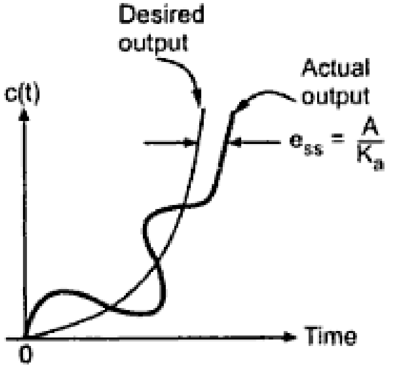
- Using Eq.(1) $$e_{ss} = \underset{s\to 0}{\mathrm{Lt}} \frac{s\left(\frac{A}{s^2}\right)}{1+G(s)H(s)} = \underset{s\to 0}{\mathrm{Lt}} \frac{A}{s+s\,G(s)H(s)} = \frac{A}{\underset{s\to 0}{\mathrm{Lt}}\; sG(s)H(s)}$$

For a selected system $\underset{s\to 0}{\mathrm{Lim}}$ sG(s)H(s) is constant and called **Velocity Error Coefficient** as K_v.

$$K_v = \underset{s\to 0}{\mathrm{Lim}}\; s\,G(s)H(s) = \text{Velocity error coefficient}$$

And corresponding error is,

$$e_{ss} = \frac{A}{K_v}$$



❑ Error for Parabolic input

- The Laplace transform for the parabolic input of magnitude A is give by,

$$R(s) = \frac{A}{S^3}$$

- Using Eq.(1)

$$e_{ss} = \underset{s\to 0}{\mathrm{Lt}} \frac{sR(s)}{1+G(s)H(s)} = \underset{s\to 0}{\mathrm{Lt}} \frac{s\left(\frac{A}{s^3}\right)}{1+G(s)H(s)}$$

$$e_{ss} = \frac{A}{\underset{s\to 0}{\mathrm{Lt}}\ s^2 G(s)H(s)}$$

- Here, $\underset{s\to 0}{\mathrm{Lt}}\ S^2 G(s)H(s) = K_a$ is constant and called acceleration error coefficient

- Substituting: $e_{ss} = \frac{A}{K_a}$

Effect of change in G(s)H(s) on Steady State Error (TYPE of a system)

- The general form G(s)H(s) is expressed in a particular form called ***time constant form*** as follows:

$$G(s)H(s) = \frac{K(1 + T_1 s)(1 + T_2 s)\ldots.}{s^j (1 + T_a s)(1 + T_b s)\ldots\ldots}$$

- Where, K = Resultant system gain and **j** =TYPE of the system

  TYPE of the system means number of poles at origin of open loop T.F.

- G(s)H(s) of the system, So, j = 0, TYPE zero system

  j = 1, TYPE one system

  j = 2, TYPE two system

  :

  :

  j = n, TYPE 'n' system

- **Note:** a popular method to asses steady state performance of servo - mechanism or a unit feedback is to find their *error co- efficient* K_p, K_v, and K_a.
- Obviously in order to find these error constants the system must be stable, b/c for unstable system there is no steady state and K_p, K_v, and K_a, are undefined.
- Thus the concept of K_p, K_v, and K_a is applicable only if,
  i) System is represented in its simple form.
  ii) Only if the system is stable

❑ **Consider the input selected as step of magnitude 'A'.**

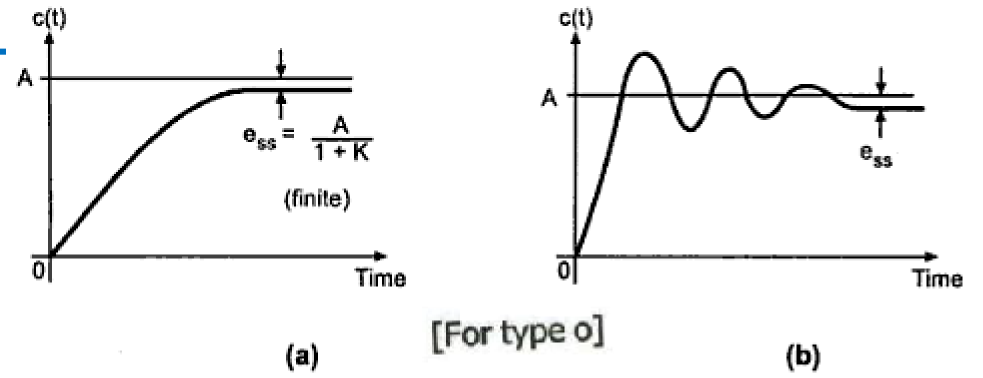
i) TYPE '0'

➢ In this case j = 0. i.e.

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots}{(1+T_a\ s)(1+T_b\ s)\ldots\ldots}$$

For step input $K_p = \lim_{s \to 0} G(s)H(s) = K$

$$\therefore \quad e_{ss} = \frac{A}{1+K_p} = \frac{A}{1+K}$$



i.e. TYPE '0' systems follow the step type of input with finite error $\frac{A}{1+K}$ which can be reduced by change in 'A' or 'k' or both as per requirement.

ii) TYPE 1

In this case j = 1

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots}{s(1+T_a s)(1+T_b s)\ldots\ldots}$$

As input is step $K_p = \lim_{s \to 0} G(s)H(s) = \infty$

$$e_{ss} = \frac{A}{1+K_p} = 0$$

iii) TYPE 2

In this case **j = 2**

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots}{s^2(1+T_a s)(1+T_b s)\ldots\ldots}$$

$$\text{As input is step, } K_p = \lim_{s \to 0} G(s)H(s) = \infty$$

$$e_{ss} = \frac{A}{1+K_p} = \frac{A}{\infty} = 0$$

Type of System	Error Coefficients			Error e_{ss} for		
	K_p	K_v	K_a	Step input	Ramp input	Parabolic input
0	K	0	0	$\frac{A}{1+K}$	∞	∞
1	∞	K	0	0	$\frac{A}{K}$	∞
2	∞	∞	K	0	0	$\frac{A}{K}$

Example 1: Find the error coefficient for a system having.

$$G(s)H(s) = \frac{(s+3)}{s(1+0.60s)(1+0.35s)}$$

Solution

$$K_P = \operatorname*{Lt}_{s\to 0} G(s)H(s) = \frac{(s+3)}{s(1+0.60s)(1+0.35s)} = \infty$$

$$K_V = \operatorname*{Lt}_{s\to 0} sG(s)H(s) = \frac{(s+3)}{(1+0.60s)(1+0.35s)} = 3$$

$$K_a = \operatorname*{Lt}_{s\to 0} s^2G(s)H(s) = \frac{s(s+3)}{(1+0.60s)(1+0.35s)} = 0$$

Example 2: Find K_P, K_V, K_a for the system having

$i)$ $$G(s) = \frac{10}{s^2} \quad and \quad H(s) = 0.7$$

$ii)$ $$G(s) = \frac{5}{s^2+3s+5} \quad and \quad H(s) = 0.6$$

$iii)$ $$G(s) = \frac{10(1+0.5s)(1+0.8s)}{s^2(s^2+3s+5)} \quad and \quad H(s) = 0.8$$

Example 2, Solutions: ***i***) $G(s) = \dfrac{10}{s^2}$ *and* $H(s) = 0.7$

$$K_P = \underset{s\to 0}{\text{Lt}}\, G(s)H(s) = \frac{7}{s^2} = \infty$$

$$K_v = \underset{s\to 0}{\text{Lt}}\, sG(s)H(s) = \frac{7}{s} = \infty$$

$$K_a = \underset{s\to 0}{\text{Lt}}\, s^2G(s)H(s) = \frac{7s^2}{s^2} = 7$$

ii) $K_P = \underset{s\to 0}{\text{Lt}}\, G(s)H(s) = \dfrac{3}{5}$

$$K_v = \underset{s\to 0}{\text{Lt}}\, sG(s)H(s) = 0$$

$$K_a = \underset{s\to 0}{\text{Lt}}\, s^2G(s)H(s) = 0$$

iii) $K_P = \underset{s\to 0}{\text{Lt}}\, G(s)H(s) = \infty$

$$K_v = \underset{s\to 0}{\text{Lt}}\, sG(s)H(s) = \infty$$

$$K_a = \underset{s\to 0}{\text{Lt}}\, s^2G(s)H(s) = \frac{8}{5}$$

Exercise: 1. *A unity feedback system has* $G(s) = \dfrac{40(s+2)}{s(s+1)(s+4)}$

Determine (i) Types of the system (ii) All error coefficients

(iii) Error for ramp input with magnitude 4.

Exercise: 2. *For a unity feedback system has* $G(s) = \dfrac{20(s+2)}{s^2(s+1)(s+5)}$

Determine (i) Types of the system (ii) All error coefficients

(iii) Steady state error for step input $1+3t+t^2/2$.

Solution : 2 *To determine type of system arrange G(s)H(s) in* ***time constant form***

(i) $G(s)H(s) = \frac{20(s+2)}{s^2(s+1)(s+5)}$

$$= \frac{8(1+0.5s)}{s^2(s+1)(1+0.2s)} \quad comparing\ this\ with\ standard\ form$$

$$Since\ \mathrm{j} = 2 \quad this\ is\ a\ type\ 2\ system$$

*(**ii**) Error coefficients:*

$$K_P = \infty,$$

$$K_v = \infty \quad \text{and} \quad K_a = 8$$

. *(iii) Since the input is non − standard, let us use the formula*

$$e_{ss} = \underset{s\to 0}{\text{Lt}}\left[\frac{sR(s)}{1+G(s)H(s)}\right]$$

$$Given: r(t) = 1 + 3t + \frac{t^2}{2}$$

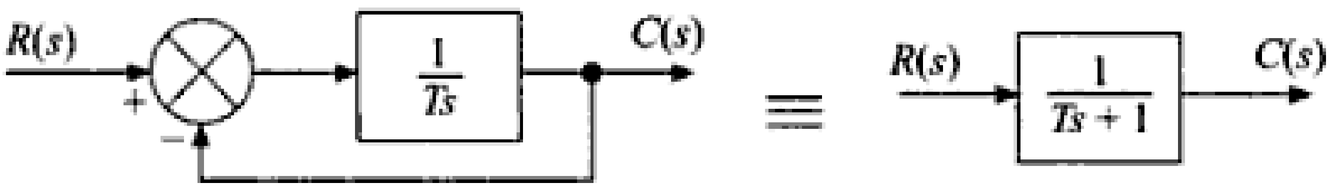
$$R(s) = \frac{1}{s} + \frac{3}{s^2} + \frac{1}{s^3} = \frac{s^2 + 3s + 1}{s^3}$$

$$e_{ss} = \underset{s\to 0}{\text{Lt}}\left[\frac{sR(s)}{1+G(s)H(s)}\right] = \underset{s\to 0}{\text{Lt}}\frac{(s^2+3s+1)}{s^2 + \frac{8(1+0.5s)}{(1+s)(1+0.2s)}}$$

$$e_{ss} = \frac{1}{8} = 0.125$$

Analysis of First Order System

- **Order:** order of system is the ***highest power of 's' in the denominator*** of a closed loop transfer function.
- Fig.(a) shows a first – order system.



(a) (b)

- From Fig. (a), we get,

$$G(s) = \frac{1}{Ts} \quad and \quad H(s) = 1$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{1}{1 + Ts} \qquad Equ.2$$

$$Now, \qquad G(s)H(s) = \frac{1}{sT}$$

- The pole of G(s)H(s) is at s = 0 and it is Type 1. It should not have steady-state for step input.

Unit- Step Time Response of First – order system

- For the Unit- step input, R(s) = 1/s. So, the output response is given by

Using Equ. 2, $$C(s) = \frac{1}{1+sT}R(s) = \frac{1}{s(1+sT)}$$

- Expanding C(s) into partial fractions gives,

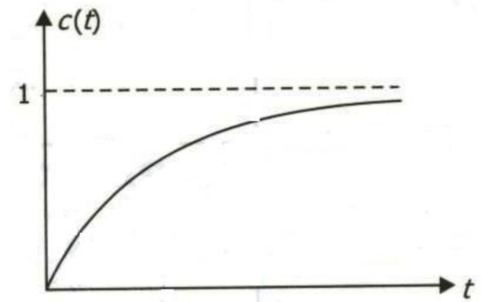
$$C(s) = \frac{1}{s(1+sT)} = \frac{\frac{1}{T}}{s(s+\frac{1}{T})} = \frac{1}{s} - \frac{1}{(s+\frac{1}{T})}$$

$$\frac{1}{s+a} = e^{-at}$$

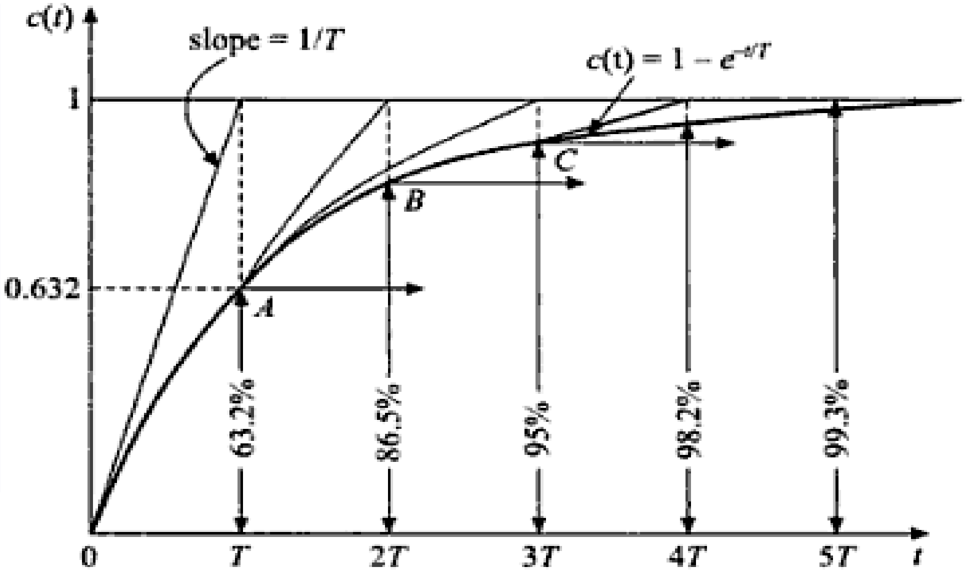
- Taking the inverse LT $C(t) = 1 - e^{-\frac{t}{T}} \quad t \geq 0$

➢ This states that the output rises exponentially from zero value to the final value of unity.

- The plot of *C(t)* versus t is,



- One important characteristic of such an exponential response curve c(t) as shown in Fig below is that at t = T, the value of c(t) is 0.632, or the response of c(t) has reached 63.2% of its total change.



Unit-step response of a first-order system

- The initial slope of the curve at t = 0 is given by $\left.\frac{dc}{dt}\right|_{t=0} = \frac{1}{T} e^{-t/T}\Big|_{t=0} = \frac{1}{T}$

- The error response of the system is given by; $e(t) = r(t) - c(t) = e^{-\frac{t}{T}}$
- For t ≥ $4T$, the response remain within 2% of the final value.
- Where T is known as the time constant of the system

Closed Loop Poles of First Order System

- The closed loop transfer function of a system is given by,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

- The equation which gives poles of system is called *characteristic equation which is,*

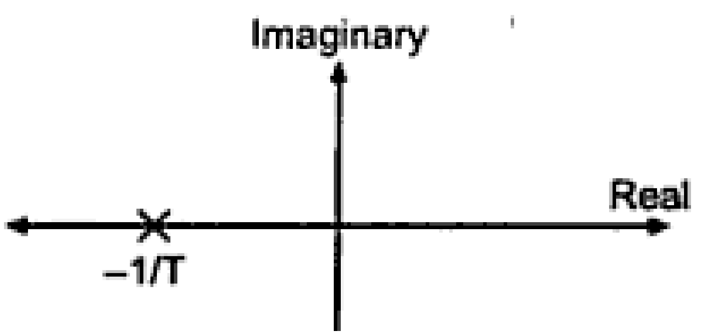
$$1 + G(s)H(s) = 0$$

$$i.e. \qquad 1 + Ts = 0$$

- Since closed loop poles are the root of characteristic equation, so for first order system there is only one closed pole i.e.

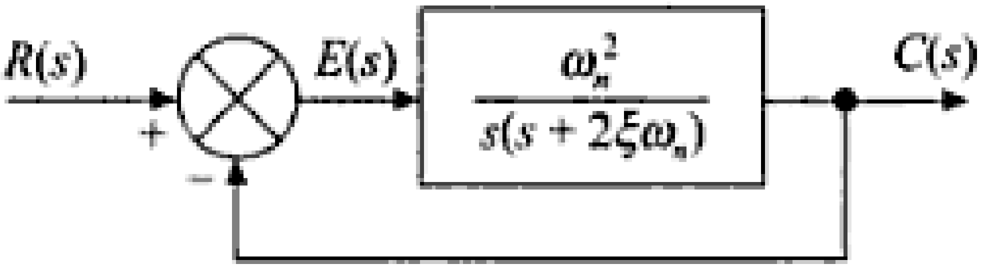
$$s = -\frac{1}{T}$$

- The pole - zero plot is as shown,



Analysis of Second Order System

- Fig. below shows a general second- order system.



Second-order system.

- The closed-loop transfer function C(s)/H(s) of the system is given by,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

- Where, $\xi =$ *Damping ratio*. *Denoted by Greek symbol* *Zeta*.

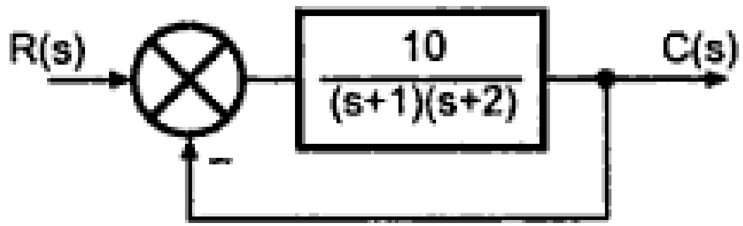
  $\omega_n =$ *Undamped natural frequency (rad/sec)*

And $\quad s^2 + 2\xi\omega_n s + \omega_n^2 = 0 \quad$ *is called characteristic equation*

- The time response of any system is characterized by the root of the denominator.

- **Note:** In practice it is not necessary that numerator must be always ω_n^2. It may be other constant or polynomial of 's' but denominator ***middle term coefficient and last term coefficient*** reflect '$2\xi\omega_n$' and 'ω_n^2' of the system respectively.
- Hence, always **denominator** (not numerator) of a T.F. must be compared with the standard form $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$ to decide the value of ξ and ω_n of the system.

- **Example:**



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{10}{(s+1)(s+2)}}{1+\frac{10}{(s+1)(s+2)}} = \frac{10}{s^2+3s+12}$$

$$\therefore \quad \omega_n^2 = 12 \qquad \text{i.e. } \omega_n = \sqrt{12} \text{ rad / sec}$$

$$\text{While} \quad 2\xi\,\omega_n = 3 \qquad \therefore \ \xi = \frac{3}{2\sqrt{12}} = 0.433$$

Effect of ξ on Second Order System

$$Damping\ ratio(\xi) = \frac{Actual\ damping}{Critical\ damping}$$

- Consider input applied to the standard second order is ***Unit step.***

$\therefore$ $$R(s) = 1/s$$

While $$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\Rightarrow \quad C(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

Finding the roots of the equation $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$

i.e. $$\frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

i.e. $$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

We can write, $$C(s) = \frac{\omega_n^2}{s\left(s + \xi\omega_n + \omega_n\sqrt{\xi^2 - 1}\right)\left(s + \xi\omega_n - \omega_n\sqrt{\xi^2 - 1}\right)}$$

- Now nature of these roots is dependent on damping ratio. Consider the following case:

(i) Case 1: $\xi = 0$ (Undamped)

- The roots are , $s_{1,2} = \pm j\omega_n$

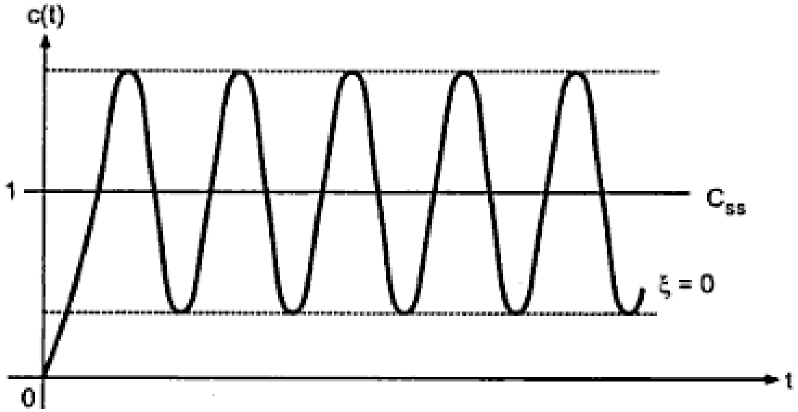
  i.e. **complex conjugates with zero real part.** i.e. purely imaginary.

$$C(s) = \frac{\omega_n^2}{s(s^2+\omega_n^2)} = \frac{1}{s} - \frac{s}{s^2+\omega_n^2} \qquad C(s) = \frac{A}{s} + \frac{Bs + C}{s^2+\omega_n^2}$$

$$c(t) = 1 - \cos \omega_n t$$

- The response is purely oscillatory, oscillating with constant frequency and amplitude.

- So, this response does not die out with time.
- The figure shows the plot of c(t) versus t.



(ii) Case 2: $0 < \xi < 1$ (Underdamped)

- The roots are, $s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

  As $\xi < 1$, the term $\sqrt{\xi^2 - 1}$ is written as $j\sqrt{1-\xi^2}$

  Hence roots are complex conjugates with negative real part.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^{\,2}}{s^2 + 2\xi\omega_n s + \omega_n^{\,2}} = \frac{\omega_n^{\,2}}{(s + \xi\omega_n + j\omega_d)(s + \xi\omega_n - j\omega_d)}$$

Where, $\omega_d = \omega_n\sqrt{1-\xi^2}$ $(rad/\sec)$ *is called damped natural frequency*

For a unit-step input, $R(s) = 1/s$.

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

$$= \frac{1}{s} - \frac{s + 2\xi\omega_n}{(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{1}{s} - \frac{s + 2\xi\omega_n}{[(s + \xi\omega_n)^2 + \omega_n^2 - \omega_n^2\xi^2]}$$

$$C(s) = \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{\omega_d}\cdot\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi}{\sqrt{1-\xi^2}}\cdot\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$\mathcal{L}^{-1}\left[\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}\right] = e^{-\zeta\omega_n t}\cos\omega_d t$$

$$\mathcal{L}^{-1}\left[\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right] = e^{-\zeta\omega_n t}\sin\omega_d t$$

Taking the inverse Laplace transform $\mathcal{L}^{-1}[C(s)] = c(t)$

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\omega_d t$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right)$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}(\sin\theta\cos\omega_d t + \cos\theta\sin\omega_d t)$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)$$

$$\theta = tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) \text{ rad}$$

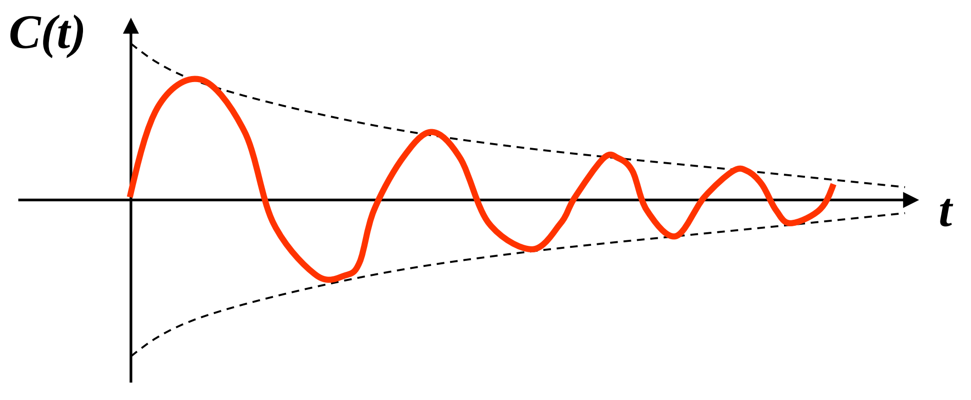
$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left(\omega_n \sqrt{1-\xi^2}\, t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right) \quad \text{for } t \geq 0$$

The error signal for this system is the difference between the input and the output and is

$$e(t) = r(t) - c(t)$$

$$= e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right), \quad \text{for } t \geq 0$$

- The error signal exhibit a damped sinusoidal oscillation. At steady-state, or **at t = ∞**, **no error** exists b/n the output and the input



(ii) Case 3: $\xi = 1$ (Critically damped)

- For critically damping the roots are ***real, equal and negative***.

$$i.e.\ s_1 = s_2 = -\xi\omega_n$$

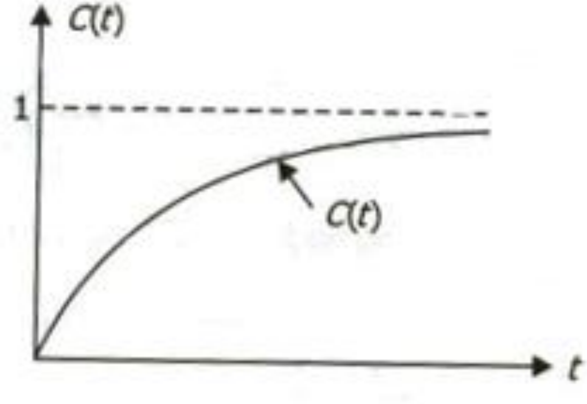
For step input,

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)(s+\omega_n)} = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}$$

$$C(t) = 1 - \omega_n e^{-\omega_n t} t - e^{-\omega_n t}$$

$$= 1 - (1+\omega_n t)e^{-\omega_n t}$$

This is purely exponential.



Critically damped

(iv) Case 4: ξ > 1 (Overdamped

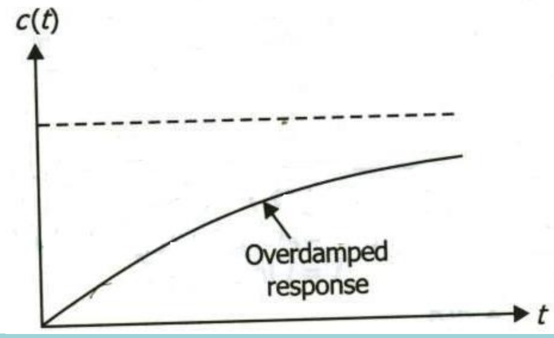
- In this case, the two poles or root of C(s)/R(s) are ***negative real and unequal.*** ***i.e.*** $s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$

- For a unit-step input, R(s) = 1/s and C(s) can be written

$$C(s) = \frac{\omega_n^2}{s(s + \xi\omega_n + \omega_n\sqrt{\xi^2 - 1})(s + \xi\omega_n - \omega_n\sqrt{\xi^2 - 1})}$$

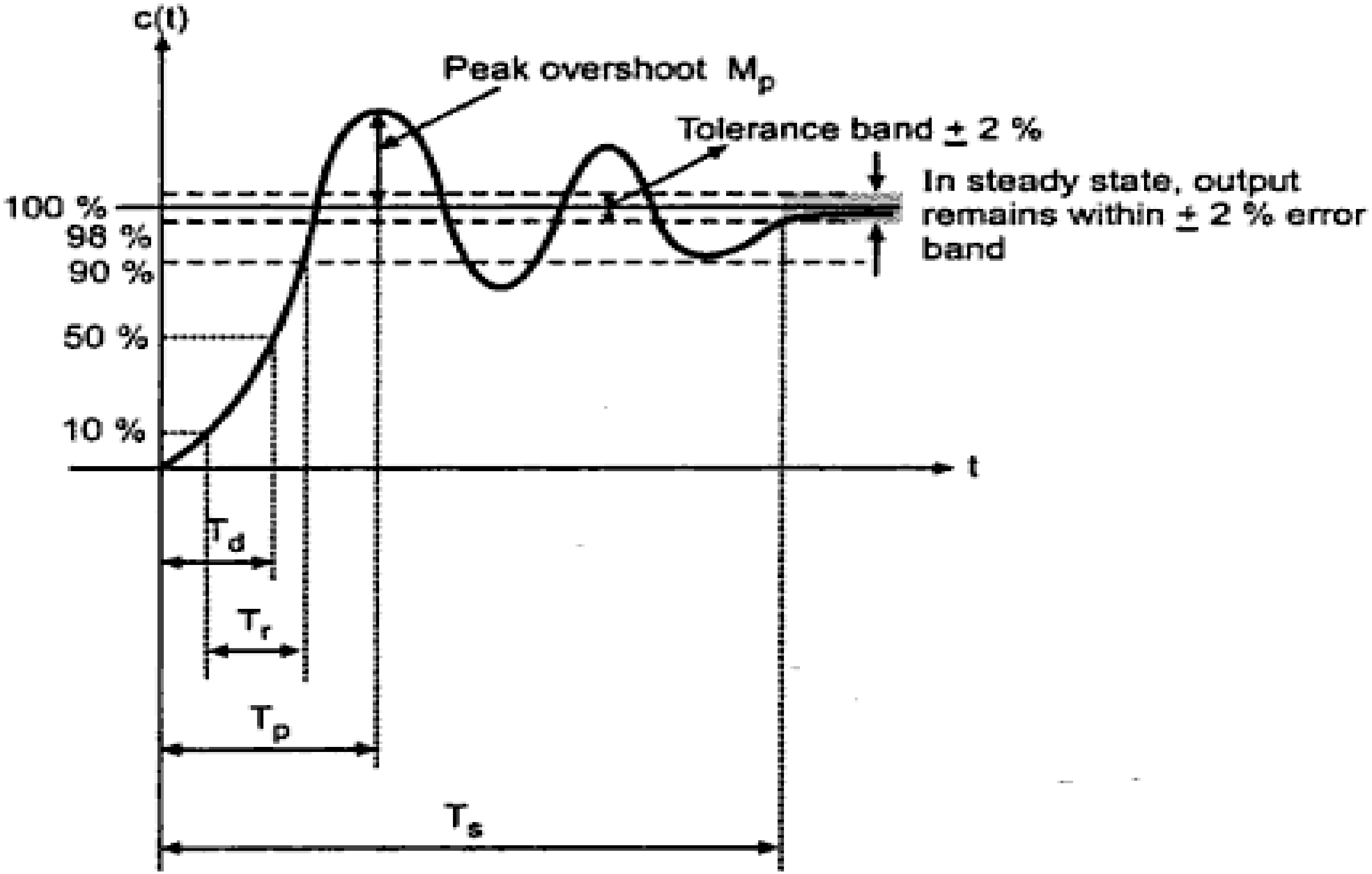
$$= \frac{1}{s} - \left[\frac{A}{s + \xi\omega_n + \omega_n\sqrt{\xi^2 - 1}} + \frac{B}{s + \xi\omega_n - \omega_n\sqrt{\xi^2 - 1}}\right]$$

$$C(t) = 1 - \left[Ae^{-\left(\xi\omega_n + \omega_n\sqrt{\xi^2 - 1}\right)t} - e^{-\left(\xi\omega_n - \omega_n\sqrt{\xi^2 - 1}\right)t}\right]$$



Transient Response Specification

- The actual output behavior according to the expression derived can be as shown.



- Let us define the various time response specifications referring to the above Figure.

1) Delay Time T_d : it is the time required for the response to reach 50% of the final value in the first attempt.

It is given by,

$$T_d = \frac{1 + 0.7\xi}{\omega_n}$$

2) Rise Time T_r: It is the time required for the response to rise from 10% to 90% of the final value for overdamped system and 0 to 100% of the final value for underdamped systems. The *rise time is reciprocal of the slope of the response at the instant*, the response is equal to 50% of the final value. It is given by;

$$\theta = tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) \text{ rad}$$

$$T_r = \frac{\pi - \theta}{\omega_d} \quad Sec$$

$$Where;\theta \;\; must\ be\ in\ radians$$

3) Peak Time T_p: It is the time required for the response to reach its peak value. It is also defined as the *time at which response under goes the first overshoot which is always peak overshoot.*

$$T_p = \frac{n\pi}{\omega_d} = \frac{n\pi}{\omega_n\sqrt{\xi^2 - 1}} \quad Sec$$

- *The first overshoot is obtained for n =1 & the second overshoot is obtained for n = 2.*

4) Peak Overshoot M_p: It is the largest error b/n reference input and output during the transient period.

- It also defined as, the *amount by which output overshoot its reference steady state value* during the first overshoot

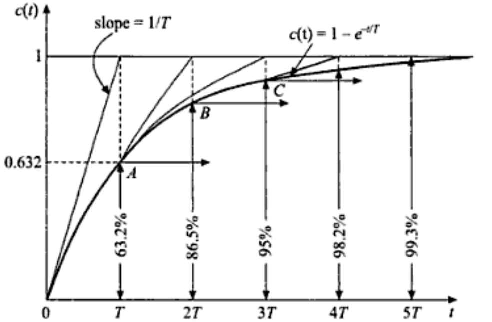
$$M_p = \left\{C(t)\,\middle|\,_{t = T_p}\right\} - 1 \qquad Or \qquad \%M_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}\, x\ 100$$

5) Setting Time T_s: it is the time required for the response to decrease and stay within specified percentage of its final value (within tolerance band).

$$Time\ constant\ of\ system = \frac{1}{\xi\omega_n} = T$$

$$T_s = \frac{4}{\xi\omega_n} = 4 \times \mathrm{T} \qquad \ldots \text{for tolerance band of} \pm 2\%\ of\ \text{steady state}$$

- **Key point:** 1 Time constant 'T' is the time required by the system output to reach 63.2% of its final value during the first attempt.
- **Reading Assignment:** Derivation of Time Domain Specification



Unit-step response of a first-order system

Example 1) For a system having $G(s) = \dfrac{25}{s(s+10)}$ and unity feedback. Find

$$(i)\omega_n \quad (ii)\,\xi \quad (iii)\,\omega_d \quad (iv)\,T_p \text{ and } (v)\,M_P$$

Solution: $G(s) = \dfrac{25}{s(s+10)}$ and $H(s) = 1$

$$G(s)H(s) = \frac{25}{s(s+10)} \qquad and \qquad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{25}{s^2+10s+25}$$

Where, Standard form of: $\dfrac{C(s)}{R(s)} = \dfrac{{\omega_n}^2}{s^2 + 2\xi\omega_n s + {\omega_n}^2}$

Comparing we get: ${\omega_n}^2 = 25 \text{ or } \omega_n = 5\dfrac{rad}{sec} \quad and \;\; 2\xi\omega_n = 10$

Therefore: $2 \times \xi \times 5 = 10$

$$\xi = 1 \quad Hence, the\ system\ is\ critically\ damped.$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 0$$

· $Again, \quad TP = \frac{\pi}{\omega_d} = \frac{\pi}{0} = \infty \quad Therefore, there\ is$ no peak

$$M_p = 100\ e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} = 100\ e^{-\frac{\pi}{\sqrt{1-1^2}}} = 0$$

Example 2) A system has $G(s) = \frac{20}{s^2 + 5s + 5}$ & H(s) = 1. Find $(i)\omega_n$, *(ii)* ξ, *(iii)* ω_d

$(iv)\ T_d \quad (v) T_r \quad (vi) T_p \quad v(ii)\ M_p\ and \quad (viii)\ T_S$

Solution,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{20}{s^2 + 5s + 5}}{1 + \frac{20}{s^2 + 5s + 5}} = \frac{20}{s^2 + 5s + 25}$$

$(i)\ Since, {\omega_n}^2$ must be compared with thedenominator of $C(s)/R(s)$ only:

$${\omega_n}^2 = 25\ or\ \omega_n = 5 rad/sec \qquad and$$

$(ii)\ \ 2\xi\omega_n = 5 \qquad or\ \xi = 0.5$

$(iii)\ \omega_d = \omega_n\sqrt{1 - \xi^2} = 5x\sqrt{1 - 0.5^2} = 4.33\ rad/sec$

$$(vi) \qquad T_p = \frac{\pi}{\omega_d} = \frac{3.14}{4.33} = 0.725\ sec$$

$$(v) \qquad M_P = 100\ e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} = 100\ e^{-\frac{0.5\pi}{\sqrt{1-0.5^2}}} = 16.32\%$$

$$(viii) \quad T_S = \frac{4}{\xi\omega_n} = \frac{4}{0.5*5} = 1.6\ sec$$

$$(iv) \quad T_d = \frac{1+0.7\xi}{\omega_n} = \frac{1+0.7x0.5}{5} = 0.27\ sec$$

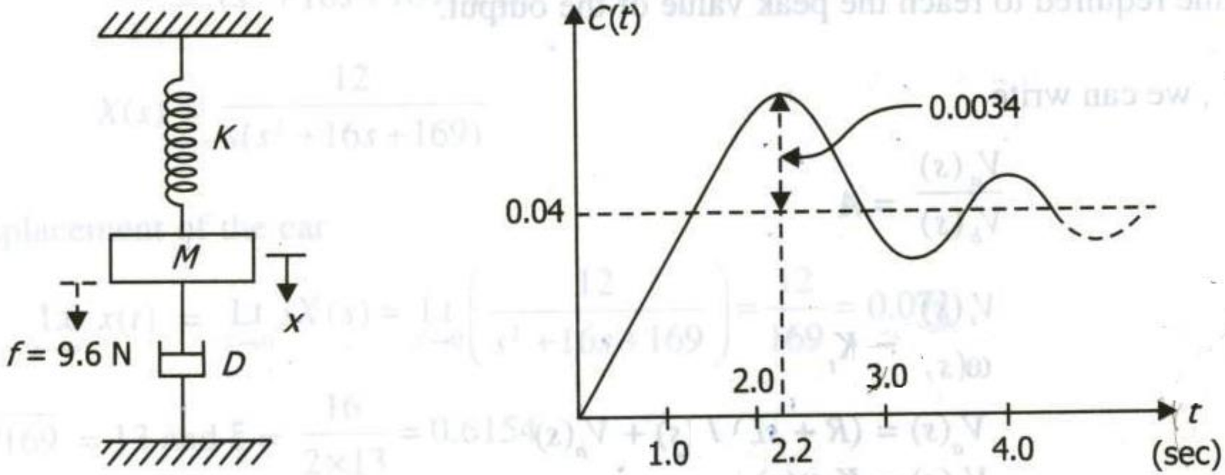
$$(v) \qquad T_r = \frac{\pi-\theta}{\omega_d} \qquad Where,\ \theta = tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = tan^{-1}\left(\frac{\sqrt{1-0.5^2}}{0.5}\right) = 1.0472rad$$

$$\therefore \quad T_r = \frac{\pi-1.0472}{4.33} = 0.484\ sec$$

Key point: *Steady state output or final value of the system,*

$$\lim_{t\to\infty} C(t) = \lim_{s\to 0} sC(s)$$

Calculate M, D and K. The system was initially relaxed.



(a) (b)

Solution

From Fig. E8.12(a), we can write

$$f(t) = M\frac{d^2x}{dt^2} + D\frac{dx}{dt} + Kx$$

Taking Laplace transform, we get

$$F(s) = s^2MX(s) + sDX(s) + KX(s) = (s^2M + sD + K)X(s)$$

or

$$X(s) = \frac{F(s)}{(s^2M + sD + K)}$$

Here $f(t) = 9.6$. Therefore

$$F(s) = \frac{9.6}{s}$$

Therefore

$$X(s) = \frac{9.6}{s(s^2M + sD + K)} = \frac{\frac{9.6}{M}}{s\left(s^2 + s\frac{D}{M} + \frac{K}{M}\right)}$$

Here $\omega_n^2 = \frac{K}{M}$

or $\omega_n = \sqrt{\frac{K}{M}}$

and $2\xi\omega_n = \frac{D}{M}$

$$\xi = \frac{D}{M} \times \frac{1}{2\omega_n} = \frac{D}{2\sqrt{MK}}$$

From Fig. E8.12. (b), the final value is 0.04. Therefore

$$\underset{s \to 0}{\text{Lt}}\, sX(s) = \underset{s \to 0}{\text{Lt}} \frac{9.6}{s^2M + sD + K} = \frac{9.6}{K}$$

$$\frac{9.6}{K} = 0.04$$

$$K = 240 \text{ N/m.}$$

From Fig. E8.12 (b), the overshoot is 0.0034 for final value 0.04. Therefore

$$\% M_p = \frac{0.0034}{0.04} \times 100 = 8.5$$

Since, we can write

$$\% M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} \times 100$$

$$8.5 = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} \times 100$$

or

$$0.085 = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$$

or

$$-2.465 = -\frac{\pi\xi}{\sqrt{1-\xi^2}}$$

or

$$\xi = 0.6175$$

Since $T_p = 2.2$ sec, we can write

$$\frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 2.2$$

$$\omega_n = \frac{3.14}{2.2 \times \sqrt{1-0.6157^2}} = 1.8145 \text{ rad/sec}$$

$$\omega_n = \sqrt{\frac{K}{M}} = 1.8145$$

$$M = \frac{K}{1.8145^2} = \frac{240}{1.8145^2} = 72.89\text{kg}$$

$$\xi = \frac{D}{2\sqrt{MK}},$$

$$D = 2\xi\sqrt{MK} = 2 \times 0.6145 \times \sqrt{72.89 \times 240} = 162.55 \text{ N/m/sec}$$