TEXTILE ECONOMICS AND COSTING

CHAPTER 2: UNIFORM SERIES OF PAYMENTS

2.1 Definitions

- A set of payments each of **equal amount** made at **equal intervals of time** is referred to as a ***uniform series or annuity.***
- The interval between successive payments is termed the ***payment period***.
- An ***ordinary uniform series*** is one in which a payment is made at the beginning or end of each interest period.

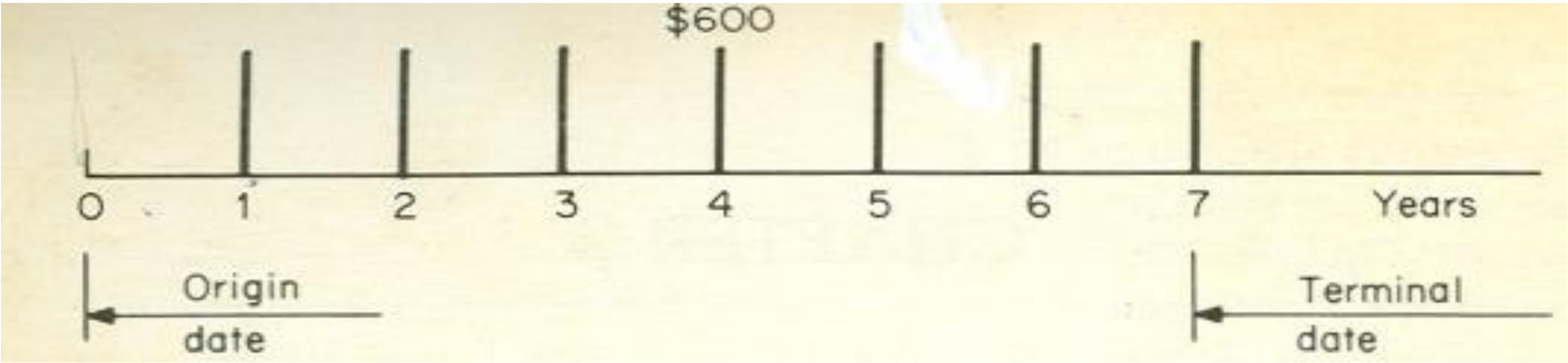


Fig. 2.1 the cash-flow diagram of a uniform series that consists of seven annual payments of $600 each.

Cont.

- ***Origin Date*** of a uniform series is placed one payment period prior to the first payment
- ***Terminal Date*** is placed at the date of the last payment.
- ***Present Worth* of the series** is the value of the entire set of payments at the origin date
- ***Future Worth*** *is* the value at the terminal date
- ***Sinking Fund*** is *a* fund in which deposits of equal amount are made at equal intervals of time,
- If a loan is to be discharged by a set of equal payments made at equal intervals of time, the series of payments are called **amortization** payments and the loan is said to be ***amortized,*** after the completion of the payments

2.2 Calculation of Present Worth & Future Worth

Let :-

A = periodic payment

Pu = present worth of the series

Fu = future worth of the series

n= number of payments

i= interest rate

- Evaluating all payments at the origin date of the series and summing the results, we obtain:

$Pu = A[(l + i)^{-1} + (1 + i)^{-2} + . . . + (1 + i)^{-n}]$. Then:

Cont.

$$P_u = A\frac{1-(1+i)^{-n}}{i} = A\frac{1-1/(1+i)^n}{i}$$

$$F_u = A\frac{(1+i)^n-1}{i}$$

$$(P_u/A,n,i) = \frac{1-(1+i)^{-n}}{i} \quad (2.1a)$$

$$(F_u/A,n,i) = \frac{(1+i)^n-1}{i} \quad (2.2a)$$

Pu=A(Pu/A,n,i)

Fu = A(Fu/A,n,i)

Cont.

E.g.2.1 A fund earns interest at the rate of 6 percent per annum. If the sum of $850 is deposited in the fund at the end of each year for 11 consecutive years, what is the principal in the fund immediately after the last deposit is made?

SOLUTION

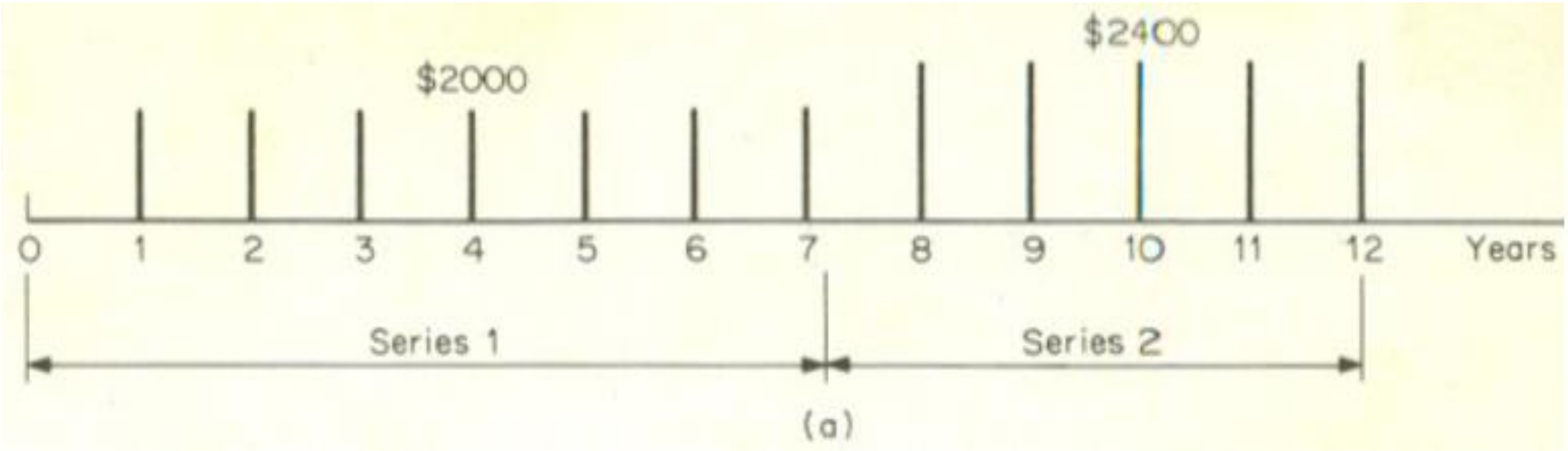
The principal at that date is the future worth of the series:

A = $850 n = 11 i = 6%

$Fu = 850(Fu/A,11,6\%) = 850(14.97164) = \underline{\$12,726}$

Cont.

E.g. 2.2 Deposits were made in a fund at the end of each year for 12 consecutive years. The annual deposit was $2000 for the first 7 years and $2400 for the remaining 5 years. If the interest rate of the fund was 5 percent per annum, what was the principal in the fund at the end of the twelfth year?



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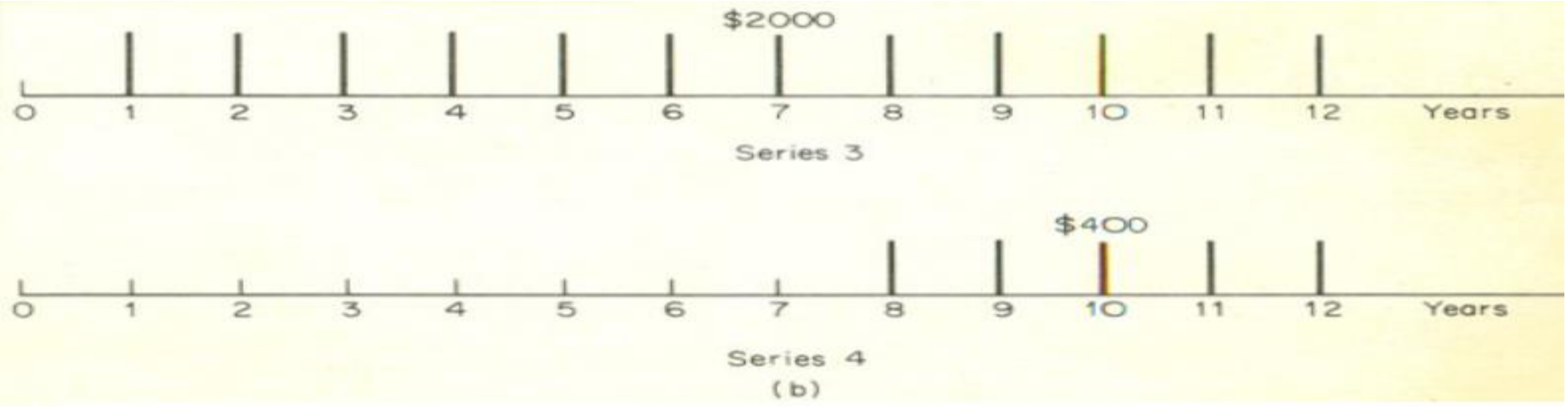

Solution

Method 1: Resolve the set of deposits into the two series shown in Figure(a). The value of series 1 at its terminal date is *2000(Fu/A,7)*, and this expression must be multiplied by the appropriate factor to find the value of this series at any other date. At the end of the 12th year,

Cont.

Value of series 1= 2,000*(Fu/A,7)(F/P,5)*

= 2,000*(8.14201)(1.27628)*

= $20,783

Value of series 2 = 2,400*(Fu/A,5)*

= 2,400(5.52563) = $13,262

Principal in fund = 20,783 + 13,262 = $34,045

Method 2: Consider the given set of deposits to be a composite of the two sets of deposits shown in Figure(b). At the end of the twelfth year,

Cont.

Value of series 3 $= 2000(Fu/A,12)$

$= 2000(15.91713)$

$= \$31,834$

Value of series 4 $= 400(Fu/A,5)$

$= 400(5.52563)$

$= \$2210$

Principal in fund = 31,834 + 2210 = $34,044

2.3 Calculation of Periodic Payment

A = periodic payment

Pu = present worth of series

Fu = future worth of series

n = number of payments

i = interest rate

$$A = P_u \frac{i}{1 - (1 + i)^{-n}} = P_u \frac{i}{1 - 1/(1 + i)^n} \qquad (2.3)$$

$$A = F_u \frac{i}{(1 + i)^n - 1} \qquad (2.4)$$

$$(A/P_u, n, i) = \frac{i}{1 - (1 + i)^{-n}} \qquad (2.3a)$$

$$(A/F_u, n, i) = \frac{i}{(1 + i)^n - 1} \qquad (2.4a)$$

Cont.

E.g.2.3 A corporation will require \$1,500,000 ten years hence to redeem its outstanding bonds. It will accumulate this amount by making 10 annual deposits in a reserve fund that earns 3 percent per annum, the first deposit being made 1 year hence. Determine the amount of the annual deposit.

SOLUTION

The set of payments has a future worth of \$1,500,000 on the basis of a 3 percent interest rate.

$Fu = \$1,500,000 \qquad n = 10 \qquad i = 3\%$

$$A = 1,500,000(A/Fu,10,3\%)$$

$$= 1,500,000(0.08723) = \underline{\$130,845}$$

Cont.

E.g. 2.4 A debt of \$30,000 is to be discharged by means of six annual payments of equal amount, the first payment to be made 1 year after the loan is consummated. If the interest rate of the loan is 8%, what annual payment is required? **Construct an amortization schedule.**

SOLUTION

The set of payments has a present worth of \$30,000 on the basis of an 8 percent interest rate.

$Pu = \$30{,}000 \qquad n = 6 \qquad i = 8\%$

$A = 30{,}000(A/Pu,6,8\%)$

$= 30{,}000(0.21632) = \underline{\$6{,}490}$

Cont.

- An *amortization schedule* is a table that traces the history of the loan in complete detail.
- It yields both the principal of the loan at the end of each payment period and the periodic interest earnings, which the lender and borrower must know for tax purposes.
- The principal of the loan at the end of each period is obtained by taking the principal at the beginning, adding the interest earning, and deducting the periodic payment.

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Year	Principal of loan at beginning, $	Interest earned, $	End-of-year payment, $	Principal of loan at end, $
1	30,000	2400	6,490	25,910
2	25,910	2073	6,490	21,493
3	21,493	1719	6,490	16,722
4	16,722	1338	6,490	11,570
5	11,570	926	6,490	6,006
6	6,006	480	6,486	0
Total		8936	38,936	

Cont.

E.g.2.5 A firm decided to construct an additional plant at the end of 10 years for an estimated cost of $1,000,000. To accumulate this sum, it established a sinking fund consisting of end-of-year deposits, the fund earning interest at 6 percent per annum. At the end of the third year, however, the firm decided to increase the size of the future plant, and the cost was now estimated to be $1,600,000. What should be the amount of the annual deposit for the remaining 7 years?

SOLUTION: cost of the plant is treated as a lump-sum payment.

Method 1: Resolve the set of payments into the two series

Cont.

The annual deposit under series 1 was:

$1{,}000{,}000(A/Fu,10)$ = 1,000,000(0.07587) = \$75,870

Select the end of the tenth year as the valuation date

Value of series 1 = 75,870$(Fu/A,3)(F/P,7)$

=75,870(3.18360)(1.50363)

= \$363,186

Value of series 2 = 1,600,000 - 363,186 = \$1,236,814

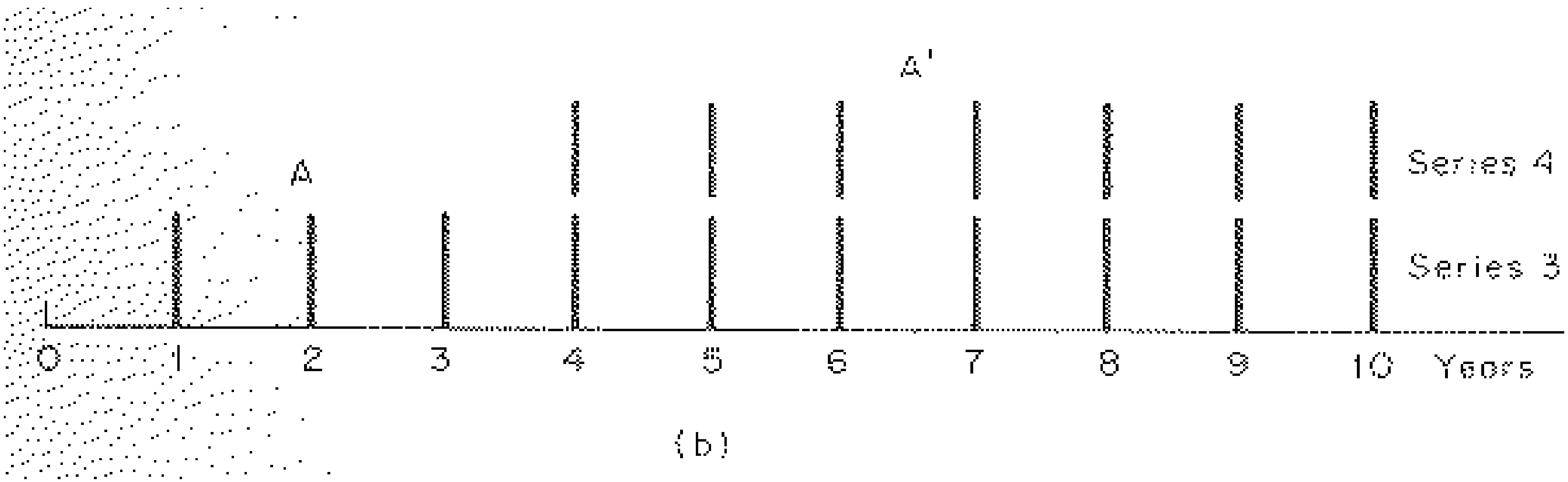
The annual deposit under series 2 is:

x = 1,236,814$(A/Fu,7)$ = 1,236,814(0.11914) = \$147,354

Cont.

Method 2:

Resolve the set of payments into the two series shown in Fig. 2.10b.



At the end of the tenth year, the value of series 3 is $1,000,000 and the value of series 4 is $600,000.

From the preceding calculations, the annual deposit under series 3 is $75,870.

Cont.

The annual deposit under series 4 is:

$600{,}000(A/Fu,7) = 600{,}000(0.11914)$

$= \$71{,}484$

Then $X = 75{,}870 + 71{,}484 = \underline{\$147{,}354}$

2.4 Series With Unequal Payment And Interest Periods

❖*Extraordinary Uniform Series* have the following characteristics:

- Each payment is made at the beginning or end of an interest period, and
- the payment period is a multiple of the interest period
- n = number of payments
- m = number of interest periods in one payment period

Cont.

$$P_u = A\frac{1-(1+i)^{-mn}}{(1+i)^m - 1} = A\frac{1-1/(1+i)^{mn}}{(1+i)^m - 1} \quad (2.7)$$

$$F_u = A\frac{(1+i)^{mn} - 1}{(1+i)^m - 1} \quad (2.8)$$

❖Dividing the numerator and denominator of each fraction by *i*, we obtain equations composed of the basic compound-interest factors. The results are as follows:

- ***Pu = A(Pu/A,mn)(A/Fu,m)*** **(2.9)**
- ***Fu = A(Fu/A,mn)(A/Fu,m)*** **(2.10)**

Cont.

- $A = Pu(A/Pu,mn)(Fu/A,m)$ **(2.11)**
- $A = Fu(A/Fu,mn)(Fu/A,m)$ **(2.12)**

E.g.2.6 Deposits of $2000 each were made in a fund at the end of each year for 7 consecutive years. If the fund earned interest at 6 percent per annum compounded quarterly, what was the principal in the fund immediately after the last deposit was made?

SOLUTION $A = \$2000$ $m = 4$ $n = 7$ $i = 1.5\%$

$$Fu = 2000(Fu/A,28)(A/Fu,4)$$

$$= 2000(34.48148)(0.24444) = \underline{\mathbf{\$16,857}}$$

Cont.

E.g.2.7 A debt of $180,000 is to be discharged by means of five equal payments made at 3-year intervals, the first payment to be made 3 years after the loan is consummated. If the interest rate of the loan is 8 percent per annum, what must be the periodic payment?

SOLUTION

Pu = $180,000 m = 3 n = 5 i = 8%

A = 180,000*(A/Pu,15)(Fu/A,3)*

= 180,000(0.11683)(3.24640) = $68,270

- **Reading assignment calculation of interest rate**

2.5 Calculation of Interest Rate

❖For an ordinary uniform series,

- Given the periodic payment and
- either the present or future worth,
- it is necessary to determine the interest rate pertaining to the series.
- However, prior equations in this chapter are not easy for directly solving the interest rate, thus the recourse must be a trial-and-error procedure.

Cont.

E.g.2.8 In purchasing a certain commodity, a firm is offered two alternative arrangements for payment. Under scheme A, it pays \$4700 at date of purchase. Under scheme B, it pays \$650 at date of purchase and pays \$250 at the end of each month for 18 consecutive months, the first payment of \$250 being made 1 month after date of purchase. If the firm chooses scheme B, what effective interest rate is it paying?

SOLUTION

- Whether the firm goes for scheme A or B, the difference falls on way of payment

Cont.

- If the firm goes for scheme A, it will pay 4700 at the date of purchase. This implies that cost of the commodity is 4700.
- Therefore, under scheme B, the firm in effect is borrowing the sum of 4700 - 650 = $4050 and discharging the debt by means of the monthly payments.
- For the uniform series under scheme B,

$Pu = \$4050$ $\qquad A = \$250$ $\qquad n = 18$

- We shall find the **monthly interest rate** i of this series and then compute the corresponding **effective interest rate**

Cont.

- Then, *from Pu = A(Pu/A, n, i), (Pu/A, n, i)=* Pu/A

  (Pu/A, 18) =4050/250= 16.2

- Reference to the tables in Appendix A reveals that the value of *i* lies between 1% and 1.25%.
- By assigning trial values to i and then applying the equation (Pu/A, n, i) = $(1 – (1 + i)^{-n})/i$
- It is found that *(Pu/A,18)* = 16.20504 when *i* = 1.13 percent, and we accept this value of *i*.
- $i_e = (1 + r/m)^m – 1$, $i_e = (1.0113)^{12} - 1$ = 14.4% per annum

2.6 Equivalent Uniform Series

- Two sets of payments are equivalent to one another if they have an identical value on any valuation date.
- For comparative purposes, it is often desirable to **transform** a given set of non-uniform payments to an equivalent set of uniform payments.
- The periodic payment under the equivalent set is termed the ***equivalent uniform payment***
- This transformation can readily be accomplished by selecting any **convenient valuation date,**

Cont.

- calculating the value of the given set of payments at that date, & on this basis calculating the equivalent uniform payment

E.g.2.9 A machine having a 5-year life has the following annual operating costs: first year, $4000; second year, $5200; third year, $6100; fourth year, $6800; fifth year, $7700. For simplicity, expenditure for operation may be treated as lump-sum payments made at the end of the year. If money is worth 10 percent, what was the equivalent uniform annual operating cost?

SOLUTION: Select the beginning of the 1st year as the valuation date.

Cont.

$4000(P/F,1)+5200(P/F,2)+6100(P/F,3)+6800(P/F,4)+7700(P/F,5)$

$= 4000(0.90909)+5200(0.82645)+6100(0.7501) + 6800(0.68301)+ 7700(0.62092) = \$21{,}942$

The equivalent uniform annual operating cost is:

$21{,}942(A/Pu,5) = 21{,}942(0.26380) = \underline{\$5788}$

2.7 Perpetuities

❖A uniform series in which the **number of payments is infinite** is termed a ***perpetuity***.

❖As in the case of a uniform series of finite duration, the origin date of a perpetuity is placed one payment period before the first payment, and the value of the perpetuity at the origin date is termed its *present worth*.

❖On the equation Pu = A(1 – (1 + i)$^{-mn}$)/ ((1 + i)^{m} – 1, when n goes to infinity, (1 + i)$^{-mn}$) goes to zero, therefore:-

- *Let*, P_{up} = present worth of perpetuity

Cont.

- m = number of interest periods contained in one payment period, then

$$P_{up} = \frac{A}{(1+i)^m - 1} \qquad\qquad P_{up} = \frac{A(A/Fu,m)}{i}$$

- In the special case where the payment period and interest period are coincident, the above equation reduces to:

$P_{up} = A/i$ when $m = 1$

Cont.

- **E.g.2.10** An endowment fund is established to provide annual scholarships of $5000 each. If the interest rate of the fund is 6 percent per annum compounded quarterly, what sum must be deposited to pay for each scholarship?

SOLUTION

- A = $5000 $m = 4$ $i = 1.5\%$
- $P_{up} = 5000/[(1.015)^4 - 1] = \$81{,}482$