CHAPTER SIX

UNSTEADY OPEN CHANNEL FLOW

6.1 INTRODUCTION

Unsteady open channel flow is a flow at which the depth of flow changes with time under consideration (i.e. $dy/dt \neq 0$).The flow of water in rivers, canals, reservoirs, lakes, pools, and free-surface flow in storm water drains, conduits, pipes , galleries, tunnels and culverts, in which the velocities change with time, is defined as unsteady flow (non -permanent, non -stationary or time-variable free-surface water flow).Flow in natural channel is always unsteady. When the discharge changes slowly with time is unsteady flow and is approximated by steady flow.

Unsteady flow in open channels differs from that in closed conduits in that the existence of a free surface allows the flow cross section to freely change, a factor which has an important influence on the rate of transient change propagation. Unsteady open channel flow is encountered in flood flow in rivers, in headrace canals supplying hydropower stations, in river estuaries, and so on.

Unsteady flow occurs where flow parameters vary with time at fixed point.

Example: - Oscillatory Sea Waves,
Predicting Waver Levels in Rivers in Flood,
Dam Break Flood Waves,
Surges due to gate operation, e.g. in irrigation canal.

WAVES –Definitions

"a wave is a temporal variation in the water surface which is propagated through a fluid medium"
the celerity of a wave is the speed of propagation of the disturbance relative to the fluid.

6.2 Waves Classification

Capillary — due to surface tension
Elastic — due to fluid compression

Gravity waves

a. **Oscillatory Wave** [e.g. sea waves) Zero net mass transport
b. **Translatory Waves** [e.g. Flood Waves] net transport of fluid in direction of wave

Solitary Wave	**Wave Train**
Rising limb,	Created by
Single peak	Sequence of
Followed & preceded	Several Waves
by steady flow	

Further definitions................................

Downstream Wave - moves down channel slope
Upstream Wave - moves up channel slope
Increase in level from steady flow - **<u>positive wave</u>**
Decrease in level from steady flow - **<u>negative wave</u>**
Monoclonal - single faced
Two faced - symmetrical or asymmetrical

Deep water waves ~ only surface layers disturbed

$$\frac{Depth}{Wavelength} = \frac{y}{L} > 0.5$$

Shallow waver waves ~ Entire depth disturbed (bottom effect)

$$\frac{y}{L} < 0.05$$

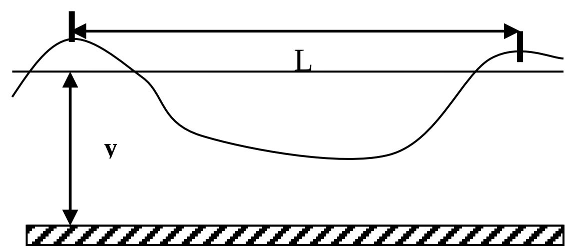
WAVE CELERITY, c

This material will not include rigorous proofs of equations, for these proofs are based on the methods of theoretical hydrodynamics, which are beyond the scope of this course. Complete treatment of the proofs is given in a number of advanced texts, for example Open Channel flow by Henderson.

In order to distort a liquid surface it is necessary to work against both gravity (by lifting liquid from below mean water level to above the level) and surface tension (by elongating free surface).

Surface tension has very little effect unless the wavelength is quite small and the effect of viscosity is small, it is distinctly second order effect. Airy theory neglects viscosity (viscous damping of oscillatory waves) and surface tension. For small amplitude 2-D flows:

$$c^2 = \frac{gL}{2\pi} \tanh \frac{2\pi y}{L}$$



Where:

L = the wave length

C = celerity (wave velocity)

The behavior of the tanh function shows that when:

$$\frac{y}{L} > 0.5\ (l \arg e)$$

$$than \left(\frac{2\pi y}{L}\right) \to 1$$

$\therefore c^2 = gL/2\pi$ ~ ***Deep water wave equation***

y/L < 0.05 (small), tanh $\left(\frac{2\pi y}{L}\right) \to \frac{2\pi y}{L}$

$$c^2 = \frac{gL}{2\pi} . \frac{2\pi y}{L} = gy$$

$ie \quad c = \sqrt{gy}$ ~ Shallow ***water***

This is known as the **LAGRANGE equation**

6.3 Basic Equations of Unsteady Flow

As in the case of closed conduits the basic equations are derived from continuity and momentum considerations. In deriving these equations the following assumptions are made:

1. Hydrostatic pressure prevails at every point in the channel.
2. Velocity is uniformly distributed over each cross-section.
3. The slope of the channel bed is small and uniform.
4. The frictional resistance is the same as for steady flow.

Fig 6.1 defines a control volume and the dimensional parameters used to develop the Continuity equation.

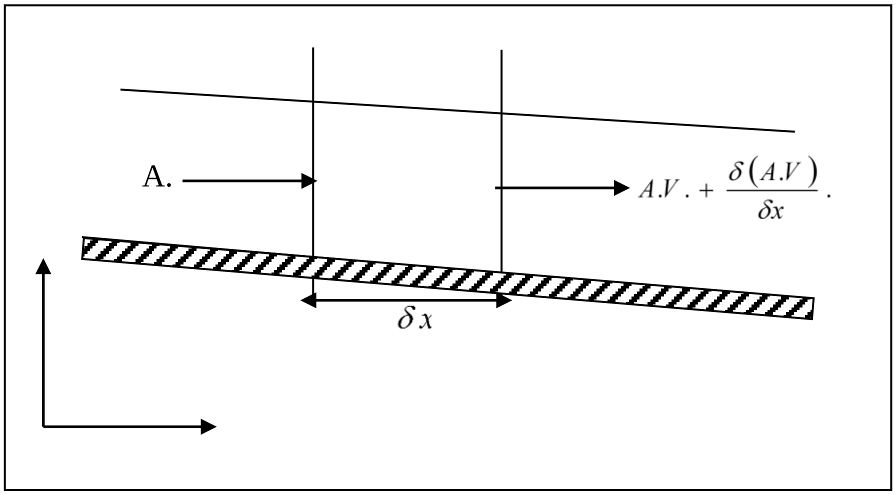


Figure 6.1: Reference diagram for the continuity equation

The continuity equation balances mass inflow and mass outflow with the rate of change of the contained mass within the control volume:

In time dt: Inflow - Outflow = Change in mass of fluid in CV

$$\rho vAdt + \rho q_1 dtdx - \rho\left(v + \frac{\partial v}{\partial x}dx\right)\left(A + \frac{\partial A}{\partial x}dx\right)dt = \rho\frac{\partial A}{\partial t}dtdx$$

Dividing across by ρ dt dx and neglecting higher order terms:

$$A\frac{\partial v}{\partial x} + v\frac{\partial A}{\partial x} + \frac{\partial A}{\partial t} = q_1$$

Equation this term can also be written in the form

$$\frac{\partial}{\partial x}(vA) + \frac{\partial A}{\partial t} = q_1$$

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q_1$$

For a rectangular channel with zero lateral inflow,) $A\frac{\partial v}{\partial x} + v\frac{\partial A}{\partial x} + \frac{\partial A}{\partial t} = q_1$ simplifies to

$$y\frac{\partial v}{\partial x} + v\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = 0$$

Hence $\frac{\partial}{\partial x}(vy) + \frac{\partial y}{\partial t} = 0$

$$\frac{\partial q}{\partial x} + \frac{\partial y}{\partial t} = 0$$

Where: q is the discharge per unit width of channel

The momentum equation relates net force to momentum change:

$$F - \left(F + \frac{\partial F}{\partial x} dx \right) + w\sin\theta - \tau_o P_r dx = \upsilon A dx \frac{dv}{dt}$$

Where the pressure force F = ρ g A $\bar{y}$; the weight force W = ρ g Ad; the wall shear stress τ_o = ρ g$R_h S_f$; Pr is the perimeter length. This equation may therefore be expressed in terms of more basic flow parameters:

$$-\rho g \frac{\partial}{\partial x}\left(A\bar{y}\right)dx + \rho g A S_o dx - \rho g A S_f dx = \rho A dx \frac{dv}{dt}$$

Dividing across by ρ g dx:

$$-\frac{\partial}{\partial x}(A\bar{y}) + AS_o - AS_f = \frac{A}{g}\left(v\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} \right)$$

For a rectangular section this simplifies to

$$-\frac{\partial y}{\partial x} + S_o - S_f = \frac{v}{g}\frac{\partial v}{\partial x} + \frac{1}{g}\frac{\partial v}{\partial t}$$

$$v\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} + g\left(S_f - S_o\right) = 0$$

The terms of the momentum equation have the dimension of acceleration or force per unit mass. The first two terms on the left hand side are the fluid acceleration terms, g $\partial y / \partial x$ represents the pressure force component, gS_f and gS_o represent the friction and gravity force components, respectively. The forms of the continuity and momentum equations, represented in the above equations respectively, are known as the ***Saint Venant equations***; they relate the dependent variables y and v to the independent space and time variables x and t, respectively.

Solutions of the St. Venant equations

- a complicated matter, even for rectangular channels.
- general solutions are only possible by numerical methods
- some remarks

3 principal numerical methods

- finite differences method (FDM)
- characteristic methods (MOC)
- finite elements method (FEM)

- It is not possible to solve it analytically.

1. Solution by the Characteristics Method

For the solution of the corresponding pair of equations for unsteady flow in pipes is applied here. Multiplying the continuity equation by the factor l and adding to the above momentum equation.

$$\left[\frac{\partial v}{\partial x}(\lambda y + v) + \frac{\partial v}{\partial t}\right] + \lambda\left[\frac{\partial y}{\partial t} + \frac{\partial y}{\partial x}\left(v + \frac{g}{\lambda}\right)\right] + g\left(S_f - S_o\right) = 0$$

This partial differential equation can be converted to a total differential equation provided that

$$\frac{dx}{dt} = \lambda y + v = v + \frac{g}{\lambda}$$

$$\text{Or } \lambda y = \frac{g}{\lambda}.$$ hence

$$\lambda = \pm\left(\frac{g}{y}\right)^{0.5}$$

and $$\frac{dx}{dt} = v \pm (gy)^{0.5} = v \pm c$$

Where c is the gravity wave speed; hence l = ±g/c. Thus equation can be written in the equivalent total differential form:

$$\frac{dv}{dt} + \frac{g}{c}\frac{dy}{dt} + g\left(S_f - S_o\right) = 0$$

Subjected to, $$\frac{dv}{dt} = v + c;$$

and $$\frac{dv}{dt} = \frac{g}{c}\frac{dy}{dt} + g\left(S_f - S_o\right) = 0$$

Subjected to

$$\frac{dx}{dt} = v - c$$

Thus, the two partial differential equations have been converted to their characteristic form i.e. linked pairs of ordinary differential equations. On integration of the latter over the time interval Δt we get a pair of C^+ characteristic equations and a pair of C^- characteristic equations:

$$v_p - v_R + g\int_{yR}^{yP}\frac{1}{c}dy + \int_{tR}^{Tp} g\left(S_f - S_o\right)dt = 0$$

C^+

$$X_p - X_R = \int_{tR}^{tP}(v + c)dt$$

$$v_p - v_s - g\int_{ys}^{yp}\frac{1}{c}dy + \int g\left(S_f - S_o\right)dt = 0$$

C^-

$$X_p - X_s = \int_{ts}^{tp}(v - c)dt$$

Where v_R and v_S are the interpolated values of v at x_R and x_S, respectively, as illustrated on the x-t plane on Fig 6.2. The foregoing integrations may be approximated to a first order accuracy by assigning their known values to v, c and Sf, giving the characteristic equations the following format:

$$v_p - v_R + \frac{g}{c_R}\left(y_p - y_R\right) + g\left(S_R - S_o\right)\Delta t = 0$$ --1

C^+

X_p-X_R= $(v_R + c_R)\,\Delta t$ ---2

$$v_p - v_s + \frac{g}{c_s}(y_p + y_s) + g\left(S_s - S_o\right)\Delta t = 0$$ --3

C^- , X_P-X_s= $(v_s$-$C_s)\,\Delta t$ ---4

Where: S_R and S_S are values of Sf at R and S, respectively.

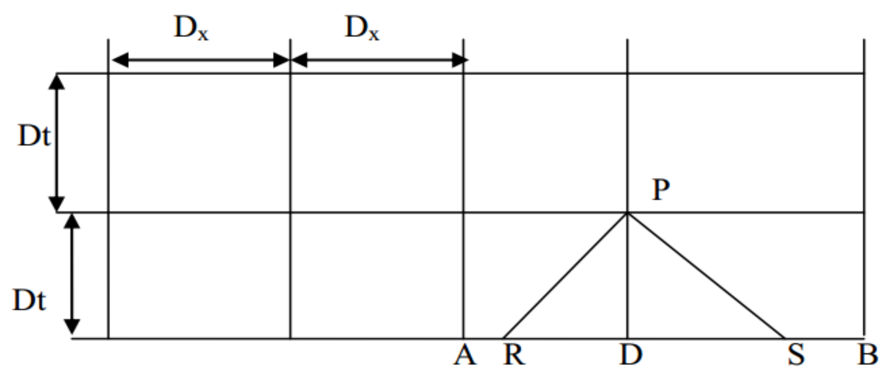


Figure 6.2: The x – t plane

The parameter values at R are found by linear interpolation in the interval AD and the parameter values at S are found by linear interpolation in the interval DB. Referring to the interval AD in Fig 6.3

$$X_D - X_R = \Delta t(v_R + C_R)$$

$$\frac{v_D - v_r}{v_D - v_A} = \frac{X_D - X_R}{X_D - X_A}$$

$$\frac{C_D - C_R}{C_D - C_A} = \frac{X_D - X_R}{X_D - X_A}$$

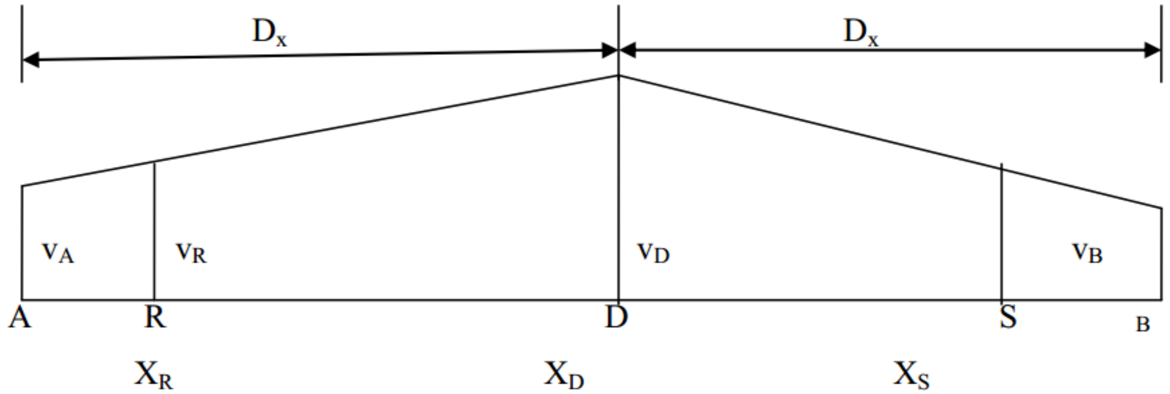


Figure 6.3: Linear interpolation

Replacing x_D by x_P and $(x_D - x_A)$ by Δx, the following are the interpolated values at R:

$$v_R = \frac{v_D - \theta(-v_D C_A + C_D v_A)}{1 + \theta(v_D - v_A + C_D - C_A)} \text{ --5}$$

$$c_R = \frac{c_D - v_R \theta(c_D - c_A)}{1 + \theta(c_D - c_A)} \text{ --6}$$

$$y_R = y_D - \theta(y_D - y_A)(v_R + c_R) \text{ --7}$$

Where $\theta = \Delta t/\Delta x$. Interpolated values are similarly established at S on the negative characteristic side of D:

$$X_D - X_S = \Delta t\ (v_s - c_S)$$

$$\frac{v_D - v_S}{v_D - v_B} = \frac{X_S - X_D}{X_B - X_D}$$

$$\frac{c_D - c_S}{c_D - c_B} = \frac{X_S - X_D}{X_B - X_D}$$

Solution of these equations gives the following interpolated values at S:

$$v_s = \frac{v_D + \theta(-v_D c_B - c_D v_B)}{1 - \theta(v_D - v_B - c_D + c_B)} \qquad (8)$$

$$c_S = \frac{c_D + v_S \theta(c_D - c_B)}{1 + \theta(c_D - c_B)} \qquad (9)$$

$$y_s = y_D + \theta(y_D - y_n)(v_s - c_s) \qquad (10)$$

2. Numerical Computation Procedure

The foregoing finite difference formulation of the characteristic form of the unsteady flow equations can be used where there are no abrupt changes in the water surface profile and where conditions are subcritical. The channel length is divided into N reaches, each of length Δx. The corresponding value of the time step Δt is set by the so -called Courant condition:

$$\Delta t \le \frac{\Delta x}{|v| + c}$$

This ensures that the characteristic curves plotted on the x-t plane (Fig 6.2) remain within a single x-t grid. At time zero the values of y and v are known at each channel node point. Their values at internal nodes, at one time interval Δt later, are found by solution of equations (1) and (3) and are as follows:

$$y_p = \frac{1}{c_R 4 + c_S}\left\{ y_S c_R + y_R c_S + c_R c_S \left[\frac{v_R - v_S}{g} - \Delta t(S_R - S_s) \right] \right\}$$

$$v_P = v_R - \frac{g(y_p - y_R)}{c_R} - g\Delta t(S_R - S_o)$$

The updated values of y (y_P) and v (v_P) at the upstream end of the channel are governed by the negative characteristic equations (3) and (4) and the prevailing upstream boundary condition equation, which is typically in the form of a defined variation of either y or Q with time. Solution of equation (4) and the boundary condition equation yields the required values for v_P and y_P.

The new values for v_P and y_P at the downstream end of the channel are found in the same manner as their corresponding values at the upstream end, the defining equations being the positive characteristic equation (5) and the prevailing downstream boundary condition equation.

The foregoing analysis relates to conditions of tranquil flow only, that is, where the Froude number Fr is less than unity. As the flow depth approaches the critical value (Fr = 1), the numerical computation becomes unstable. At critical depth, v = c and hence the negative characteristic on the x-t plane becomes vertical, that is, points S and D are coincident.

Simplification of the St Venant equations: - The Saint Venant equations can be made more amenable to solution by omitting selected terms from the momentum equation. The latter may be written in the form

$$S_o = S_f + \frac{\partial y}{\partial x} + \frac{g}{v}\frac{\partial v}{\partial x} + \frac{1}{g}\frac{\partial v}{\partial t}$$

Henderson has pointed out that the acceleration terms (3rd. and 4th. on the right-hand side of)) are usually two orders of magnitude less than the gravity (S_o) and friction (S_f) terms and one or two orders of magnitude less than the remaining term $\partial y/\partial x$. This suggests that the solution of the simplified equation obtained by dropping the acceleration terms may provide a good approximation to the solution based on the full equations. The resulting simplified momentum equation becomes

$$\frac{dy}{dx} = S_o - S_f$$

On combination with the continuity equation the resulting unsteady open channel flow equation for a rectangular channel has the form

$$\frac{y}{v}\frac{\partial v}{\partial t} + \frac{1}{v}\frac{\partial y}{\partial t} = S_f - S_o$$

A further simplification of the momentum equation is obtained by omission of the dy/dx term (this term represents the unbalanced pressure force component), reducing the momentum equation to its steady uniform flow form:

$$Sf = So$$

Using the Manning expression of friction slope equation S_o becomes

$$S_o = \left(\frac{nQ}{AR_h^{0.67}}\right)^2$$

and hence we can write

$$Q = f(A) and \frac{dQ}{dx} = \frac{dQ}{dA}\frac{dA}{dx} = F(A)\frac{dA}{dx}$$

Combining this form of simplified momentum equation with the continuity equation, where $q_1 = 0$, the resulting open channel unsteady flow equation becomes:

$$F(A)\frac{\partial A}{\partial x} + \frac{\partial A}{\partial t} = 0$$

This equation is known as the kinematic wave equation because the dynamic terms of the momentum equation have been omitted in its development. The solution of equation is clearly of the form:

$$A = \varphi\left(t - \frac{x}{F(A)}\right)$$

Where the form of the function φ is determined by the boundary condition for x=0.