Chapter 2

ONE-DIMENSIONAL, STEADY-STATE CONDUCTION

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Temperature gradient exists along only a single coordinate direction, and heat transfer occurs exclusively in that direction.

T=T(x,t)

Steady-state implies that at any given location, the temperature does not change with time.

$$\frac{\partial T(x,t)}{\partial t} = 0$$

The two conditions above will make T=T(x). Initially heat generation will not be considered.

3.1 The Plane Wall

3.1.1 Temperature Distribution

The two surfaces of the plane wall are at temperatures $T_{s,1}$ and $T_{s,2}$, where due to the existing temperature gradient conduction heat transfer will occur as shown in **fig-chp3\fig.3.1.pptx**. At the surfaces convective heat transfer will also occur.

The appropriate equation to be used for one-dimensional steady-state conduction with no heat generation will be

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0$$

For constant k, the general solution is

$$T(x) = C_1 x + C_2$$

Two boundary conditions will be required to determine the two constants.

$$T(0) = T_{s,1}, \quad \text{and} \quad T(L) = T_{s,2}$$

The above will give

$$C_1 = \frac{T_{s,2} - T_{s,1}}{L} \quad \text{and} \quad C_2 = T_{s,1}$$

Substitution will give

$$T(x) = T_{s,1} + (T_{s,2} - T_{s,1})\frac{x}{L}$$

The above shows T varying linearly with x.

Heat transfer rate or flux can be determined as:

$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}(T_{s,1} - T_{s,2})$$

$$q_x'' = \frac{q_x}{A} = \frac{k}{L}(T_{s,1} - T_{s,2})$$

3.1.2 Thermal Resistance

Thermal Ohm’s law can be formed as

$$q_x = \frac{T_{s,1} - T_{s,2}}{(L/kA)} \qquad R_t = \frac{L}{kA}; \quad R_e = \frac{L}{\sigma A}$$

$\rho = \sigma^{-1}$ = resistivity of material (Ω.m)

For convection heat transfer

$$q = hA(T_s - T_\infty)$$

Corresponding thermal resistance

$$R_{t,conv} = \frac{T_s - T_\infty}{q} = \frac{1}{hA}$$

Since q_x is constant, the thermal circuit diagram will **fig-chp3\fig.3.1.pptx** give

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2A}$$

For an overall temperature difference, ($T_{\infty,1}$-$T_{\infty,2}$)

i.e. thermal potential difference

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}} \qquad R_{tot} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

Also for radiation

$q_{rad} = h_r A(T_s - T_{sur})$ and

$$R_{t,rad} = \frac{T_s - T_{sur}}{q_{rad}} = \frac{1}{h_r A}$$

3.1.3 The Composite Wall

Composite walls can be analyzed as series and parallel thermal network. The schematic is seen in **fig-chp3\fig.3.2.pptx** where the series thermal circuit is also shown for both conduction and convection heat transfer.

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\sum R_i}$$

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\frac{1}{h_1 A} + \frac{L_A}{k_A A} + \frac{L_B}{k_B A} + \frac{L_C}{k_C A} + \frac{1}{h_4 A}}$$

For constant q_x

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{(1/h_1 A)} = \frac{T_{s,1} - T_2}{(L_A / k_A A)} = \frac{T_2 - T_3}{(L_B / k_B A)} = \frac{T_3 - T_{s,4}}{(L_C / k_C A)} = \frac{T_{s,4} - T_{\infty,4}}{(1/h_4 A)}$$

An overall heat transfer coefficient U is defined as

$q_x = UA\,\Delta T_{overall}$ and $R_{tot} = 1/UA$

$$R_{tot} = \frac{1}{h_1 A} + \frac{L_A}{k_A A} + \frac{L_B}{k_B A} + \frac{L_C}{k_C A} + \frac{1}{h_4 A} = \frac{1}{UA}$$

In general

$$R_{tot} = \sum R_i = \frac{\Delta T_{overall}}{q} = \frac{1}{UA}$$

Series-parallel arrangement can also be analyzed as shown in **fig-chp3\fig.3.3.pptx**. Although the heat flow is now multidimensional, it is often reasonable to assume one-dimensional conditions. Two equivalent thermal circuits are presented. In (a) surfaces normal to the x direction are isothermal and in (b) being symmetric to the x-axis, the surfaces can be considered adiabatic.

3.1.4 Contact Resistance

At the interface of composite materials, due to surface roughness, there is what is called thermal contact resistance. This results in a temperature drop across the interface as shown in **fig-chp3\fig.3.4.pptx**. This thermal contact resistance is designated by $R_{t,c}$ and expressed by

$$R''_{t,c} = \frac{T_A - T_B}{q''_x}$$

The contact resistance may be viewed as two parallel resistances:

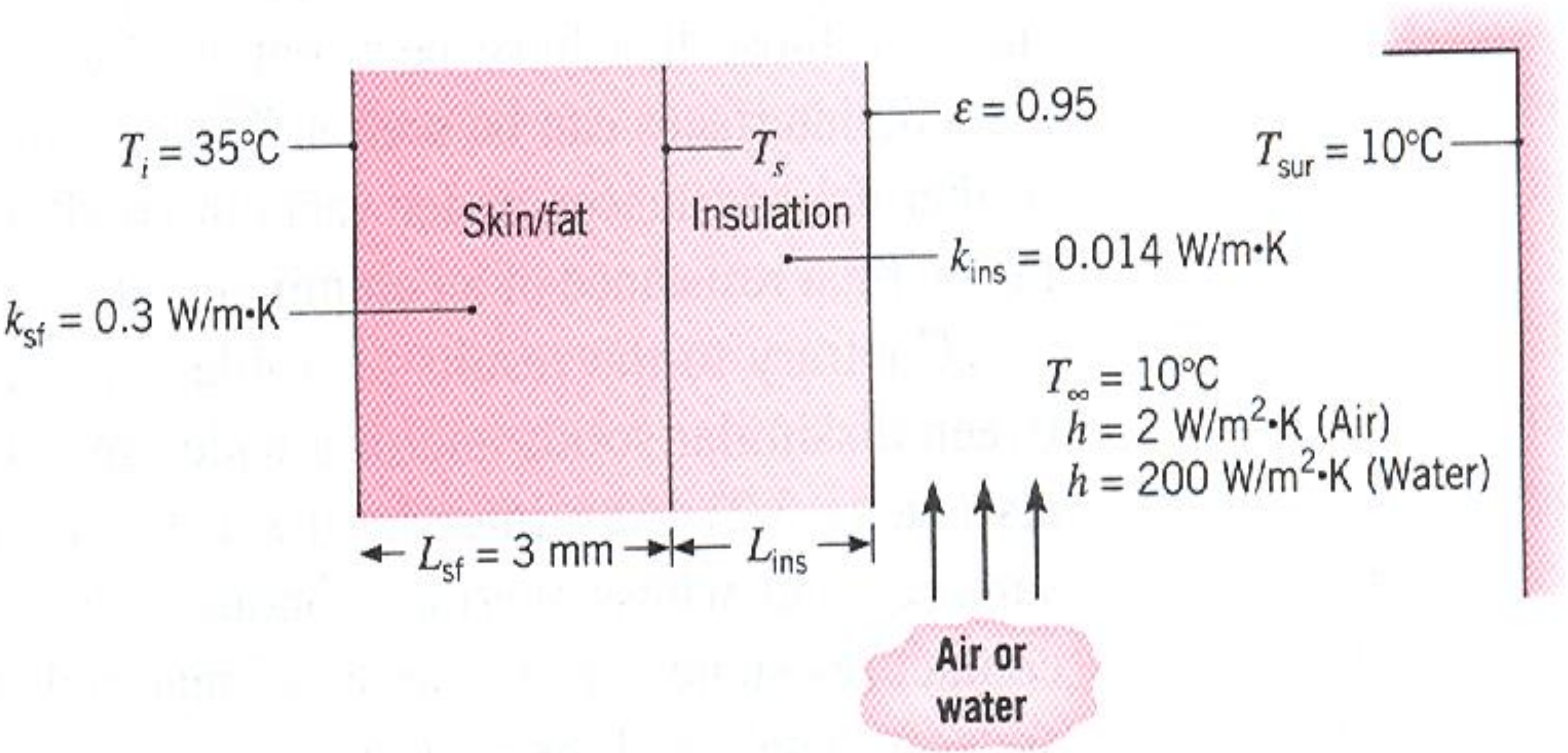
- Due to contact spots
- Due to gaps

For rough surfaces, the contact area is typically small and the major contribution is from the gaps.

By increasing the joint pressure the contact area can be increased thus decreasing the contribution from the gap.

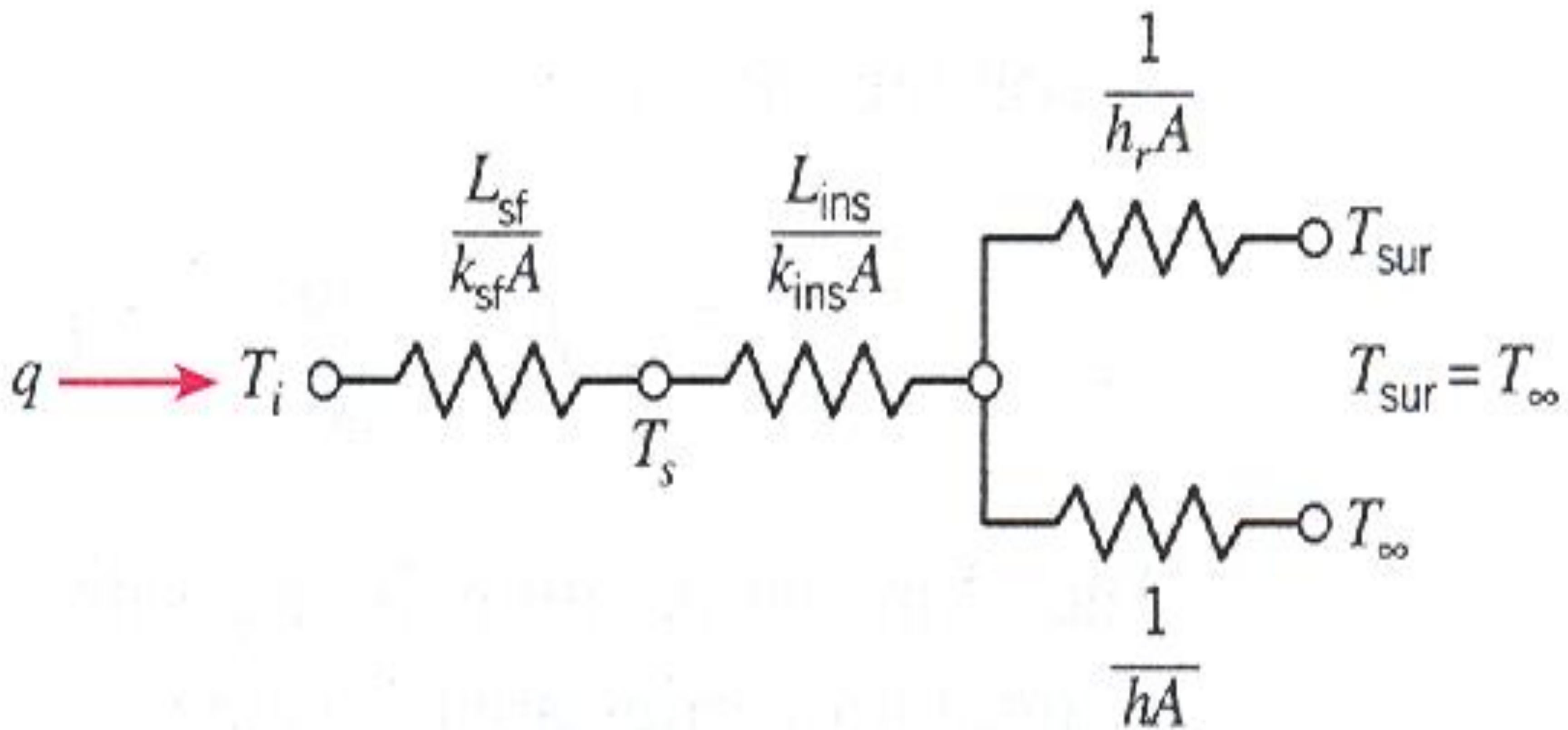
Example 3.1

To reduce the heat loss rate, a person wears special sporting clothes (insulation, k=0.014 W/m.K) against an environment of air or water at 10^{o}C. The emissivity of the cloth is 0.95. What thickness of insulation is needed to reduce the heat loss rate to 100 W (a typical metabolic heat generation rate) in air and water? What are the resulting skin temperatures? Take surface area as 1.8 m^2.



Solution

The thermal circuit showing conduction through the skin fat and insulation, and convection and radiation from the insulation surface



The total thermal resistance will be

$$R_{tot} = \frac{\Delta T_{overall}}{q} = \frac{(35-10)}{100} = 0.25\,K/W$$

From the thermal circuit diagram

$$R_{tot} = \frac{L_{sf}}{k_{sf}A} + \frac{L_{ins}}{k_{ins}A} + \left(\frac{1}{1/hA} + \frac{1}{1/h_r A}\right)^{-1} = \frac{1}{A}\left(\frac{L_{sf}}{k_{sf}} + \frac{L_{ins}}{k_{ins}} + \frac{1}{h+h_r}\right)$$

Air

To determine h_r, iteration will be required. Assume of T_o=291K (T_{ins}) and use

$$h_r = \varepsilon\sigma(T_o + T_{sur})(T_o^2 + T_{sur}^2)$$

Substitution of the values will give

h_r = 0.95 x 5.67 x 10^{-8} (291 +283)(291^2+283^2) = 5.09

To check on the heat transfer from the outer surface, we will use

$$q = \frac{T_o - T_{sur}}{\left(\dfrac{1}{1/hA} + \dfrac{1}{1/h_r A}\right)^{-1}} = \frac{T_o - T_{sur}}{\left(A(h + h_r)\right)^{-1}} = \frac{291 - 283}{\left(1.8\{2 + 5.09\}^{-1}\right)} = 102\text{W}$$

It can further be refined to give q = 100 W.

Using

$$L_{ins} = k_{ins}\left[AR_{tot} - \frac{L_{sf}}{k_{sf}} - \frac{1}{h + h_r}\right]$$

$$= 0.014\left[1.8\text{x}0.25 - \frac{0.003}{0.3} - \frac{1}{(2 + 5.09)}\right]$$

L_{ins} = 0.0042 m = 4.2 mm

This will give a temperature of 34.6°C on the skin which will be comfortable.

Water

The convection resistance is much lower than the radiation resistance. So it will be assumed that all the heat transfer from the skin will be by convection.

This will give

$$L_{ins} = k_{ins}\left[AR_{tot} - \frac{L_{sf}}{k_{sf}} - \frac{1}{h}\right] = 0.014\left[1.8x0.25 - \frac{0.003}{0.3} - \frac{1}{200}\right]$$

$$= 0.0061\text{m} = 6.1\text{mm}$$

This will give a skin temperature of 34.6^{o}C which is about the same as that due to the air stream the air.

Had it not been for the insulation the temperatures on the skin surface would have been 18^{o}C and almost 10^{o}C for the air and water respectively. You can imagine how cold you can feel in the case of water!

Example 3.2

The thermal conductivity of a D=14-nm carbon nanotube is measured with an instrument that is fabricated of a wafer of silicon nitride at a temperature of T_{∞}=300K. The 20-μm-long nanotube rests on two 0.5-μm-thick, 10 μm x 10 μm square islands that are separated by a distance

s=5 μm. A thin layer of platinum is used as an electrical resistor on the heated island (at temperature T_h) to dissipate $q=11.3\ \mu W$ of electrical power. On the sensing island, a similar layer of platinum is used to determine its temperature T_s. The platinum's electrical resistance, $R(T_s) = E/I$, is found by measuring the voltage drop and electrical current across the platinum layer. The temperature of the sensing island, T_s, is then determined from the relationship of the platinum electrical resistance to its temperature. Each island is suspended by two $L_{sn}=250$-μm-long silicon nitride beams that are w_{sn}=3 μm wide and t_{sn}=0.5 μm thick. A platinum line of width w_{pt}=1 μm and thickness t_{pt}=0.2 μm is deposited within each silicon nitride beam to power the heated island. The entire experiment is performed in a vacuum at a steady-state temperature of T_s=308.4 K. Estimate the thermal conductivity of the carbon nanotube.

k_{pt}=71.6 W/m.K k_{sn}=15.5 W/m.K

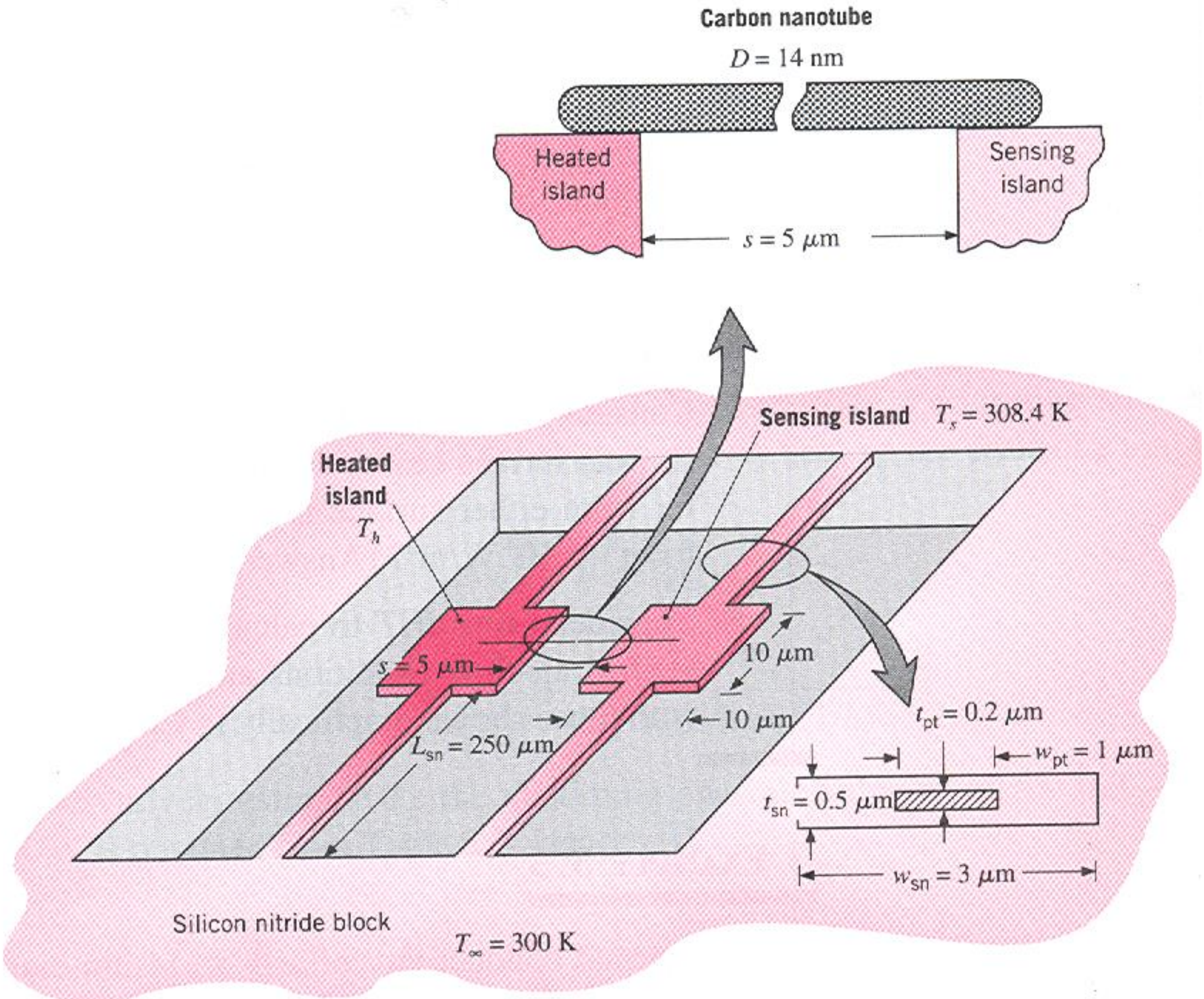
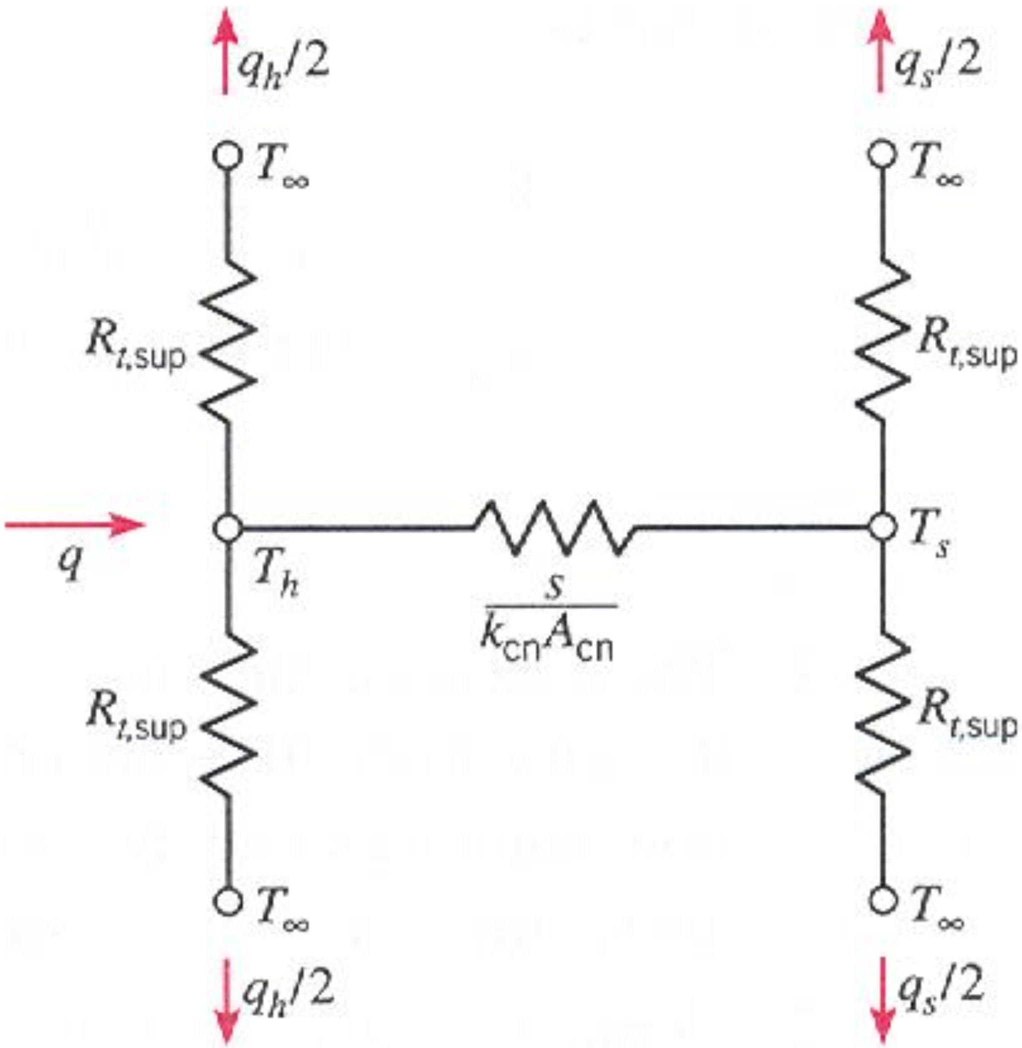


Figure for example 3.2

Solution

The thermal circuit diagram is shown below.



The cross-sectional areas of the materials in the support beams are

$A_{pt}=w_{pt}\ t_{pt} = (1x10^{-6})\ x\ (0.2x10^{-6}) =2\ x\ 10^{-13}m^2$

$A_{sn}=w_{sn}\ t_{sn} – A_{pt} = (3x10^{-6})\ x\ (0.5x10^{-6})^{-}\ ^2\ x\ 10^{-13}=$

$= 1.3\ x\ 10^{-12}\ m^2$

Cross-sectional area of the carbon nanotube is

$A_{cn} = \pi D^2/4 = \pi(14x10^{-9})^2/4=1.54\ x\ 10^{-16}\ m^2$

The thermal resistance of each support is

$$R_{t,sup} = \left[\frac{k_{pt}A_{pt}}{L_{pt}} + \frac{k_{sn}A_{sn}}{L_{sn}}\right]^{-1} = \left[\frac{71.6x2x10^{-13}}{250x10^{-6}} + \frac{15.5x1.3x10^{-12}}{250x10^{-6}}\right]^{-1}$$

$= 7.25\ x\ 10^6\ K/W$

The combined heat loss through both sensing island supports is

$q_s = 2\,(T_s - T_\infty)/R_{t,sup} = 2 \times (308.4 - 300) / (7.25 \times 10^6)$

$= 2.32 \times 10^{-6}$ W = 2.32 μW

$q_h = q - q_s = 11.3 - 2.32 = 8.98$ μW

And from

$$\frac{q_h}{2} = \frac{T_h - T_\infty}{R_{t,sup}} \qquad T_h = T_\infty + \frac{1}{2} q_h R_{t,sup} = 300 + \frac{8.98\text{x}10^{-6}\,\text{x}7.25\text{x}10^{6}}{2}$$

$\mathbf{T_h = 332.6\ K}$

Using the circuit connecting T_h and T_s

$$q_s = \frac{T_h - T_s}{s/(k_{cn} A_{cn})}$$

$$k_{cn} = \frac{q_s s}{A_{cn}(T_h - T_s)} = \frac{2.32x10^{-6} x5x10^{-6}}{1.54x10^{-16} x(332.6 - 308.4)} = 3113 W/m.K$$

Approximate heat transfer due to radiation shows a value of 0.047 μW which is negligible on a comparative basis.

3.2 ALTERNATIVE CONDUCTION ANALYSIS

For steady-state, no heat generation and no heat loss from the sides (**fig-chp3\fig.3.5.pptx**) q_x is constant and independent of x while A(x),dT/dx, and k(T) may vary.

T may be T(x,y), but the y-coordinate effect may be neglected. This makes it a one dimensional analysis.

Starting with Fourier's law

$$q_x = -kA\frac{dT}{dx} \quad \text{or} \quad q_x\frac{dx}{A(x)} = -k(T)dT$$

$$q_x\int_{x_o}^{x}\frac{dx}{A(x)} = -\int_{T_o}^{T}k(T)dT$$

For A(x) and k(T) known, the integration results in a functional form of T(x).

3.3 RADIAL SYSTEMS

Cylindrical and spherical systems can be analyzed by using the standard method and alternative method.

For the one dimensional case T=T(r)

3.3.1 The cylinder (standard method)

The appropriate equation for steady-state with no generation is

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right)=0 \quad \text{or} \quad kr\frac{dT}{dr}=\text{constant}$$

Also

$$q_r = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{dT}{dr} = (2\pi L)kr\frac{dT}{dr}$$

which shows q_r to be constant (independent of r).

The cylindrical configuration is shown in **fig-chp3\fig.3.6.pptx**

Integrating for constant k gives

$T(r) = C_1 \ln r + C_2$

Two boundary conditions

$T(r_1)=T_{s,1}$ and $T(r_2) = T_{s,2}$

The final solution becomes

$$T(r) = \frac{T_{s,1} - T_{s,2}}{\ln \frac{r_1}{r_2}} \ln \frac{r}{r_2}$$

$$q_r = -kA \frac{dT}{dr} = -k(2\pi rL) \frac{T_{s,1} - T_{s,2}}{\ln \frac{r_1}{r_2}} \frac{1}{r} = \frac{2\pi Lk(T_{s,1} - T_{s,2})}{\ln \frac{r_2}{r_1}}$$

This gives

$$R_t = \frac{\ln(r_2 / r_1)}{2\pi Lk}$$

The thermal circuit is also shown in the figure.

For composite cylinders shown in **fig-chp3\fig.3.7.pptx**

Using the thermal circuits

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{\frac{1}{2\pi r_1 L h_1} + \frac{\ln(r_2/r_1)}{2\pi k_A L} + \frac{\ln(r_3/r_2)}{2\pi k_B L} + \frac{\ln(r_4/r_3)}{2\pi k_C L} + \frac{1}{2\pi r_4 L h_4}}$$

Using the overall heat transfer coefficient U

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{tot}} = UA(T_{\infty,1} - T_{\infty,4}); \qquad U = (AR_{tot})^{-1}$$

where U is defined based on arbitrary area A.

It could also be defined on the basis of other areas such that

$U_1A_1 = U_2A_2 = U_3A_3 = U_4A_4 = (\sum R_t)^{-1}$

If U is defined on $A_1=2\pi r_1 L$, then

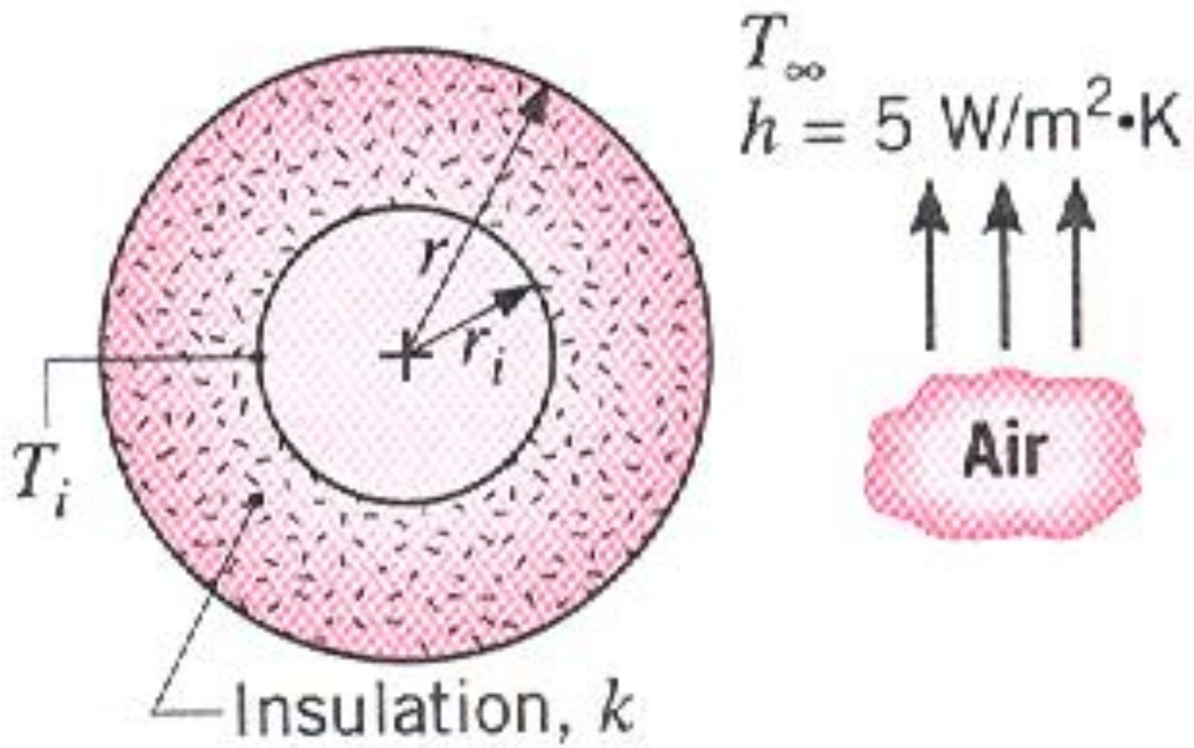
$$U_1 = \frac{1}{\frac{1}{h_1} + \frac{r_1}{k_A}\ln\frac{r_2}{r_1} + \frac{r_1}{k_B}\ln\frac{r_2}{r_1} + \frac{r_1}{k_C}\ln\frac{r_4}{r_3} + \frac{r_1}{r_4}\frac{1}{h_4}}$$

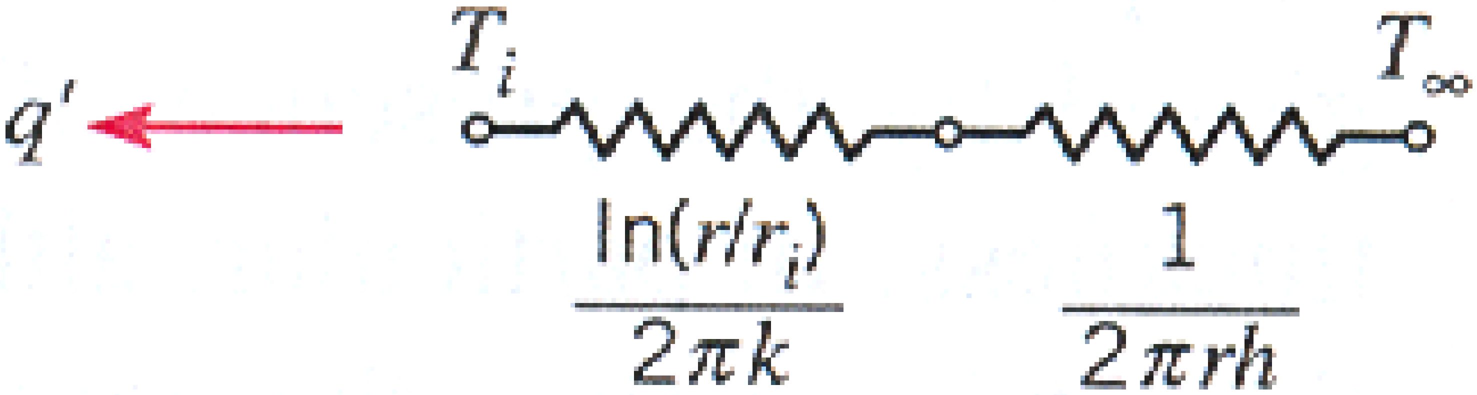
Example 3.3

A thin-walled copper tube of radius r_i is used to transport a low-temperature refrigerant and is at a temperature T_i that is less than that of the ambient air at T_∞ around the tube. Is there an optimum thickness associated with application of insulation to the tube? Use graph for justification.

Solution

The arrangement is shown in the figure below. Use cellular glass for insulation k=0.055 W/m.K and h= 5 W/m^2.K and r_i=5 mm.





Thermal circuit for example 3.3

Using the thermal circuit per unit length of the tube

$$R'_{tot} = \frac{\ln(r/r_i)}{2\pi k} + \frac{1}{2\pi r h} \quad \text{and} \quad q' = \frac{T_\infty - T_i}{R'_{tot}}$$

The above equation shows that with increase in r the conduction resistance increases while that of convection decreases. This indicates that there may be an optimum thickness of insulation that will minimize the heat transfer. This can be achieved by using

$$\frac{dR'_{tot}}{dr} = 0 \quad \text{which gives} \quad \frac{1}{2\pi k r} - \frac{1}{2\pi r^2 h} = 0$$

The above gives

$$r = \frac{k}{h}$$

Checking the second derivative gives

$$\frac{d^2 R'_{tot}}{dr^2} = \frac{1}{2\pi k^3 / h^2} > 0 \quad \text{at} \quad r = k/h$$

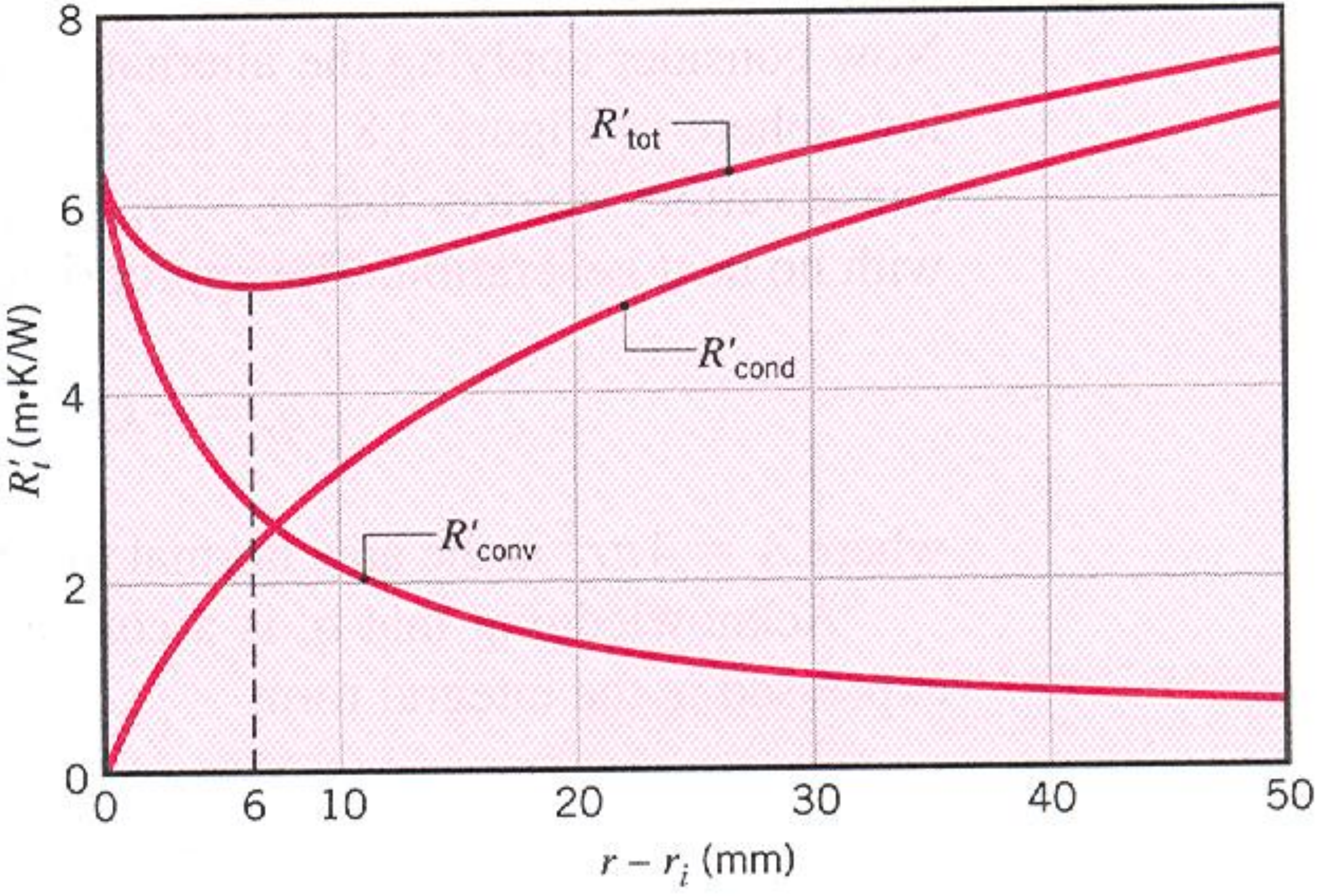
which suggests that the resistance determined is the minimum or gives maximum heat transfer. However this benchmark ($r=r_c$) which is the indicator of maximum heat transfer, can still be used to get information on the heat transfer characteristics with increase of insulation thickness.

The figure shows a plot of R'_{tot} against insulation thickness ($r-r_i$).

$r_c=k/h=0.055/5=0.011m$ $r_c-r_i=(0.011-0.005)=0.006m$

The graph shows that insulation thicknesses below this critical radius results in increased heat transfer with increase in insulation thickness. This is desirable for electrical wire insulation (good cooling effect).

For insulation thicknesses above the critical radius, heat transfer decreases with increase in thickness of insulation. (But make a note that the bare tube has the same resistance up to a certain insulation thickness) This is the usual setup in practical insulation. The thickness of the insulation will be optimized with respect to the cost of the insulation material and the saved energy cost.



Thermal resistance vs. insulation thickness

3.3.2 The Sphere

To diversify our approach, here we will use the alternative method.

Considering **fig-chp3\fig.3.8.pptx**, steady-state conduction with no generation gives q_r=constant.

$$q_r = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{dT}{dr}$$

$$\frac{q_r}{4\pi}\int_{r_1}^{r_2}\frac{dr}{r^2} = \int_{T_{s,1}}^{T_{s,2}} k(T)dT$$

For constant k

$$q_r = \frac{4\pi k(T_{s,1} - T_{s,2})}{\frac{1}{r_1} - \frac{1}{r_2}}$$

The thermal resistance will be

$$R_{th} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Example 3.4

A spherical, thin-walled metallic container is used to store liquid nitrogen at 77 K. The container has a diameter of 0.5 m and is covered with an evacuated, reflective insulation composed of silica powder. The insulation is 25 mm thick, and its outer surface is exposed to ambient air at 300 K. The convection coefficient is known to be 20 $W/m^2.K$. The latent heat of vaporization and the density of liquid nitrogen are 2×10^5 J/kg and 804 kg/m^3, respectively. (1) What is the rate of heat transfer to the liquid nitrogen? (b) What is the rate of liquid boil-off?

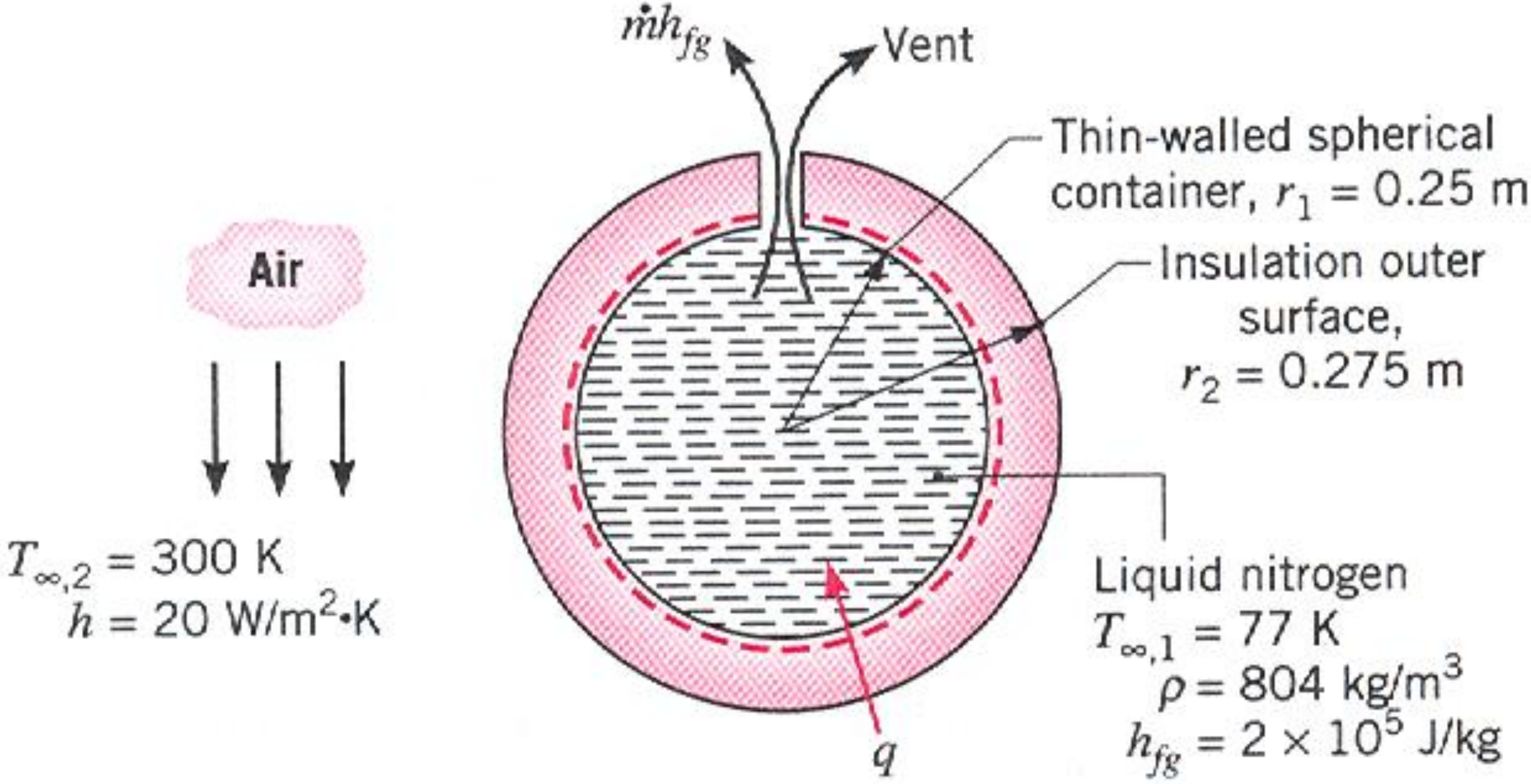


Figure for example

Solution

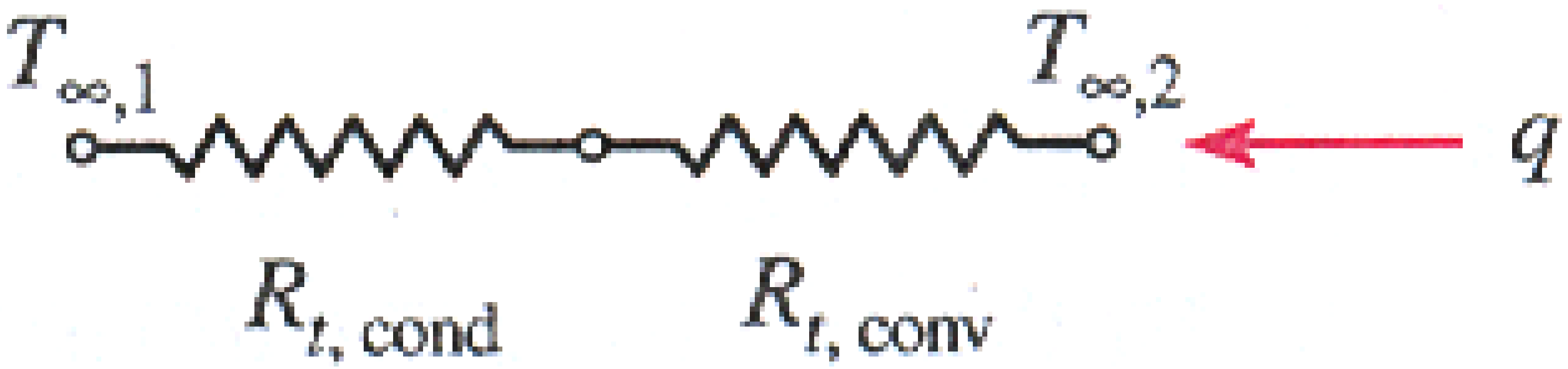
(1) The thermal circuit is shown below

$$R_{t,cond} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad R_{t,conv} = \frac{1}{h(4\pi r_2^2)}$$

Hence

$$q = \frac{T_{\infty,2} - T_{\infty,1}}{(1/4\pi k)[(1/r_1) - (1/r_2)] + [1/h(4\pi r_2^2)]}$$

$$= \frac{300 - 77}{\left[\frac{1}{4\pi(0.0017)}\right]\left(\frac{1}{0.25} - \frac{1}{0.275}\right) + \frac{1}{[20(4\pi)(0.275)^2]}}$$



Thermal circuit for example 3.4

which gives

$$q = \frac{223}{17.02 + 0.05} = 13.06\,\mathrm{W}$$

(2) Energy balance on the control volume (liquid N_2)

$$\dot{E}_{in} - \dot{E}_{out} = 0 \qquad q - \dot{m} h_{fg} = 0$$

and the boil-off $\dot{m}$ is

$$\dot{m} = \frac{q}{h_{fg}} = \frac{13.06}{2\mathrm{x}10^{5}} = 6.53\mathrm{x}10^{-5}\,\mathrm{kg/s} = 5.64\,\mathrm{kg/day}$$

$$V = \frac{\dot{m}}{\rho} = \frac{5.64}{804} = 0.007\,\mathrm{m}^3/\mathrm{day} \;\; (= 10.8\%\ \text{of container})$$

3.4 CONDUCTION WITH THERMAL ENERGY GENERATION

Internal energy generation could be from current, nuclear, or exothermic chemical reaction where the generation will be assumed to be uniform. Typical for electric current is

$$\dot{E}_g = I^2 R$$

and the energy generation per unit volume will be

$$\dot{q}''' = \frac{\dot{E}_g}{V} = \frac{I^2 R}{V}$$

3.4.1 The Plane Wall

The applicable equation for constant k is

$$\frac{d^2T}{dx^2}+\frac{\dot{q}'''}{k}=0$$

And the surfaces are maintained at $T_{s,1}$ and $T_{s,2}$ as shown in **fig-chp3\fig.3.9.pptx**

The general solution is

$$T=-\frac{\dot{q}'''}{2k}x^2+C_1x+C_2$$

The boundary conditions to determine C_1 and C_2 are

$T(-L)=T_{s,1}$ and $T(L)=T_{s,2}$

Applying the boundary conditions will give

$$C_1 = \frac{T_{s,2} - T_{s,1}}{2L} \quad \text{and} \quad C_2 = \frac{\dot{q}'''}{2k} L^2 + \frac{T_{s,2} + T_{s,1}}{2}$$

Substitution will give the temperature distribution as

$$T(x) = \frac{\dot{q}''' L^2}{2k}\left(1 - \left(\frac{x}{L}\right)^2\right) + \frac{(T_{s,2} - T_{s,1})}{2}\frac{x}{L} + \frac{(T_{s,2} + T_{s,1})}{2}$$

Unlike the previous equations, heat transfer is no more independent of x as the derivative dT/dx shows.

For symmetric boundary conditions $T_{s,1}=T_{s,2}=T_s$

$$T(x) = \frac{\dot{q}''' L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$$

Maximum temperature exists at the midplane (x=0) with a magnitude of

$$T(0) = T_o = \frac{\dot{q}'''L^2}{2k} + T_s \quad \Rightarrow \quad \frac{\dot{q}'''L^2}{2k} = T_o - T_s$$

This will give the dimensionless temperature distribution as

$$\frac{T(x) - T_s}{T_o - T_s} = 1 - \left(\frac{x}{L}\right)^2 \Rightarrow \frac{T(x) - T_o}{T_s - T_o} = \left(\frac{x}{L}\right)^2$$

The midplane temperature distribution has (dT/dx)=0 at x = 0. This acts like an adiabatic surface. Implication is that same equation applies to plane walls that are perfectly insulated on one side (x=0) and maintained at a fixed temperature T_s on the other side (x=L).

It is better to relate to the adjoining fluid, since it may not be convenient to measure T_s.

Neglecting radiation, energy balance at the surface, x=L

$$-kA\frac{dT}{dx} = hA(T_s - T_\infty)$$

$$-k\frac{dT}{dx} = -k\left(\frac{\dot{q}'''L^2}{2k}\left[-\frac{2x}{L^2}\right]\right)_{x=L} = h(T_s - T_\infty)$$

This will give

$$T_s = T_\infty + \frac{\dot{q}'''L}{h}$$

The same can be derived from

$$\dot{E}_g = \dot{E}_{out} \quad ; \text{for a unit area} \quad \dot{q}'''L = h(T_s - T_\infty)$$

The temperature distribution, then, becomes

$$T(x) = \frac{\dot{q}'''L^2}{2k}\left(1 - \left[\frac{x}{L}\right]^2\right) + T_\infty + \frac{\dot{q}'''L}{h}$$

Example 3.5

A plane wall is a composite of two materials, A and B. The wall of material A has uniform heat generation of 1.5×10^6 W/m^3, k_A = 75 W/m.K, and thickness L_A = 50 mm. The wall material B has no generation with k_B=150 W/m.K and thickness L_B=20 mm. The inner surface of material A is well insulated, while the outer surface of material B is cooled by a water stream with T_∞ =30^{o}C and h = 1000W/m^2.K.

1. Sketch the temperature distribution that exists in the composite under steady-state conditions.
2. Determine the temperature T_o of the insulated surface and the temperature T_2 of the cooled surface.

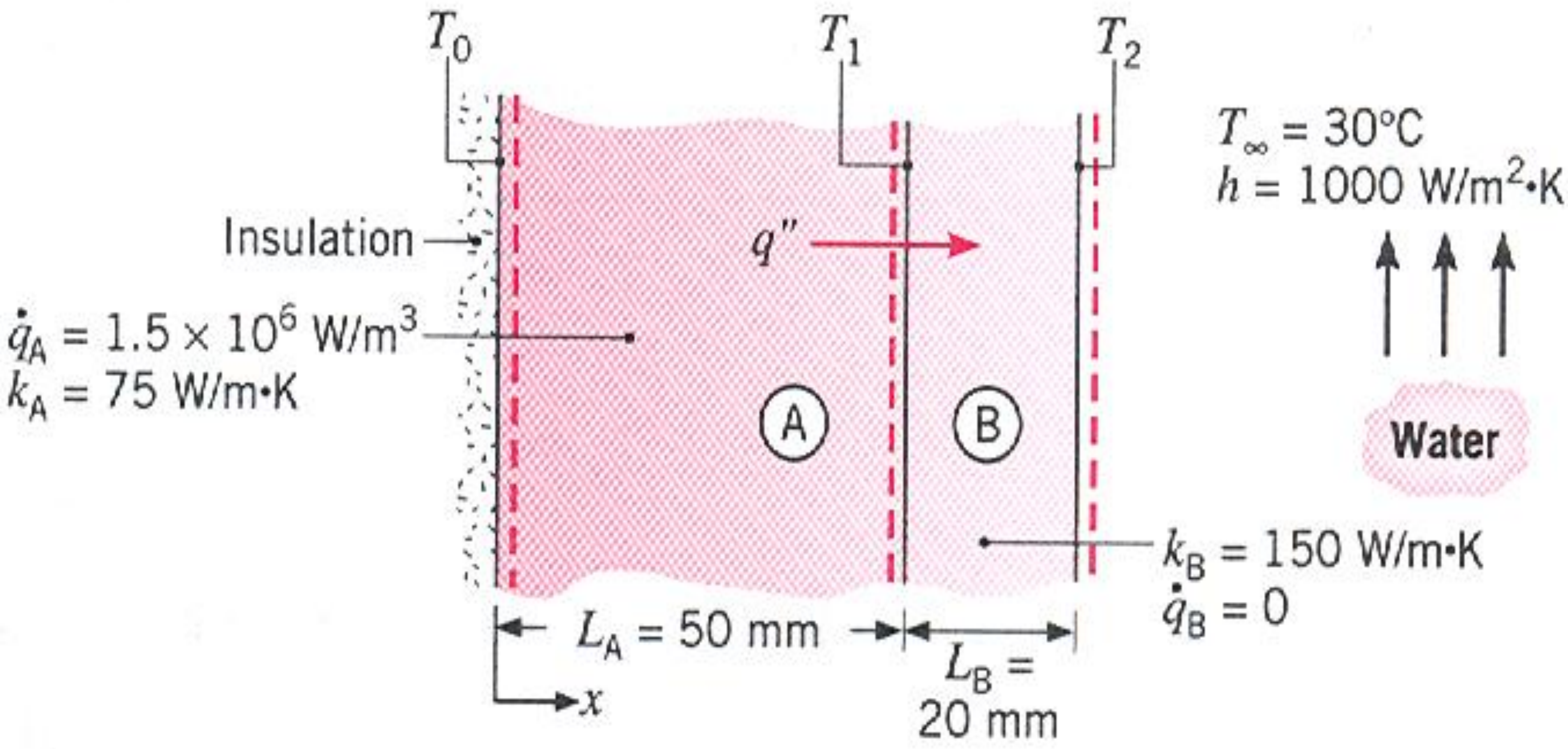
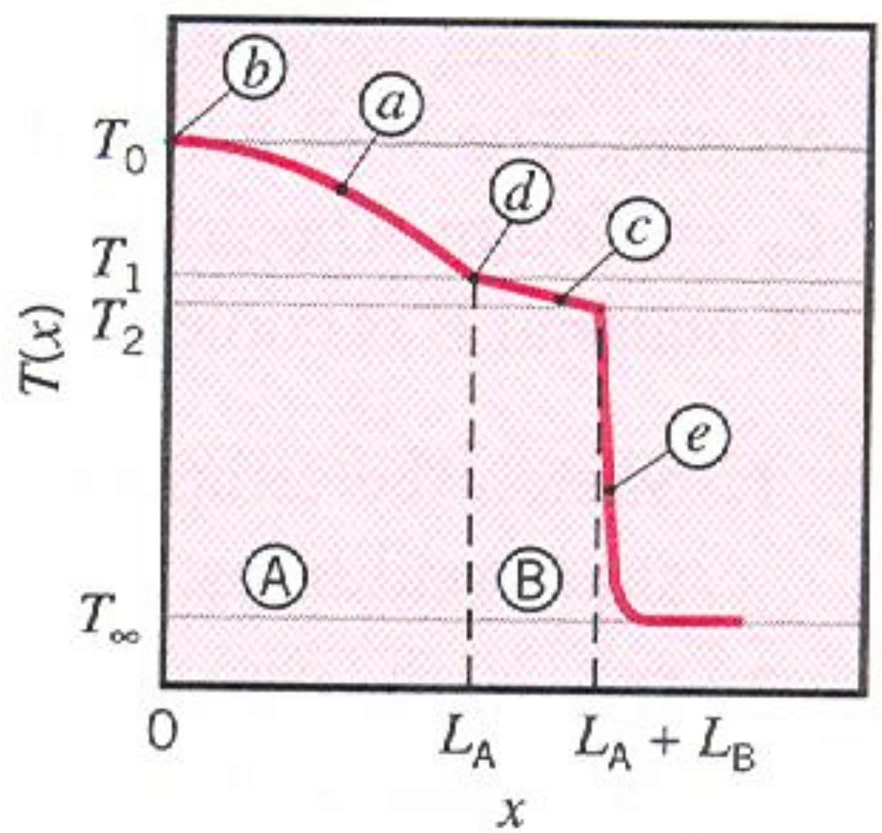


Figure for the example

Solution

1. Features of temperature distribution

   Parabolic in material A ; Zero slope at insulated boundary; Linear in material B; Slope change = k_B/k_A = 2 at interface; Large gradients near the surface (water).



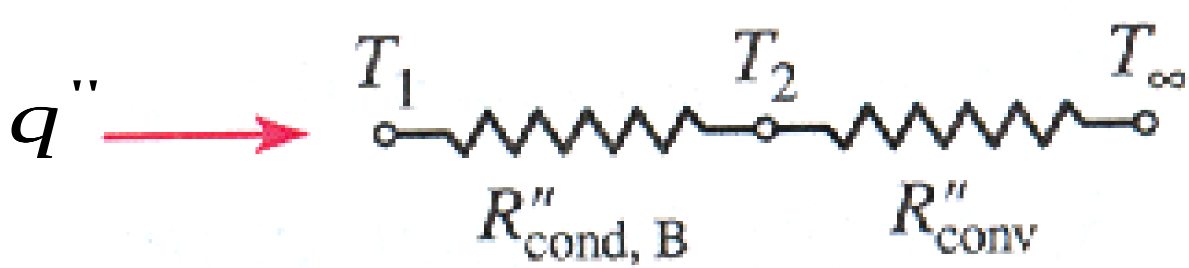
2. At the outer surface, $x=L_A+L_B$

$$q'' = h\,(T_2 - T_\infty)$$

Also $\dot{q}'''L_A = q''$ Equating the two gives

$$T_2 = T_\infty + \frac{\dot{q}'''L_A}{h} = 30 + \frac{1.5\text{x}10^6\text{x}0.05}{1000} = 105^{o}C$$

Since T_o is given in terms of T_1, using the circuit diagram shown below



Using

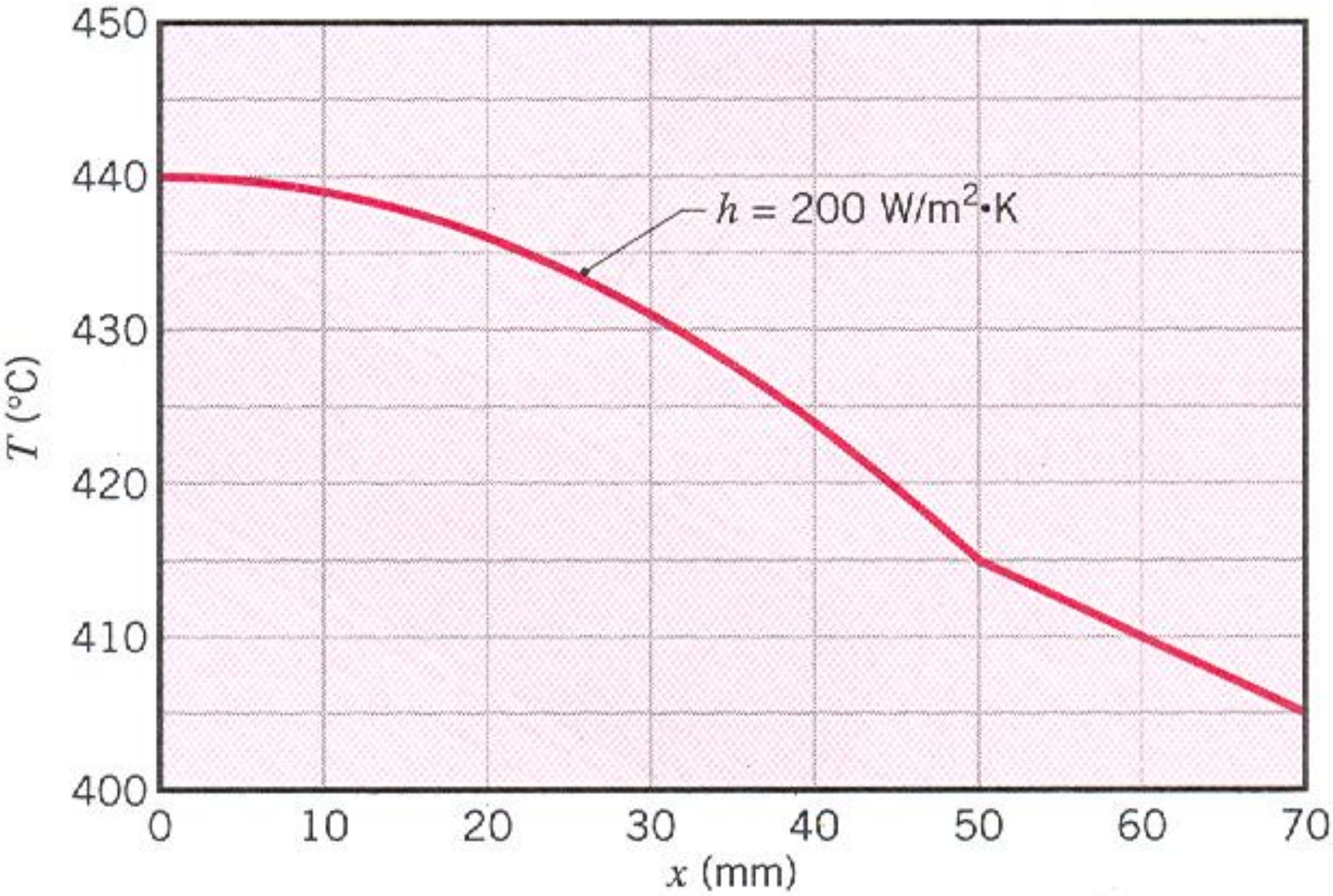
$$q'' = \frac{T_1 - T_\infty}{R''_{cond,B} + R''_{conv}}; \quad T_1 = T_\infty + (R''_{cond,B} + R''_{conv,B})q''$$

$$R''_{cond,B} = \frac{L_B}{k_B} \qquad R''_{conv} = \frac{1}{h}$$

$$T_1 = 30 + \left(\frac{0.02}{150} + \frac{1}{1000}\right) x1.5x10^6 x0.05 = 30 + 85 = 115^oC$$

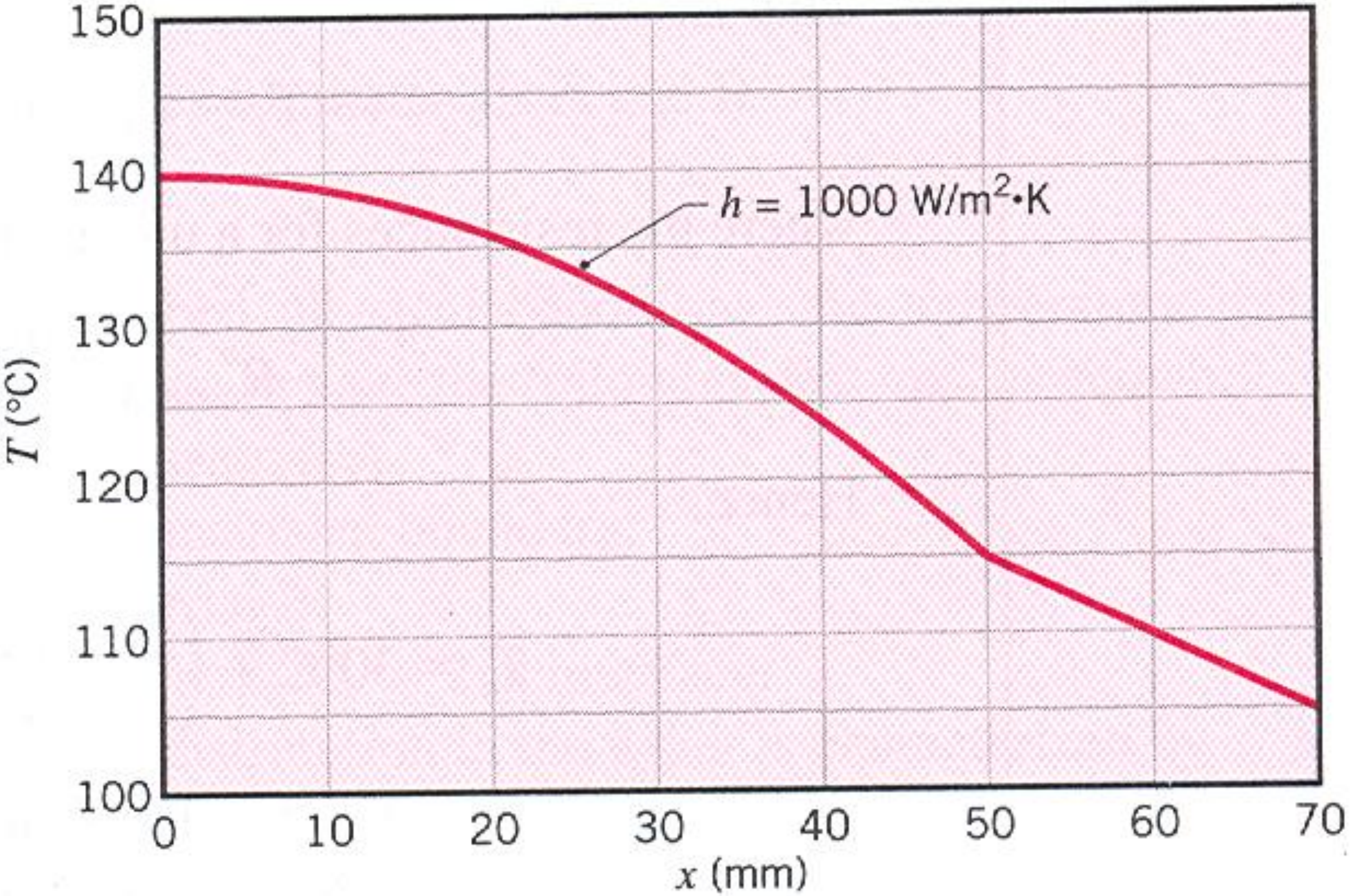
Substitution gives

$$T_o = \frac{1.5x10^6 x0.05}{2x75} + 115 = 25 + 115 = 140^oC$$



Thermal circuit for example 3.4

Temperature distribution for h = 200 W/m^2.K



Thermal circuit for example 3.4

Temperatur distribution for h = 1000 W/m^2.K

3.4.2 Radial Systems

For the cylinder shown in **fig-chp3\fig.3.10.pptx** , the heat generated is convected from the surface of the cylinder.

The appropriate equation for analysis is (constant k)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}'''}{k}=0$$

Integrating twice gives

$$T(r)=-\frac{\dot{q}'''}{4k}r^{2}+C_{1}\ln r+C_{2}$$

Boundary conditions are

$$\frac{dT(0)}{dr} = 0 \quad \text{and} \quad T(r_o) = T_s$$

The first gives $C_1 = 0$ and the second gives

$$C_2 = T_s + \frac{\dot{q}'''}{4k} r_o^2$$

This gives the temperature distribution as

$$T(r) = \frac{\dot{q}''' r_o^2}{4k}\left(1 - \frac{r^2}{r_o^2}\right) + T_s$$

Using the centerline temperature $T(0) = T_o$

$$\frac{T(r) - T_s}{T_o - T_s} = 1 - \left(\frac{r}{r_o}\right)^2$$

To relate T_s to T_∞, use is made of the equality of heat generated and heat convected from surface

$$\dot{q}'''(\pi r_o^2 L) = h(2\pi r_o L)(T_s - T_\infty)$$

$$T_s = T_\infty + \frac{\dot{q}''' r_o}{2h}$$

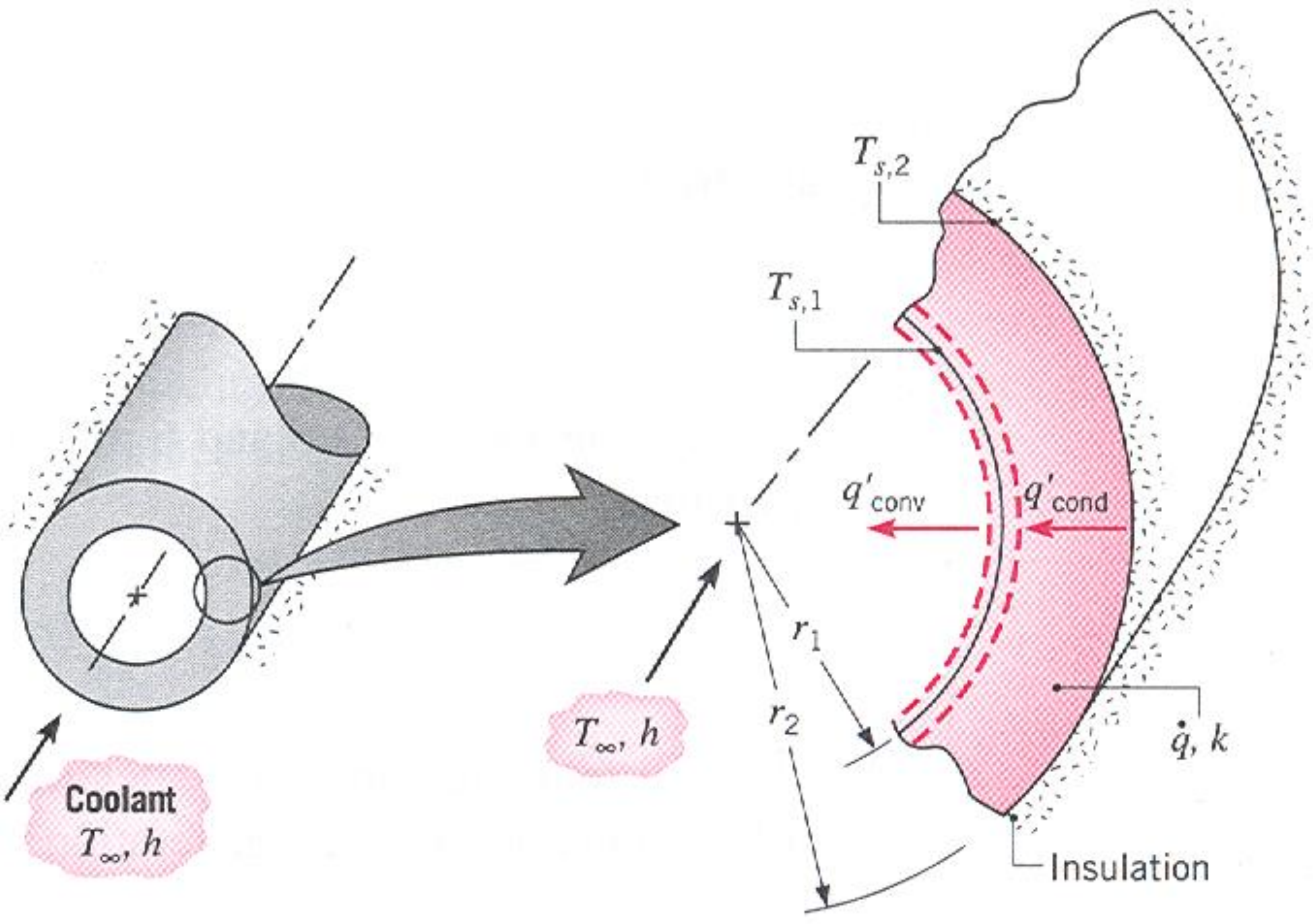
One needs to make a note that the thermal resistance approach has not been used as the heat transfer rate with heat generation is not constant.

Example 3.6

Consider a long solid tube, insulated at the outer radius r_2 (adiabatic surface with a prescribed temperature $T_{s,2}$)and cooled at the inner radius r_1, with uniform heat generation within the solid.

1. Obtain the general and particular solution for the temperature distribution in the tube.
2. Determine the heat removal rate per unit length of tube.
3. If the coolant is available at a temperature T_∞, obtain an expression for the convection coefficient that would have to be maintained at the inner surface to allow for operation at prescribed

values of $T_{s,2}$ and $\dot{q}'''$.



Solution

(1) The general solution is

$$T(r) = -\frac{\dot{q}'''}{2k} r^2 + C_1 \ln r + C_2$$

Boundary conditions $\dfrac{dT(r_2)}{dr} = 0$

$T(r_2) = T_{s,2}$ and

These give

$$C_1 = \frac{\dot{q}'''}{2k} r_2^2 \quad \text{and} \quad C_2 = T_{s,2} + \frac{\dot{q}'''}{4k} r_2^2 - \frac{\dot{q}'''}{2k} r_2^2 \ln \frac{r_2}{r}$$

And the temperature distribution after substitution

$$T(r) = T_{s,2} + \frac{\dot{q}'''}{4k}(r_2^2 - r^2) - \frac{\dot{q}'''}{2k} r_2^2 \ln \frac{r_2}{r}$$

(2) Heat removal rate is the conduction heat rate at $r=r_1$

$$q_r^{'}(r_1) = -k(2\pi r_1)\frac{dT(r_1)}{dr} = -k2\pi r_1\left(-\frac{\dot{q}^{'''}}{2k}r_1 + \frac{\dot{q}^{'''}}{2k}\frac{r_2^2}{r_1}\right)$$

$$q_r^{'}(r_1) = -\dot{q}^{'''}\pi(r_2^2 - r_1^2)$$

$$q_{r-tot} = -\dot{q}^{'''}\pi(r_2^2 - r_1^2)L$$

Alternatively, the same result could be found by equating the heat generated with the removal at r_1.

$$\dot{E}_g = \dot{q}^{'''}\pi(r_2^2 - r_1^2)L = E_{out}$$

(3) At the inner surface

$$q_{cond}^{'} = q_{conv}^{'} \qquad \text{or}$$

$$\pi\dot{q}^{'''}(r_2^2 - r_1^2) = h2\pi r_1(T_{s,1} - T_\infty)$$

Hence

$$h = \frac{\dot{q}'''(r_2^2 - r_1^2)}{2r_1(T_{s,1} - T_\infty)}$$

3.5 HEAT TRANSFER FROM EXTENDED SURFACES

The characteristics of an extended surface is shown in **fig-chp3\fig.3.11.pptx** . For $T_1>T_2$, heat transfer by conduction occurs in the strut. For $T_1>T_2>T_\infty$, in addition to heat conduction there will be convection heat transfer from the surfaces of the strut thus decreasing q_x continuously and hence $\left|\frac{dT}{dx}\right|$. When compared with pure conduction the slope at the base is increased thus increasing the conduction rate.

Thus extended surfaces are used to enhance the heat transfer rate by increasing the surface area. The term ***fin*** is used for extended surfaces.

Consider the heat transfer from surface given in **fig-chp3\fig.3.12.pptx** (a). For fixed T_s, T_∞ and A, heat transfer can be maximized by increasing h, which will require a higher velocity of fluid (need of blower or pump)-becomes costly. The other option is to increase the surface area for convection as shown (b) where fins or extended surfaces are used.

- k has pronounced effect on the heat transfer effect

Infinite k-no temperature gradient, max q

fig-chp3\fig.3.13.pptx and **fig-chp3\fig.3.14.pptx**

Selection depends on space, weight, manufacturing and cost reduction as well as the reduction of h and increase of ΔP.

3.5.1 A General Conduction Analysis

Temperature distribution along the fin is to be obtained. This will allow the heat transfer rate to be determined from the surface and tip of fin shown in **fig-chp3\fig.3.15.pptx**.

The heat transfer originates from the base and finally transferred through the surface and tip of the fin. Since the fins involved are very thin it can safely be assumed the heat transfer is one-dimensional.

Energy balance on the differential CV gives

$$q_x = q_{x+dx} + dq_{conv}$$

$$q_x = -kA_c \frac{dT}{dx} \qquad q_{x+dx} = q_x + \frac{dq_x}{dx} dx$$

$$q_{x+dx} = -kA_c \frac{dT}{dx} - k \frac{d}{dx}\left(A_c \frac{dT}{dx} \right) dx$$

$$q_{conv} = h\, dA_s (T - T_\infty)$$

Substitution in the energy balance equation gives

$$\frac{d}{dx}\left(A_c \frac{dT}{dx} \right) - \frac{h}{k} \frac{dA_s}{dx} (T - T_\infty) = 0$$

After performing the differentiation, the final general fin equation becomes

$$\frac{d^2T}{dx^2}+\left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx}-\left(\frac{1}{A_c}\frac{h}{k}\frac{dA_s}{dx}\right)(T-T_\infty)=0$$

This equation will be used on specific fin geometries to get the particular fin equations.

3.5.2 Fins of Uniform Cross-Sectional Area

The rectangular and the pin fin are shown in **fig-chp3\fig.3.16.pptx** .

A_c=constant, $dA_c/dx=0$ $A_s=Px$ $dA_s/dx=P$

Substitution in the general equation gives

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0$$

The above equation can be further simplified if we define $\theta(x) = T(x) - T_\infty$

$d\theta/dx = dT/dx$ θ is also called ***excess temperature***

This will give

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad where \quad m^2 \equiv \frac{hp}{kA_c}$$

The above is a linear, homogeneous, second order differential equation with constant coefficients

General solution

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} \qquad \frac{d\theta}{dx} = C_1 m e^{mx} - C_2 m e^{-mx}$$

Boundary Conditions

(1) Common $\theta(0) = T_b - T_\infty = \theta_b$

(2) There are four possible cases

I. Convection heat transfer from the tip (T_L=T(L))

fig-chp3\fig.3.17.pptx

$$hA_c[T(L) - T_\infty) = -kA_c \left.\frac{dT}{dx}\right|_{@\,x=L} \qquad or$$

$$h\theta(L) = -k \left.\frac{d\theta}{dx}\right|_{@\,x=L}$$

Substitution will give

$$\theta_b = C_1 + C_2$$

$$h\left(C_1 e^{mL} + C_2 e^{-mL}\right) = km\left(C_2 e^{-mL} - C_1 e^{mL}\right)$$

Final solution (**details.docx**)

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$$

Particular interest is the heat transfer rate from such fins. (a) from base (b) from surface of fins

(a) is easier

$$q_f = q_b = -kA_c \frac{dT(0)}{dx} = -kA_c \frac{d\theta(0)}{dx}$$

This will give

$$q_f = \sqrt{hPkA_c}\,\theta_b \frac{\mathbf{sinh}\,mL + (h/mk)\,\mathbf{cosh}\,mL}{\mathbf{cosh}\,mL + (h/mk)\,\mathbf{sinh}\,mL}$$

II. Convection heat transfer from the tip is negligible in which case the tip may be treated as adiabatic.

$(dT/dx)_{x=L}=0$ or $(d\theta/dx)_{x=L}=0$

This will give the second equation as

$C_1e^{mL} - C_2e^{-mL} = 0$ $C_1 + C_2 = \theta_b$ (common)

The above will give the solution as

$$\frac{\theta}{\theta_b} = \frac{\mathbf{cosh}\,m(L-x)}{\mathbf{cosh}\,mL}$$

The heat transfer in this case is determined as

$$q_f = \sqrt{hPkA_c}\,\theta_b \tanh mL = M \tanh mL; \; M = \sqrt{hPkA_c}\,\theta_b$$

This is the most popular form

III. Temperature is prescribed at the tip,

θ(L)=θ$_L$=T$_L$-T$_\infty$

Solution will be

$$\frac{\theta}{\theta_b} = \frac{(\theta_L/\theta_b)\sinh mx + \sinh m(l-x)}{\sinh mL} \qquad and$$

$$q_f = M\,\frac{\cosh mL - (\theta_L/\theta_b)}{\sinh mL}$$

IV. The fin is very long such that as

$L \to \infty$, $T \to T_\infty$ or $\theta_L \to 0$

The solution

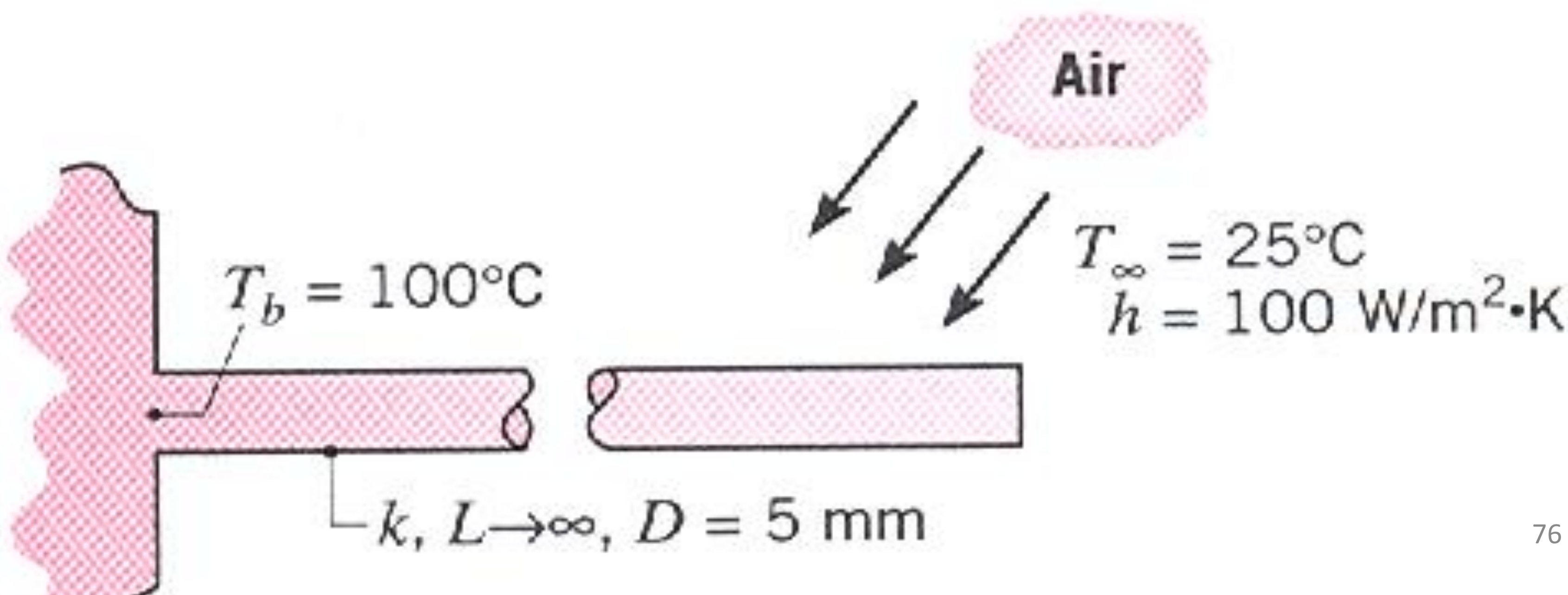
$$\frac{\theta}{\theta_b} = e^{-mx} \qquad\qquad q_f = M$$

table3.1.docx summarizes the solutions for different BC.

Example 3.7

A very long rod 5 mm in diameter has one end maintained at 100°C. The surface of the rod is exposed to ambient air at 25°C with a convection heat transfer coefficient of 100 W/m^2.K.

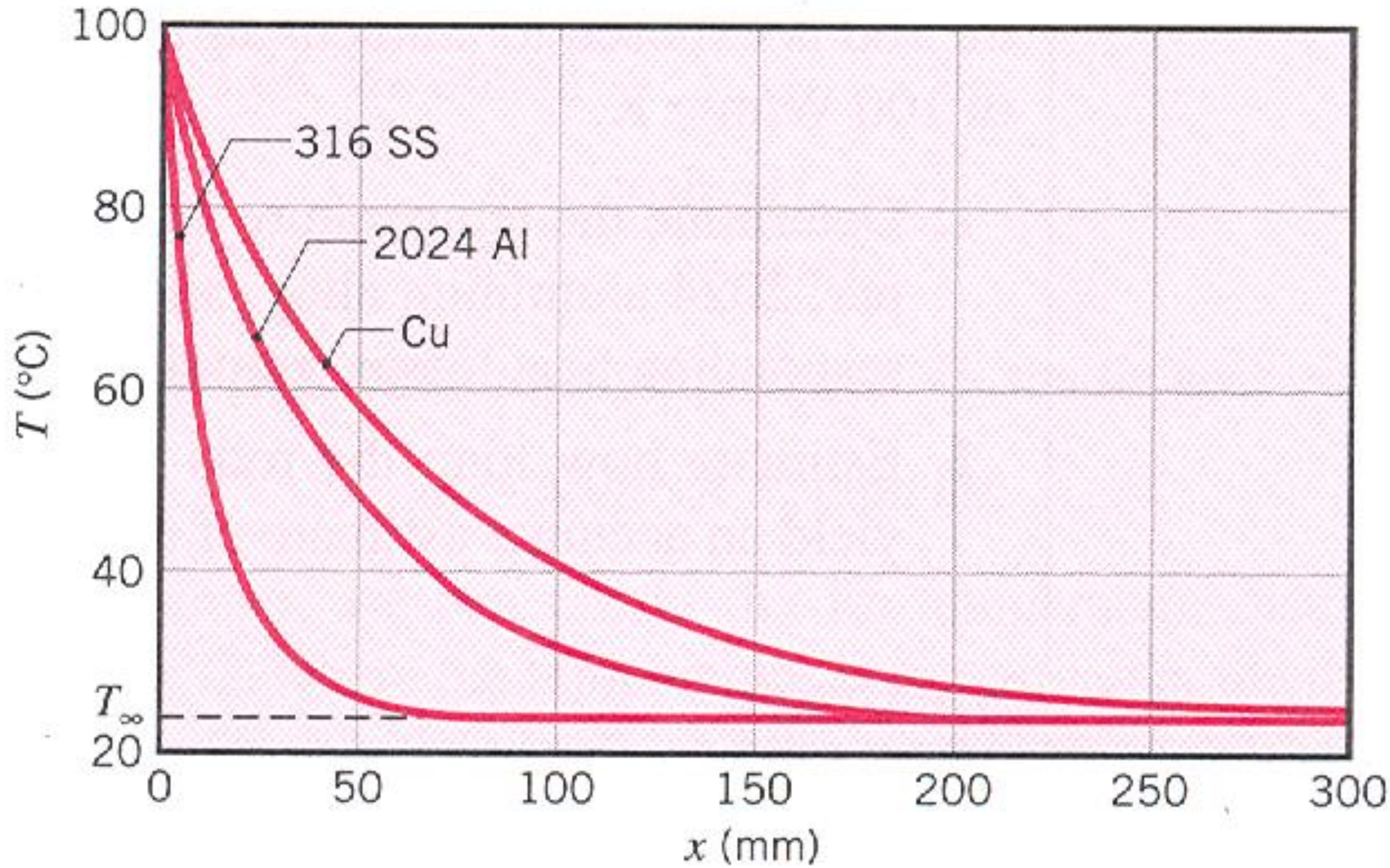
1. Determine the temperature distributions along rods constructed from Cu, Al and SS with k=398, 180, 14 W/m.K, respectively. What are the corresponding heat losses from the rods?
2. Estimate how long the rods must be for the assumption of infinite length to yield an accurate estimate of the heat loss.



Solution

1. $\theta/\theta_b = e^{-mx}$ or $T = T_\infty + (T_b - T_\infty)e^{-mx}$

where $m = (hP/kA_c)^{1/2} = (4h/kD)^{1/2} = 14.2,\ 21.2,\ 75.6\ m^{-1}$. T(x) shown below



$q_f = (hPkA_c)^{1/2}\theta_b$ $=(100\pi$ x 0.005 x 398 x $(\pi/4)$ x $0.005^2)^{1/2}$ (75)= 8.3W

For Al and SS q_f = 5.6 W and 1.6 W respectively.

2. Since no heat transfer at the tip, comparing cases II and IV one can approximate with

$$\mathbf{tanh}\, mL \geq 0.99 \quad or \quad mL \geq 2.65$$

Hence a rod may be assumed to be infinitely long if

$$L \geq L_\infty \equiv \frac{2.65}{m} = 2.65\left(\frac{kA_c}{hP}\right)^{1/2}$$

For copper

$$L_\infty = \left(\frac{398(\pi/4)(0.005)^2}{100(\pi\, x\ 0.005)}\right) = 0.19\,m$$

Corresponding values for Al and SS are 0.13 and 0.04 m respectively.

If the approximation is to accurately predict the temperature distribution one may use

$\theta(L)/\theta_b = \exp(-mL) < 0.01$

This gives $mL > 4.6$ in which case $L_\infty \approx 0.33$, 0.23, and 0.07 for the copper, aluminum, and stainless steel. This is consistent with the graphical representation

3.5.3 Fin Performance

As the fin itself does have conduction resistance, there may be no assurance that the heat transfer will be increased by using fins. The evaluation uses ***fin effectiveness***, ε_f defined by

$$\boldsymbol{\varepsilon_f = \frac{q_f}{hA_{c,b}\theta_b}}$$

where $A_{c,b}$ is the fin cross-sectional area at the base. Usage of fins may not be justified unless $\varepsilon_f \geq 2$.

As an example for case IV

$$\boldsymbol{\varepsilon_f = \frac{\sqrt{hPkA_c}\,\theta_b}{hA_{c,b}\theta_b} = \left(\frac{kP}{hA_c}\right)^{1/2}}$$

since $A_c = A_{c,b}$ for uniform cross-section. The above relation gives sufficient information for the important parameters

- ε_f increases with k-with respect to Al and Cu, Al preferred due to low weight and low cost
- ε_f increases with P/A_c- Use of thin fins recommended
- ε_f increases with small h-the need for fins is stronger on the gas side than on the liquid side or the natural convection side than the forced convection side.

Fin performance may also be quantified in terms of thermal resistance.

Thermal resistance due to convection at the exposed base

$$\mathbf{R_{t,b} = \frac{1}{hA_{c,b}}}$$

The conduction/convection resistance of a fin is given by

$$\mathbf{R_{t,f} = \frac{\theta_b}{q_f}}$$

Performing division gives

$$\mathbf{\frac{R_{t,b}}{R_{t,f}} = \frac{(1/hA_{c,b})}{\theta_b/q_f} = \frac{q_f}{hA_{c,b}\theta_b} = \varepsilon_f}$$

Need low $R_{t,f}$ to increase ε_f. Also $R_{t,f} < R_{t,b}$ (a must)

A commonly used measure of fin performance is by using ***fin efficiency, η_f***. This compares q_f with the maximum possible heat transfer (when all the fin surface is at base temperature

$T = T_b$ or $\theta = \theta_b = (T_b - T_\infty)$

Based on this

$$\eta_f = \frac{q_f}{q_{max}} = \frac{q_f}{hA_f\theta_b}$$

For a straight fin of uniform cross-section and adiabatic tip

$$\eta_f = \frac{M\,\mathbf{tanh}\,mL}{hPL\theta_b} = \frac{\mathbf{tanh}\,mL}{mL} \qquad (M = \sqrt{hPkA_c}\,\theta_b)$$

η_f: $1 \rightarrow 0$ as L: $0 \rightarrow \infty$

For a fin with an active tip, the length can be slightly increased (corrected) so that we have an adiabatic tip, and use the simplified relation of case II. The corrected lengths are given as

$L_c = L + (t/2)$ - rectangular fin

$L_c = L + (D/4)$ - pin fin

Errors associated with these approximations are negligible if

$$\frac{ht}{k} \text{ or } \frac{hD}{2k} \leq 0.0625$$

For w >> t which is usually the case $P \approx 2w$

And $(P/A_c) = (2w/wt) = 2/t$

$$mL_c = \left(\frac{hP}{kA_c}\right)^{1/2} L_c \qquad = \left(\frac{2h}{kt}\right)^{1/2} L_c$$

Multiplying numerator and denominator by $(L_c)^{1/2}$ and introducing a corrected fin profile area $A_p = L_c t$, it follows that

$$mL_c = \left(\frac{2h}{kA_p}\right)^{1/2} L_c^{3/2}$$

η_f is plotted as a function of $\left(\frac{2h}{kA_p}\right)^{1/2} L_c^{3/2}$ in

fig-chp3\fig.3.18.pptx for rectangular, triangular, and parabolic profiles.

3.5.4 Fins of Nonuniform Cross-Sectional Area

Consider the annular fin shown in **fig-chp3\fig.3.19.pptx** . While the thickness is constant, the cross-sectional area A_c varies with r.

$A_c = 2\pi rt$ and $A_s = 2\pi(r^2 - r_1^2)$

Substituting in the general fin equation will give

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} - \frac{2h}{kT}(T - T_\infty) = 0$$

With $m^2 = (2h/kt)$ and $\theta = T - T_\infty$

The equation becomes

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr} - m^2\theta = 0 \qquad r^2\frac{d^2\theta}{dr^2}+r\frac{d\theta}{dr} - m^2r^2\theta = 0$$

This is a modified Bessel equation of order zero whose general solution is given by (Hyperbolic Functions Error Function.docx

$$\theta(r) = C_1 I_o(mr) + C_2 K_o(mr)$$

Where I_o and K_o are modified, zero order Bessel functions of the first and second kind respectively.

B.C. used are $\theta(r_1) = \theta_b$ and adiabatic tip

$$\frac{d\theta(r_2)}{dr} = 0$$

will finally give the solution for the temperature distribution as

$$\frac{\theta}{\theta_b} = \frac{I_o(mr)K_1(mr_2) + K_o(mr)I_1(mr_2)}{I_o(mr_1)K_1(mr_2) + K_o(mr_1)I_1(mr_2)}$$

Where $I_1(mr)$ and $K_1(mr)$ are modified first order Bessel functions of the first and second kinds.

The heat transfer by conduction from the base

$$q_f = -kA_{c,b}\frac{dT(r_1)}{dr} = -k(2\pi r_1 t)\frac{d\theta(r_1)}{dr} \quad \text{gives}$$

$$q_f = 2\pi r_1 t\theta_b m\frac{K_1(mr_1)I_1(mr_2) - I_1(mr_1)K_1(mr_2)}{K_o(mr_1)I_1(mr_2) + I_o(mr_1)K_1(mr_2)}$$

The fin efficiency is given by

$$\eta_f = \frac{q_f}{h[2\pi(r_2^2 - r_1^2)]\theta_b}$$

$$= \frac{2r_1}{m(r_2^2 - r_1^2)} \frac{K_1(mr_1)I_1(mr_2) - I_1(mr_1)K_1(mr_2)}{K_o(mr_1)I_1(mr_2) + I_o(mr_1)K_1(mr_2)}$$

For an active convection tip, use $r_{2c} = r_2 + t/2$

The fin efficiency is given graphically in **fig-chp3\fig.3.20.pptx** .

For determination of fin effectiveness, ε_f , the conduction/convection fin resistance is given by

$R_{t,f} = 1/(hA_f\eta_f)$

For equivalent heat transfer the triangular fin requires much less volume than a rectangular profile. Slightly larger (q/V) is parabolic. But when considering the manufacturing cost triangular is the more preferred one. The annular fin of rectangular profile is commonly used to enhance heat transfer in circular tubes.

Expressions for the efficiency and surface area of several common fin geometries are summarized in **table3.2.docx**

3.5.5 Overall Surface Efficiency

Unlike η_f which considers a single fin, the ***overall surface efficiency, η_o*** considers the array of fins and the base surface. **fig-chp3\fig.3.21.pptx** shows typical arrays of fins.

In each case the overall efficiency is defined as

$$\eta_o = \frac{q_t}{q_{max}} = \frac{q_t}{hA_t\theta_b}$$

q_t = total heat transfer rate = $q_f + q_b$

A_t = total surface area = $NA_f + A_b$

The total heat transfer rate from fins and bare surface by convection is given by and then substitution of

$A_b = A_t - NA_f$

$q_t = N\,\eta_f\,h\,A_f\,\theta_b + h\,A_b\,\theta_b$

$$q_t = h[N\eta_f A_f + (A_t - NA_f)]\theta_b = hA_t\left[1 - \frac{NA_f}{A_t}(1-\eta_f)\right]\theta_b$$

Substitution in the expression for η_o gives

$$\eta_o = 1 - \frac{NA_f}{A_t}(1-\eta_f)$$

Using thermal fin resistance, the overall thermal resistance can be expressed as

$$R_{t,o} = \frac{\theta_b}{q_t} = \frac{1}{\eta_o h A_t}$$

For fins machined as an integral part of the wall, the total resistance can be seen to be one of parallel circuit shown in **fig-chp3\fig.3.22(a).pptx** . This circuit will give the same overall efficiency as determined earlier.

But if the fins are manufactured separately and are attached (metallurgical, adhesive joint) or press fit to the wall, (**fig-chp3\fig.3.22(b).pptx**) there will be a thermal resistance, $R_{t,c}$, to be considered.

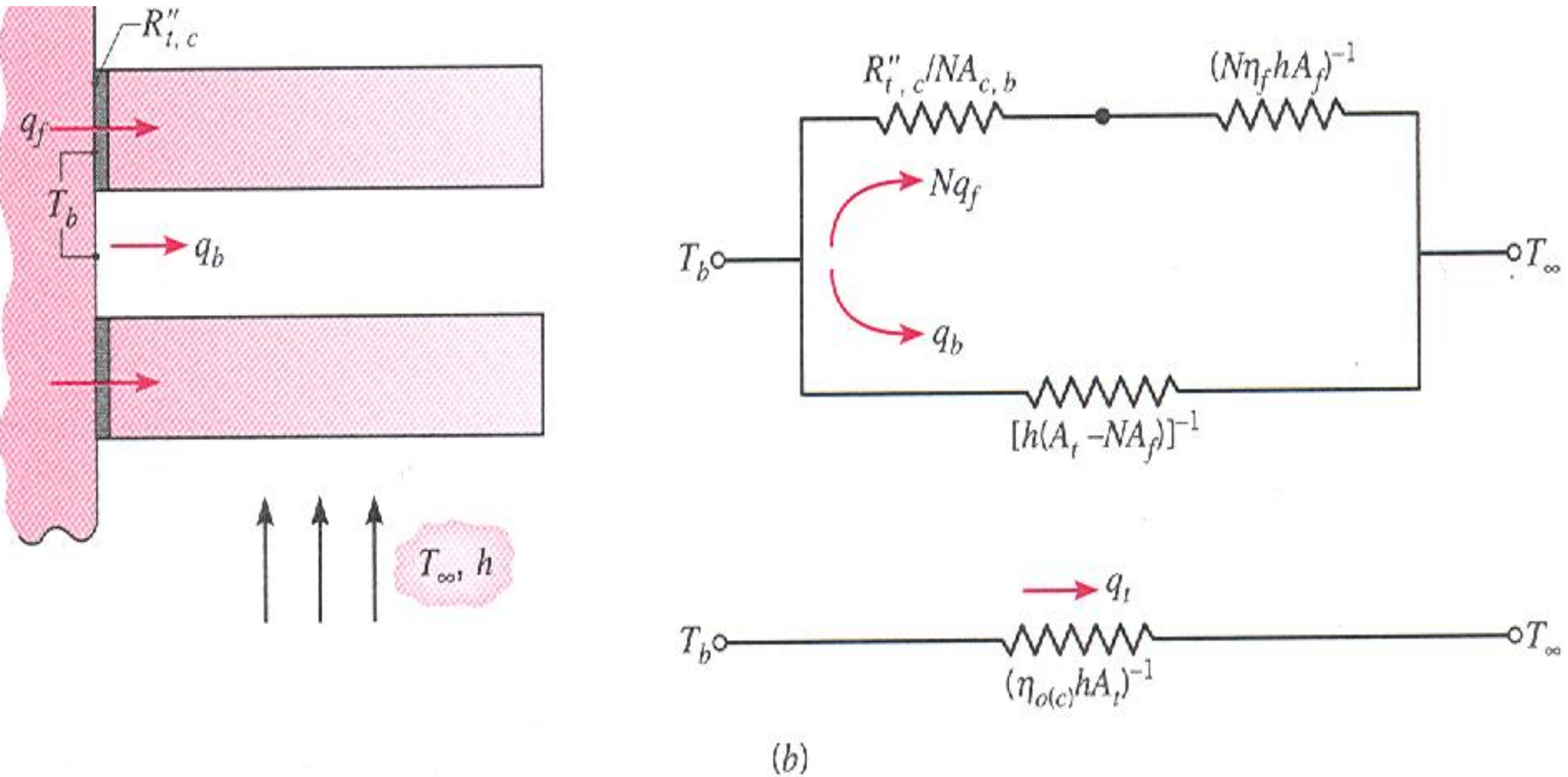


Fig.3.22(b) Fin array and thermal circuit-fins attached to base

The effective circuit resistance will be

$$\eta_{o(c)} = \frac{q_t}{q_{max}} = \frac{q_t}{hA_t\theta_b} \Rightarrow \frac{\theta_b}{q_t} = \frac{1}{\eta_{o(c)}hA_t} = R_{t.o(c)}$$

Using the parallel/series circuit shown the effective overall efficiency can be determined as

$$\eta_{o(c)} = 1 - \frac{NA_f}{A_t}\left(1 - \frac{\eta_f}{C_1}\right)$$

where

$$C_1 = 1 + \eta_f hA_f (R''_{t,c} / A_{c,b})$$

While manufacturing make sure $R_{tc} << R_{tf}$

Example 3.8

The engine cylinder of a motor cycle is constructed of 2024-T6 aluminum alloy and is of height H = 0.15 m and outside diameter D = 50 mm. Under typical operating conditions the outer surface of the cylinder is at a temperature of 500 K and is exposed to ambient air at 300 K, with a convection coefficient of 50 $W/m^2.K$. Annular fins are integrally cast with the cylinder to increase heat transfer to the surroundings. Consider five such fins, which are of thickness t = 6 mm. length L = 20 mm, and equally spaced. What is the increase in heat transfer due to use of the fins?

Solution

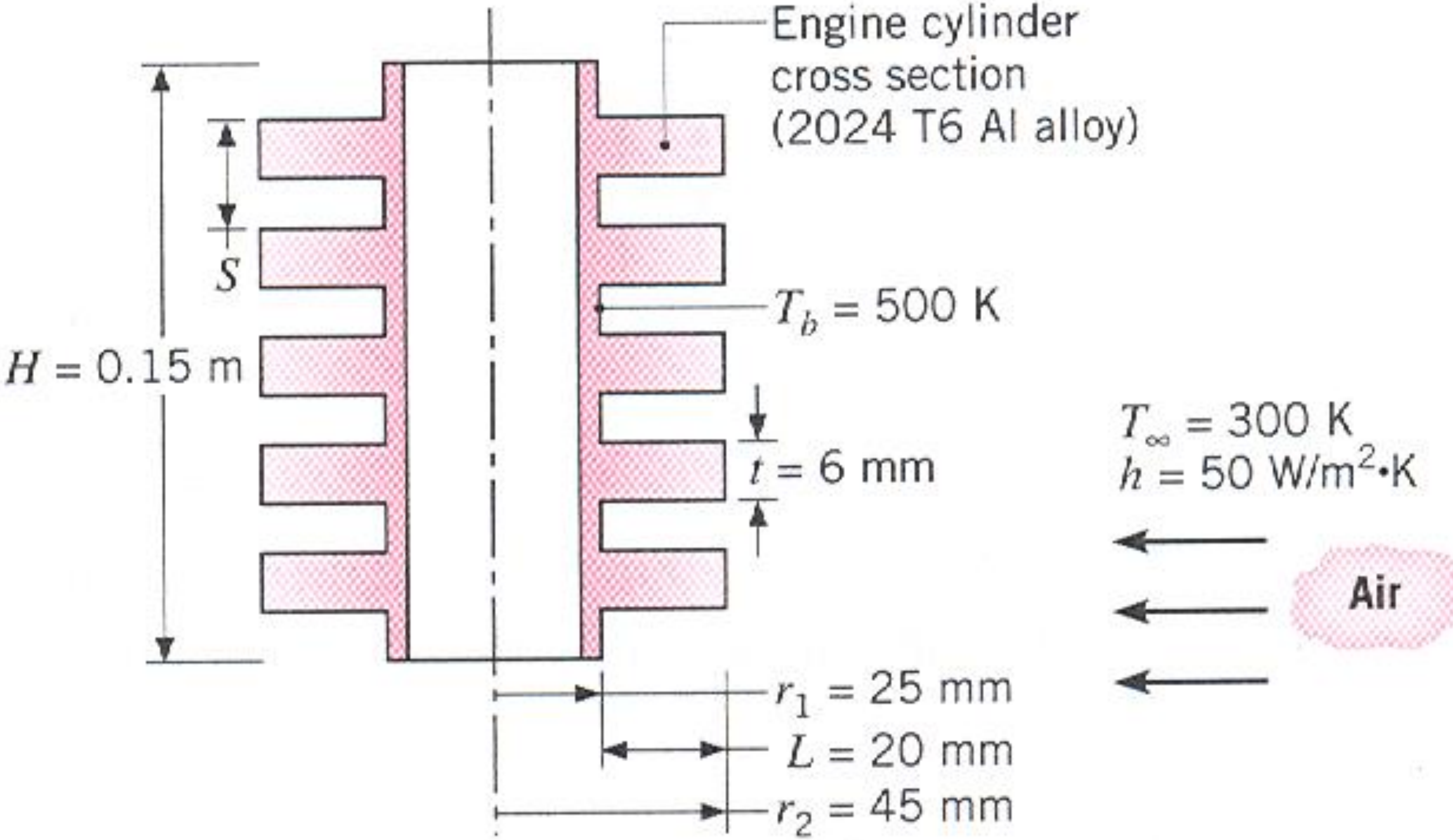


Figure for example 3.8

$$q_t = hA_t\left[1-\frac{NA_f}{A_t}(1-\eta_f)\right]\theta_b$$

$$A_f = 2\pi(r_{2c}^2 - r_1^2) = 2\pi(0.048^2 - 0.025^2) = 0.0105m^2$$

$$A_t = NA_f + 2\pi r_1(H - Nt)$$

$$= 5x0.0105 + 2\pi(0.025)[0.15 - 5x0.006] \quad = 0.0716m^2$$

$$\frac{r_{2c}}{r_1} = 1.92,\ \ L_c = 0.023m,\ \ A_p = 1.380x10^{-4}m^2 \ \ gives$$

$$\left(\frac{h}{kA_p}\right)L_c^{3/2} = 0.15$$

From Fig.3.19, $\eta_f \approx 0.95$

With the fins the total heat transfer rate becomes

$$q_t = 50 x 0.0716 \left[1 - \frac{0.0527}{0.0716}(1-0.95)\right](200) = 690W$$

Without fins

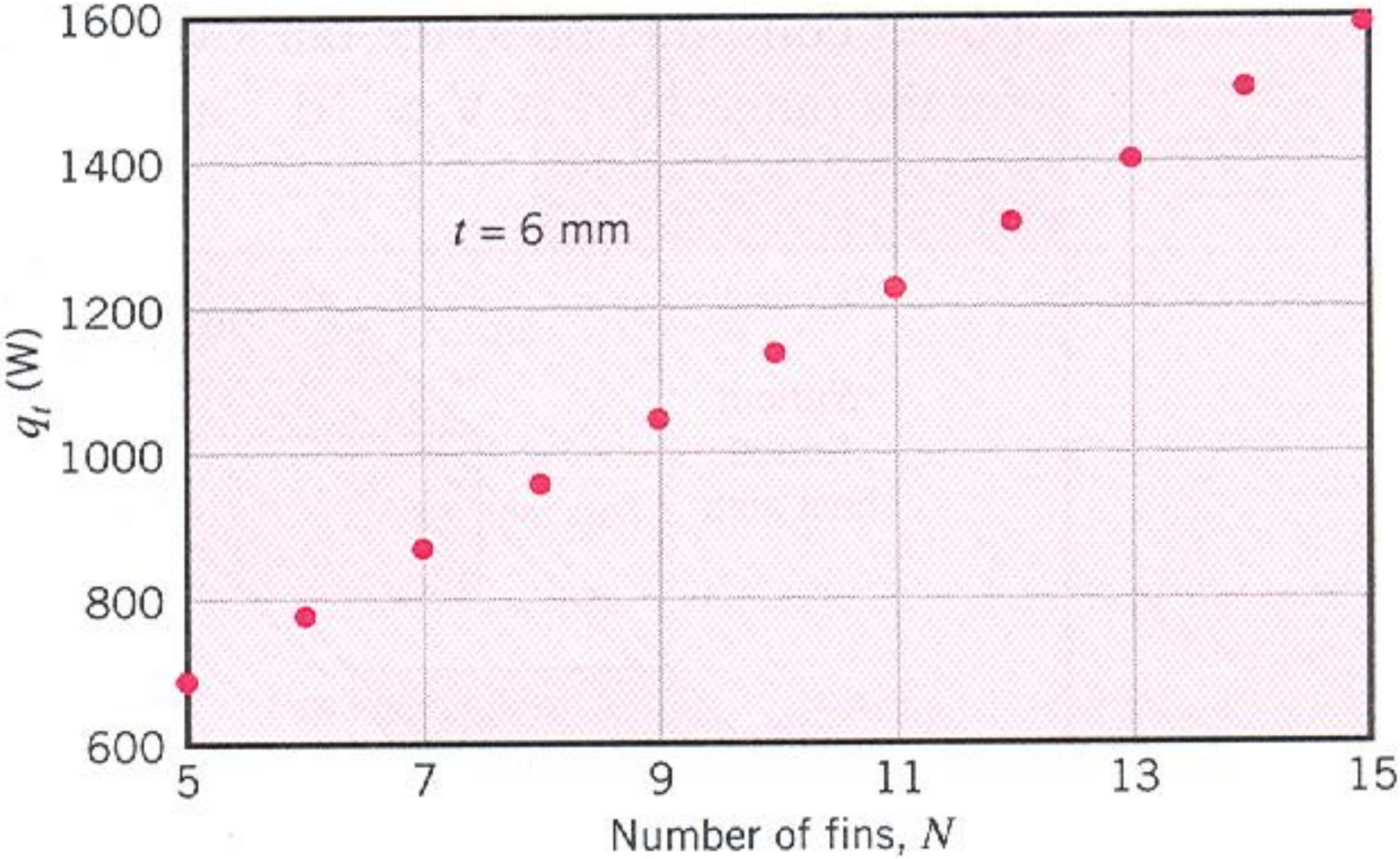
$q_{wo} = h(2\pi r_1 H)\theta_b = 50(2\pi \text{ x } 0.025 \text{ x } 0.15)(200)$

$=236W$

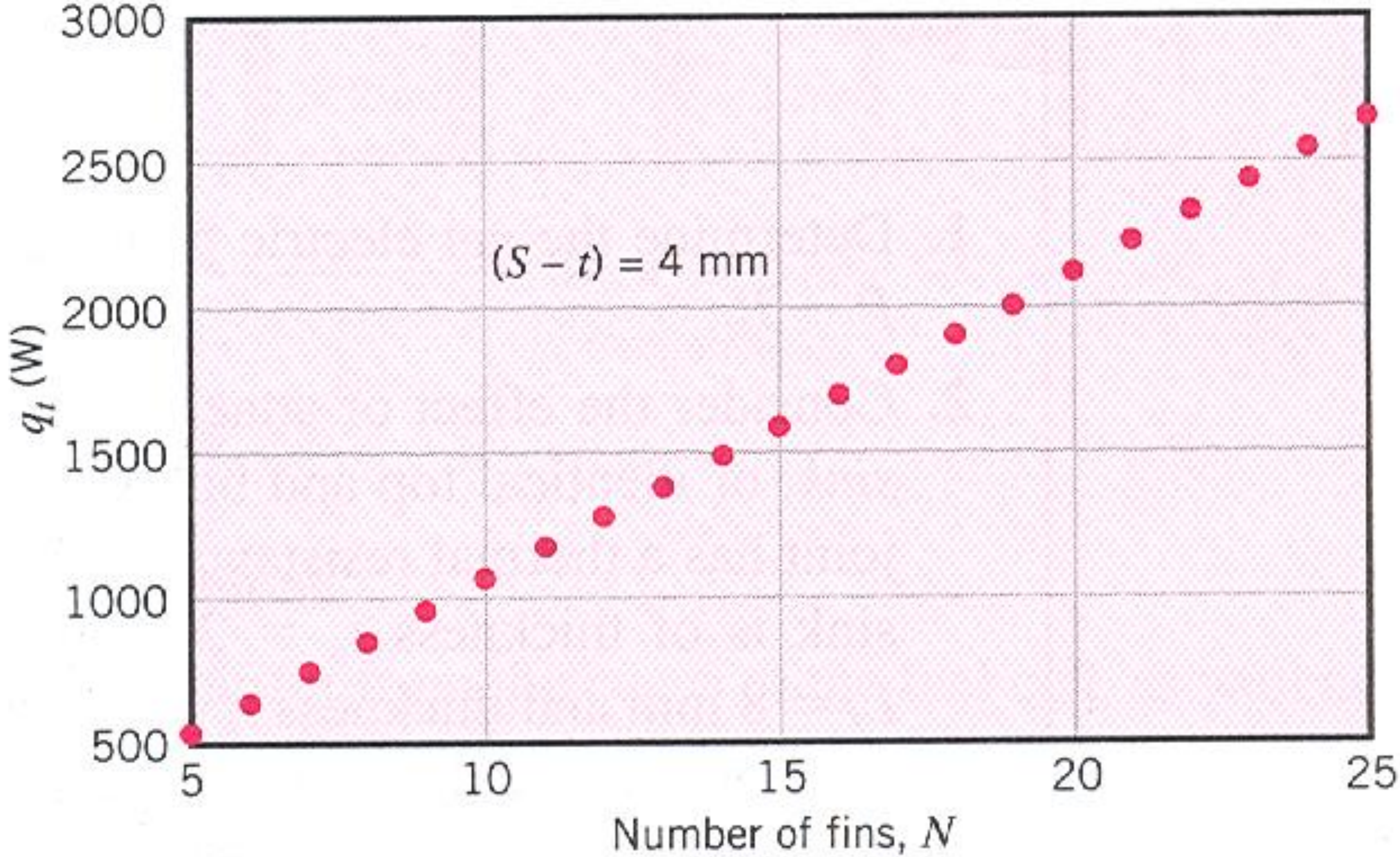
Increase

$\Delta q = q_t - q_{wo} = 454$ W (nearly 300% increase)

The increase could even be much higher if number of fins are increased for the same fin thickness or fin thicknesses are decreased to accommodate more number of fins in a given space. (see figures)



t = 6 mm No of fins are increased



In the given H for a fin thickness of 2mm, 25 fins could be accommodated

Example 3.9

In the example on PEM on chapter 1, we saw that to generate an electrical power P = 9 W, the temperature of the PEM fuel cell had to be maintained at $T_c \approx 56.4^oC$, which required total removal of 11.25 W from the fuel cell and a cooling air velocity of V = 9.4 m/s for $T_\infty = 25^oC$. To provide these convective conditions, the fuel cell is centered in a 50 mm x 26 mm rectangular duct, with 10 mm gaps between the exterior of the 50 mm x 50 mm x 6 mm fuel cell and the top and bottom of the well-insulated duct wall. A small fan, powered by the fuel cell, is used to circulate the cooling air. Inspection of a particular fan vendor's data

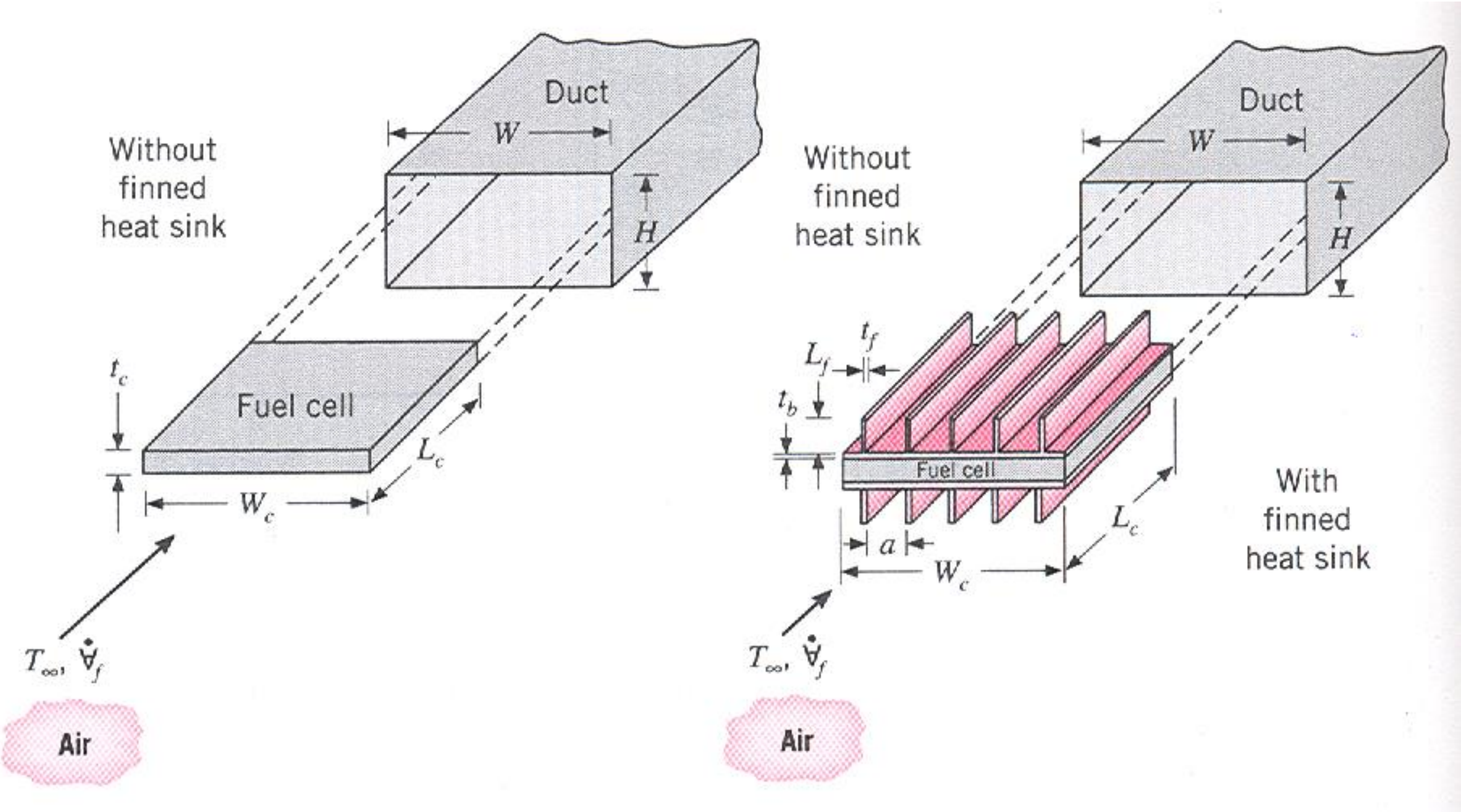


Figure for example 3.9

sheets suggest that the ratio of the fan power consumption to the fan's volumetric flow rate is

$P_f/\dot{V}_f = C = 1000\ W/(m^3/s)$ for the range

$10^{-4} \le \dot{V}_f \le 10^{-2}\ m^3/s$.

1. Determine the net electric power produced by the fuel-cell-fan system, $P_{net} = P - P_f$.
2. Consider the effect of attaching an aluminum (k=200 W/m.K) finned heat sink, of identical top and bottom sections, onto the fuel cell body. The contact joint has a thermal resistance of $R''_{t,c} = 10^{-3} m^2.K/W$, and the base of the heat sink is of thickness t_b = 2 mm. Each of the N rectangular fins is of length L_f = 8 mm and thickness

of 1 mm, and spans the entire length of the fuel cell,

L_c = 50 mm. With the heat sink in place, the radiation losses are negligible and the convective heat transfer coefficient may be related to the size and geometry of a typical air channel by an expression of the form

$h = 1.78\, k_{air} (L_f + a)/(L_f \,.\, a)$

where a is the distance between fins. Draw an equivalent thermal circuit for part 2 and determine the total number of fins needed to reduce the fan power consumption to half the value found in part 1.

Solution

1. Volumetric flow rate air for cooling

$$\dot{V} = VA_c \quad \text{and} \quad A_c = W(H - t_c)$$

Substitution gives

$$\dot{V} = V[W(H - t_c) = 9.4[0.05(0.026 - 0.006)]$$

$$= 9.4x10^{-3} m^3 / s$$

This enables us to use the formula for power consumption by the fan

$$P_{fan} = C\dot{V} = 1000x9.4x10^{-3} = 9.4W$$

$$P_{net} = P_{fc} - P_{fan} = 9 - 9.4 = -0.4W$$

As the fan consumes more power than is generated, the system cannot produce net power.

2. To reduce the fan power by 50%, the volumetric flow rate of air must be reduced to $\dot{V} = 4.7x10^{-3} m^3/s$

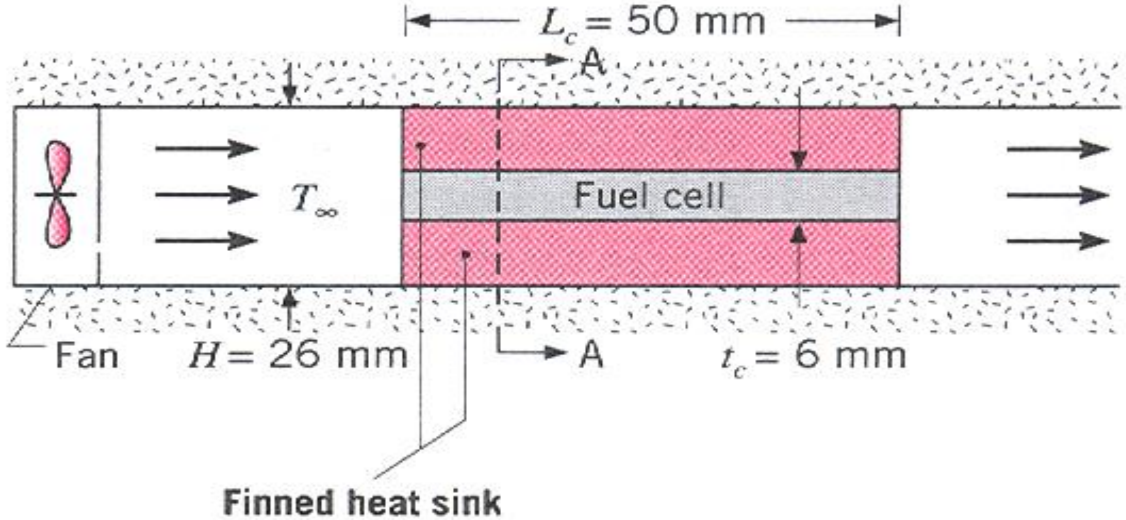
If we consider half of the fins the thermal resistances can be written as follows:

Contact resistance of the fin base:-

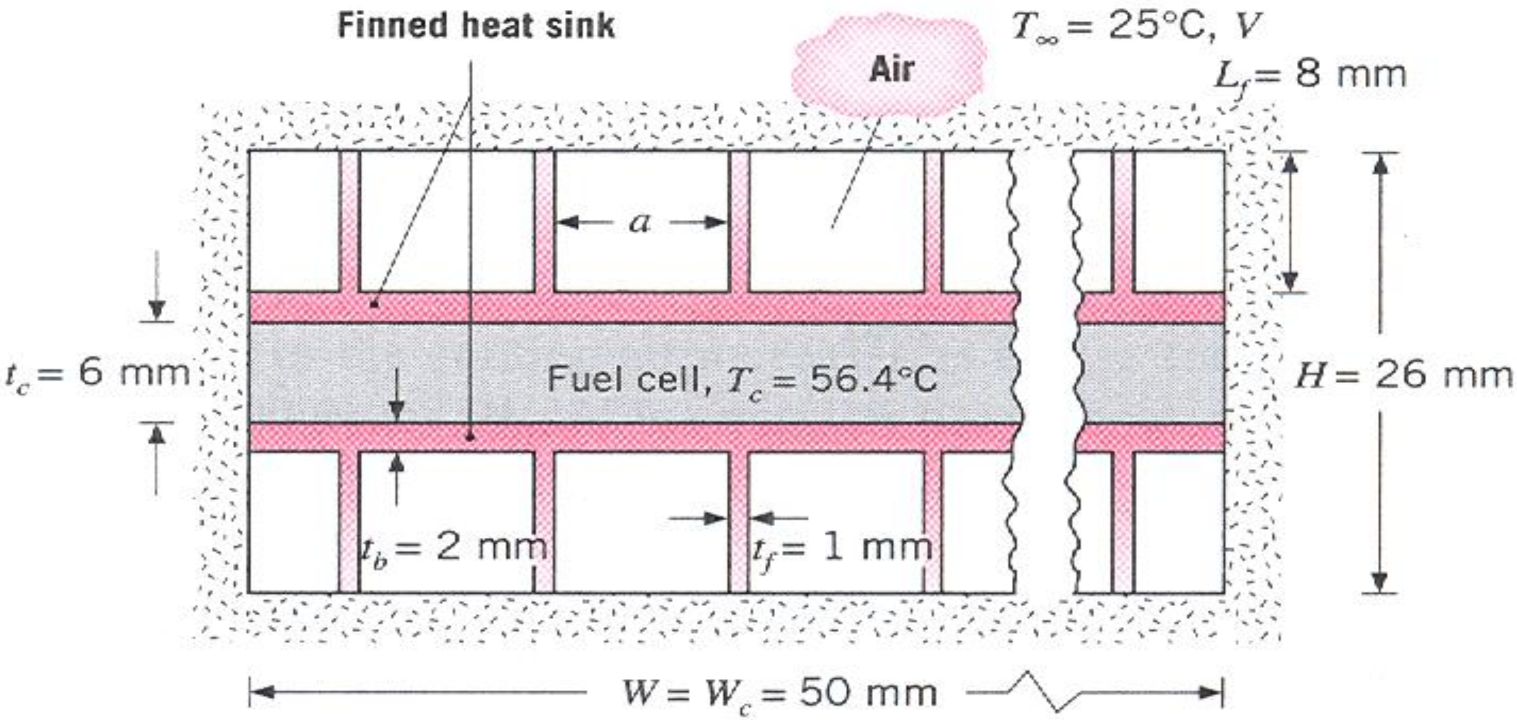
$$R_{t,c} = R''_{t,c} / L_c W_c = 10^{-3} /(0.05x0.05) = 0.4K/W$$

Fin base resistance:-

$$R_{t,base} = t_b / kL_c W_c = 0.002/(200x0.05x0.05) = 0.004K/W$$

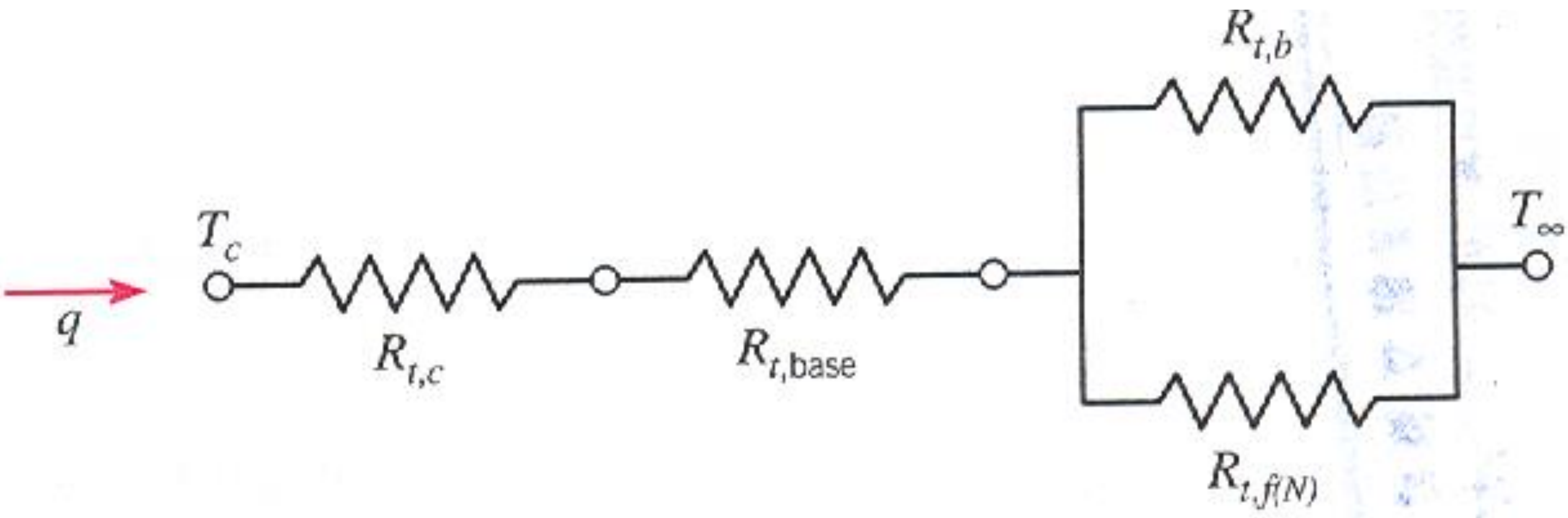


$L_c = 50$ mm
A
Fuel cell
T_∞
Fan
$H = 26$ mm
A
$t_c = 6$ mm
Finned heat sink



Finned heat sink
Air
$T_\infty = 25°C$, V
$L_f = 8$ mm
a
$t_c = 6$ mm
Fuel cell, $T_c = 56.4°C$
$H = 26$ mm
$t_b = 2$ mm
$t_f = 1$ mm
$W = W_c = 50$ mm

Section A–A



Thermal circuit for example 3.9

Bare surface of the base:-

$$R_{t,b} = 1/[h(W_c - Nt_f)L_c = 1/[h(0.05 - Nx0.001)0.05]$$

The above cannot be evaluated until N and h are determined

Fin resistance:-

For a single fin (insulated tip)

$$q_f = \sqrt{hPkA_c}\,\theta_b \tanh mL \qquad R_{t,f} = \theta_b / q_f = 1/\sqrt{hPkA_c}\tanh mL_f$$

$$P = 2(L_c + t_f) = 2(0.05+0.001) = 0.102 \text{ m}$$

$$A_c = L_c t_f = 0.05x0.001 = 0.00005 \text{ m}^2$$

$$m = \sqrt{hP/kA_c} = \sqrt{hx0.102/(200x0.00005)}$$

Substitution will give

$$R_{t,f} = \frac{(hx0.102x200x0.000005)^{-1/2}}{\tanh(mx0.008)}$$

And for N fins, $q_{f(N)} = Nq_f$ this will give

$R_{t,f(N)} = \theta_b/Nq_f = R_{t,f}/N$

$a = (W_c - Nt_f)/N = (0.05 - Nx0.001)/N$

$$R_{tot} = R_{t,c} + R_{t,base} + R_{equiv} \text{ and } R_{equiv} = [R_{t,b}^{-1} + R_{t,f(N)}^{-1}]^{-1}$$

R_{equiv} can be determined from

$$q = \frac{T_c - T_\infty}{R_{tot}} = \frac{T_c - T_\infty}{R_{t,c} + R_{t,base} + R_{equiv}}$$

In which case

$$R_{equiv} = \frac{T_c - T_\infty}{q} - (R_{t,c} + R_{t,base})$$

=(56.4-25)/5.625-(0.4+0.004)=5.2 K/W

The solution requires iteration which can be initiated by assuming N. Let N=11

Then this will give a=0.0035 m, h=19.1 W/m^2.K, m=13.9 m^{-1}, $R_{t,f(N)}$=5.88 K/W, $R_{t,b}$=26.8 K/W

This will give R_{equiv} = 4.82 K/W and R_{tot}= 5.224 resulting in a fuel cell temperature of 54.4^{o}C.

N=10 and N=12 give fuel cell temperatures of 58.9^{o}C and 50.7^{o}C which are far from the given fuel cell temperature.

So the total number of fins required will be 22.

$$P_{net} = P - P_f = 9.0 - 4.7 = 4.3 \text{ W}$$

In actual cases h is not sensitive to the velocity of the air when the flow is confined to passages. This will be seen in detail in convection heat transfer topic.