

Chapter 3

TWO-DIMENSIONAL, STEADY-STATE CONDUCTION

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In many situations the one-dimensional approach of heat conduction is oversimplification. Multidimensional effects must be accounted for.

4.1 Alternative Approaches

Here the two-dimensional, steady state heat conduction will be treated. Again the two major objectives in conduction analysis is determining the temperature field, $T(x,y)$ and then the heat transfer.

For two-dimensional steady-state conduction with no generation and constant k , the PDE to be solved is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Methods of solving include *analytical*, *graphical*, and *numerical* (*finite difference*, *finite-element*, or *boundary element*) approaches.

The analytical gets the exact solution of the PDE for only a restricted set of simple geometries and boundary conditions. Often the solutions are in series form. The method of *separation of variables* will be used for this solution.

Graphical and numerical solutions give approximate results at discrete points.

Graphical solution is restricted to two-dimensional problems involving adiabatic and isothermal boundaries.

The method is based on the fact that isotherms must be perpendicular to heat flow lines [figchp-4\fig4.1.pptx](#). It requires good construction skills.

Numerical methods may be used to obtain accurate results for complex, two- or three-dimensional geometries involving a variety of boundary conditions.

4.2 The method of Separation of variables

Consider the system shown in [figchp-4\fig4.2.pptx](#).

Assumption made is that heat transfer from end surfaces is negligible or $(\partial^2 T / \partial z^2) \approx 0$. This will make it a two dimensional problem.

To simplify the solution, the following transformation is used

$$\theta \equiv \frac{T - T_1}{T_2 - T_1}$$

Substituting gives

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

For the complete solution of the above PDE two boundary conditions are required for each of the coordinates and these are

$$T(0, y) = T_1 \text{ and } T(x, 0) = T_1 \text{ or } \theta(0, y) = 0 \text{ and } \theta(x, 0) = 0$$

$$T(L, y) = T_1 \text{ and } T(x, W) = T_2 \text{ or } \theta(L, y) = 0 \text{ and } \theta(x, W) = 1$$

Because of the transformation, the BC's have been restricted to 0's and 1.

The method of separation of variables assumes a product solution of the form

$$\theta(x,y) = X(x) \cdot Y(y)$$

$$\frac{\partial \theta}{\partial x} = X'(x)Y(y) \quad \text{and} \quad \frac{\partial^2 \theta}{\partial x^2} = X''(x)Y(y)$$

Similarly

$$\frac{\partial \theta}{\partial y} = X(x)Y'(y) \quad \text{and} \quad \frac{\partial^2 \theta}{\partial y^2} = X(x)Y''(y)$$

Substitution in the PDE of θ gives

$$X''Y + XY'' = 0 \quad \text{Or} \quad -\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{Y} \frac{d^2 Y}{dy^2}$$

Indeed the differential equation is separable. Another observation is that the left hand side expression is only a function of x and the right one only a function of y . The equality can be satisfied only if the expression is independent of x and y . Only a constant can satisfy this condition. Let this constant be λ^2 . This makes the equation

$$-\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{Y} \frac{d^2 Y}{dy^2} = \lambda^2$$

0 or $-\lambda^2$ could have been the constants. But λ^2 satisfies the prescribed boundary conditions.

The resulting differential equations are

$$\frac{d^2X}{dx^2} + \lambda^2 = 0 \qquad \frac{d^2Y}{dy^2} - \lambda^2 = 0$$

And the solutions are

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x$$

$$Y = C_3 e^{\lambda y} + C_4 e^{-\lambda y}$$

This will give θ as

$$\theta = XY = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 e^{\lambda y} + C_4 e^{-\lambda y})$$

Using the boundary conditions

$$(1) \quad \theta(0,y) = 0 \quad \rightarrow C_1 = 0$$

$$(2) \quad \theta(x,0) = 0 \quad \rightarrow (C_2 \sin \lambda x)(C_3 + C_4) = 0$$

The only possibility is $(C_3 + C_4) = 0$. If $C_2 = 0$

It will result in $\theta = 0$ (not the required solution)

This will then give $C_3 = -C_4$

(3) $\theta(L, y) = 0$ Gives $C_2 C_4 \sin \lambda L (e^{\lambda y} - e^{-\lambda y}) = 0$

As C_2 and C_4 can not be zeros, the only possibility is

$$\sin \lambda L = 0 \quad \rightarrow \quad \lambda L = n\pi \quad n=0, 1, 2, 3, \dots$$

$n=0$ makes $\theta(x, y) = 0$ which is not acceptable. So $n=0$ will be precluded or $n = 1, 2, 3, \dots$

The solution then becomes

$$\theta = C_2 C_4 \sin \frac{n\pi x}{L} \left(e^{n\pi y/L} - e^{-n\pi y/L} \right)$$

Combining the two constants, knowing that the new constant may depend on n will give

$$\theta(x, y) = C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}$$

Since $\left(e^{n\pi y/L} - e^{-n\pi y/L} \right) = 2 \sinh (n\pi y/L)$

As the sum of the solutions is also a solution, then the general solution becomes

$$\theta(x, y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}$$

To obtain C_n , we will apply the last boundary condition

$$(4) \quad \theta(x, W) = 1 = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi W}{L}$$

To determine C_n , some manipulation is required. This requires the knowledge of *orthogonality*.

An infinite set $g_1(x), g_2(x), \dots, g_n(x)$ is said to be orthogonal in the domain $a \leq x \leq b$ if

$$\int_a^b g_m(x)g_n(x)dx = 0 \quad m \neq n$$

Most functions exhibit orthogonality including trigonometric functions

$$\sin \frac{n\pi x}{L} \text{ and } \cos \frac{n\pi x}{L} \text{ for } 0 \leq x \leq L$$

If $f(x)$ is expressed in terms of series of orthogonal functions as

$$f(x) = \sum_{n=1}^{\infty} A_n g_n(x)$$

Then manipulating the above equation as follows

$$\int_a^b f(x) g_m(x) dx = \int_a^b g_m(x) \sum_{n=1}^{\infty} A_n g_n(x) dx$$

And then using the principles of orthogonality, on the right side all the expressions will be zero except one term and the result is given by

$$\int_a^b f(x) g_m(x) dx = A_m \int_a^b g_m^2(x) dx$$

Solving for A_m and switching the subscript to n will give

$$A_n = \frac{\int_a^b f(x)g_n(x)dx}{\int_a^b f(x)g_n^2(x)dx}$$

For our situation $g_n(x) = \sin(n\pi x/L)$, $f(x)=1$ and substitution will give

$$A_n = \frac{\int_0^L \sin \frac{n\pi x}{L} dx}{\int_0^L \sin^2 \frac{n\pi x}{L} dx} = \frac{2}{\pi} \frac{(-1)^{n+1} + 1}{n}$$

Substitution in the series representation of $f(x)$ gives

$$1 = \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{(-1)^{(n+1)} + 1}{n} \sin \frac{n\pi x}{L}$$

Recalling the equation of the result of the fourth boundary condition and comparing gives

$$C_n = \frac{2[(-1)^{n+1} + 1]}{n\pi \sinh(n\pi W / L)} \quad n = 1, 2, 3, \dots$$

And the resulting final solution will be

$$\theta(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n} \sin \frac{n\pi x}{L} \frac{\sinh \frac{n\pi y}{L}}{\sinh \frac{n\pi W}{L}}$$

Isotherms and heat flow lines are shown in [figchp-](#)

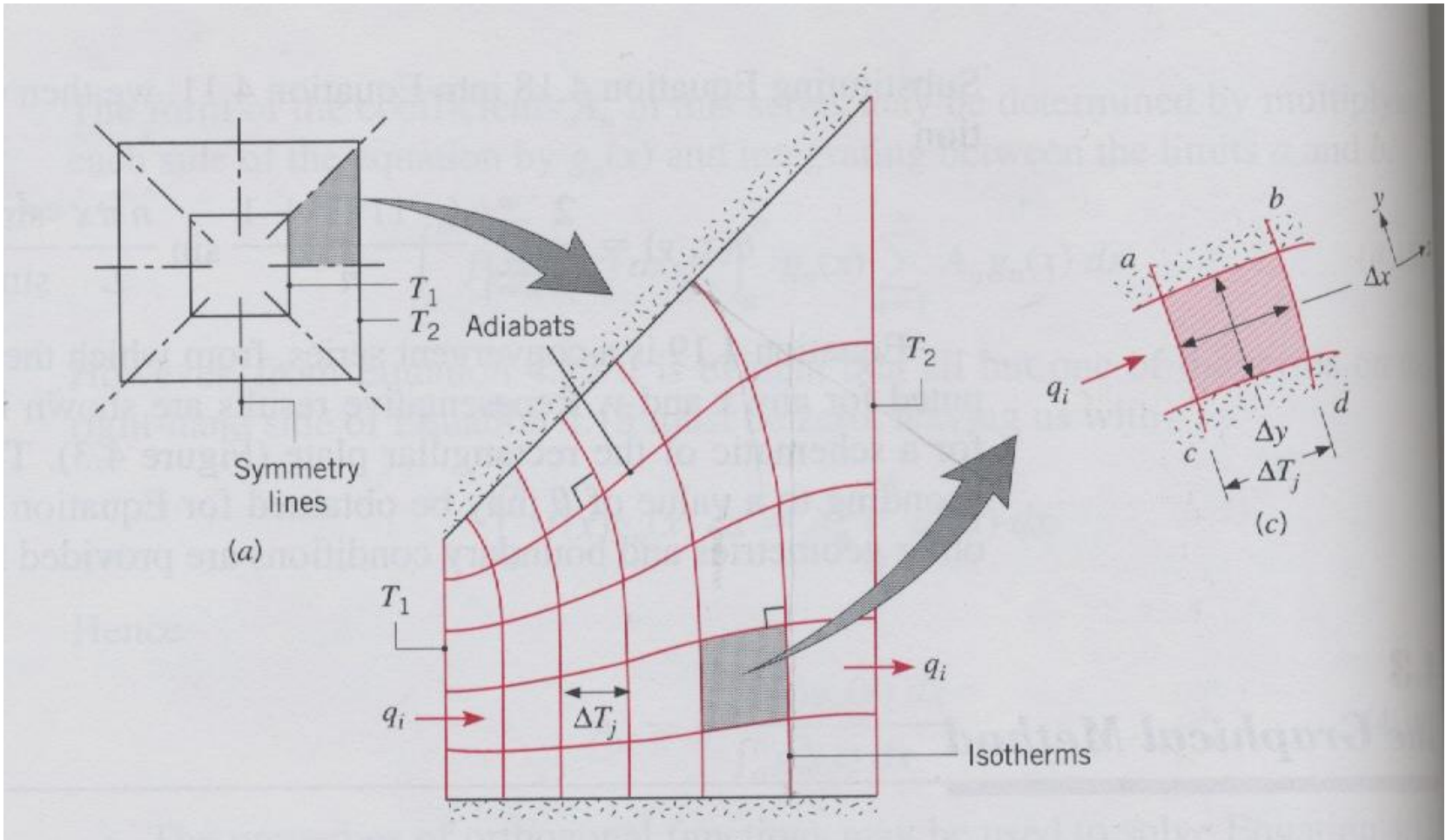
4.3 THE CONDUCTION SHAPE FACTOR AND DIMENSIONLESS CONDUCTION HEAT RATE

Conduction *shape factors*, S are used to report the existing solutions of heat conduction equations. The steady-state heat transfer rate is expressed in dimensionless conduction heat rate, q_{ss}^* .

Heat transfer rate in terms of shape factors may be expressed as

$$q = Sk\Delta T_{1-2}$$

Where ΔT_{1-2} is the temperature difference between boundaries.



Two dimensional conduction in a square channel of length L . (a) Symmetry planes. (b) Flux plot. (c) Typical curvilinear square.

If properly constructed, q_i will be the same for all lanes

$$q = \sum_{i=1}^M q_i = M q_i$$

M = Number of lanes

Application of Fourier's law for the approximate square in a lane

$$q_i \approx k A_i \frac{\Delta T_j}{\Delta x} \approx k (\Delta y \cdot l) \frac{\Delta T_j}{\Delta x}$$

For equal temperature increments across the isotherms

$$\Delta T_{1-2} = \sum_{j=1}^N \Delta T_j = N \Delta T_j$$

If the construction is made with $\Delta x \approx \Delta y$ then the heat transfer can be estimated by

$$q \approx \frac{Ml}{N} k \Delta T_{1-2}$$

The above equation gives the two dimensional conduction shape factor, S , as

$$S = Ml/N$$

Accordingly the two-dimensional conduction resistance as

$$R_{t,cond(2D)} = \frac{1}{Sk}$$

Shape factors have been obtained analytically for numerous two- and three-dimensional systems and the results are summarized in [tables.docx](#). Case 9 is a three dimensional case.

For one dimensional cases, shape factors may also be defined for the familiar shapes as follows:

Plane wall: A/L Cylinder: $2\pi L / \ln \frac{r_2}{r_1}$

Sphere: $4\pi r_1 r_2 / (r_2 - r_1)$

Cases 12 through 15 are for conduction heat rates from isothermal objects at T_1 embedded within an infinite medium of uniform temperature (T_2) and to facilitate the procedure a *characteristic length* is defined as

$$L_c \equiv (A_s/4\pi)^{1/2}$$

where A_s is the surface area of the object.

Heat transfer rate to the infinite medium is given in terms *dimensionless conduction heat rate* as

$$q_{ss}^* \equiv qL_c / [kA_s(T_1 - T_2)]$$

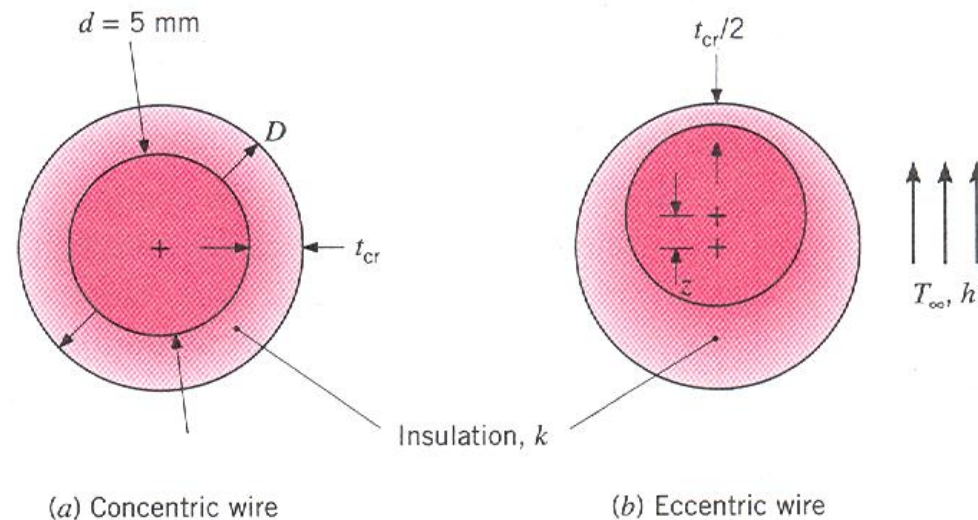
Values which have been determined analytically and numerically are given as cases (12)-(15) in

Table 4.1.

Example 4.1

A metallic electrical wire of diameter $d=5\text{mm}$ is to be coated with insulation of thermal conductivity $k=0.35\text{ W/m.K}$. It is expected that, for the typical installation, the coated wire will be exposed to conditions for which the total coefficient associated with convection and radiation is $h=15\text{W/m}^2\text{.K}$. To minimize the temperature rise of the wire due to ohmic heating, the insulation thickness is specified so that the critical insulation radius is achieved. During the wire coating process, however, the insulation thickness sometimes varies around the periphery of the wire, resulting in eccentricity

of the wire relative to the coating. Determine the change in the thermal resistance of the insulation due to an eccentricity that is 50% of the critical insulation thickness.



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Figure for example 4.1

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Solution

$$r_{cr} = k/h = 0.35/15 = 0.023 \text{ m} = 23 \text{ mm}$$

Critical insulation thickness

$$t_{cr} = r_{cr} - d/2 = 0.023 - 0.005/2 = 0.021 \text{ m} = 21 \text{ mm}$$

For concentric geometry the thermal resistance is given by

$$R'_{t,cond} = \frac{\ln[r_{cr} / (d/2)]}{2\pi k} = \frac{\ln[0.023 / (0.005/2)]}{2\pi(0.35)} = 1.0 \text{ m.K / W}$$

For the eccentric wire, use case 7 where

$$z = 0.5t_{cr} = 0.5 \times 0.021 = 0.010 \text{ m}$$

$$\begin{aligned}
 R'_{t,\text{cond}} &= \frac{1}{Sk} = \frac{\cosh^{-1}\left(\frac{D^2 + d^2 - 4z^2}{2Dd}\right)}{2\pi k} \\
 &= \frac{\cosh^{-1}\left(\frac{(2 \times 0.023)^2 + 0.005^2 - 4 \times 0.010^2}{2 \times (2 \times 0.023) \times 0.005}\right)}{2\pi \times 0.35}
 \end{aligned}$$

$$= 0.91 \text{ m.K/W}$$

The reduction in the thermal resistance of the insulation is 0.09 m.K/W or 9%.

4.4 FIFNITE DIFFERENCE EQUATIONS

Large number of two-dimensional problems involve geometries and/or boundary conditions which makes analytical solutions impossible. The best alternative, in such cases is to use *numerical* technique such as *finite difference*, *finite element* or *boundary element* method. Due to its ease of application, finite difference method will be treated.

4.4.1 The Nodal Network

Numerical solution enables determination of temperature at only discrete points. This requires subdividing the medium of interest into a number of small regions with reference points at the centers

of the regions as shown in [figchp-4\fig4.4.pptx](#). This reference point is termed as *nodal point* or *node*, and the aggregate of points is termed a *nodal network, grid, or mesh*.

m, n represents the node and the temperature at m, n is a measure of the average temperature of the region (shaded area), in this case represented by $T_{m,n}$. The temperature distribution shown in (b) is discontinuous. Accuracy of the method depends on how fine the mesh is.

4.4.2 Finite Difference Form of the Heat Equation

Considering the second derivative ($\partial^2 T / \partial x^2$) of an interior node, the value of this derivative at the nodal point m, n may be approximated as

$$\frac{\partial^2 T}{\partial x^2} @_{m,n} \approx \frac{\partial T / \partial x_{m+1/2,n} - \partial T / \partial x_{m-1/2,n}}{\Delta x}$$

The above can be determined as

$$\frac{\partial T}{\partial x_{m+1/2,n}} \approx \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\frac{\partial T}{\partial x_{m-1/2,n}} \approx \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

Substitution in the 2nd derivative gives

$$\frac{\partial^2 T}{\partial x^2} @_{m,n} = \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

Using similar procedures for the y-coordinate will give

$$\frac{\partial^2 T}{\partial y^2} @_{m,n} \approx \frac{\partial T / \partial y_{m,n+1/2} - \partial T / \partial y_{m,n-1/2}}{\Delta y}$$

$$\approx \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

Substitution in the differential equation gives

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \approx \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2} + \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

Using a network for which $\Delta x = \Delta y$ will give

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$$

Thus the PDE is reduced to an algebraic equation which gives the temperature at a node as the average of the temperatures of the four neighboring nodes.

4.4.3 The Energy Balance Method

This is an alternative method to develop the finite difference equations for nodes. The approach is more versatile than the one seen before. It uses energy conservation principle and assumes that all heat transfer is into the node as shown in [figchp-4\fig4.5.pptx](#). The analysis will consider heat generation too. For a unit depth

$$\dot{E}_{in} + \dot{E}_g = 0 \quad \sum_{i=1}^4 \dot{q}_{(i) \rightarrow m,n} + \dot{q}''' [\Delta x \Delta y(1)] = 0$$

The conduction heat rate from each node can be written as

$$\dot{q}_{(m-1,n) \rightarrow (m,n)} = k (\Delta y \cdot 1) \frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

$$\dot{q}_{(m+1,n) \rightarrow (m,n)} = k (\Delta y \cdot 1) \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\dot{q}_{(m,n+1) \rightarrow (m,n)} = k (\Delta x \cdot 1) \frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$\dot{q}_{(m,n-1) \rightarrow (m,n)} = k (\Delta x \cdot 1) \frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

Using $\Delta x = \Delta y$ and substituting in the summation equation will give

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} + \frac{\dot{q}(\Delta x)^2}{k} - 4T_{m,n} = 0$$

With no heat generation it gives the same equation.

Equations must also be developed for nodes lying on the boundary surface of the object. The surface may enjoy convection heat transfer as shown in [figchp-4/fig4.6.pptx](#) which is an internal corner.

The equations can be developed as

$$\dot{Q}_{(m-1,n) \rightarrow (m,n)} = k (\Delta y \cdot 1) \frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

$$\dot{Q}_{(m+1,n) \rightarrow (m,n)} = k \left(\frac{1}{2} \Delta y \cdot 1 \right) \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\dot{Q}_{(m,n+1) \rightarrow (m,n)} = k (\Delta x \cdot 1) \frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$\dot{Q}_{(m,n-1) \rightarrow (m,n)} = k \left(\frac{1}{2} \Delta x \cdot 1 \right) \frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

Convection:

$$\dot{Q}_{(\infty) \rightarrow (m,n)} = \left(h \frac{\Delta x}{2} \cdot 1 \right) (T_{\infty} - T_{m,n}) + \left(h \frac{\Delta y}{2} \cdot 1 \right) (T_{\infty} - T_{m,n})$$

With no energy generation and steady-state and network with $\Delta x = \Delta y$

$$\dot{E}_{in} = 0$$

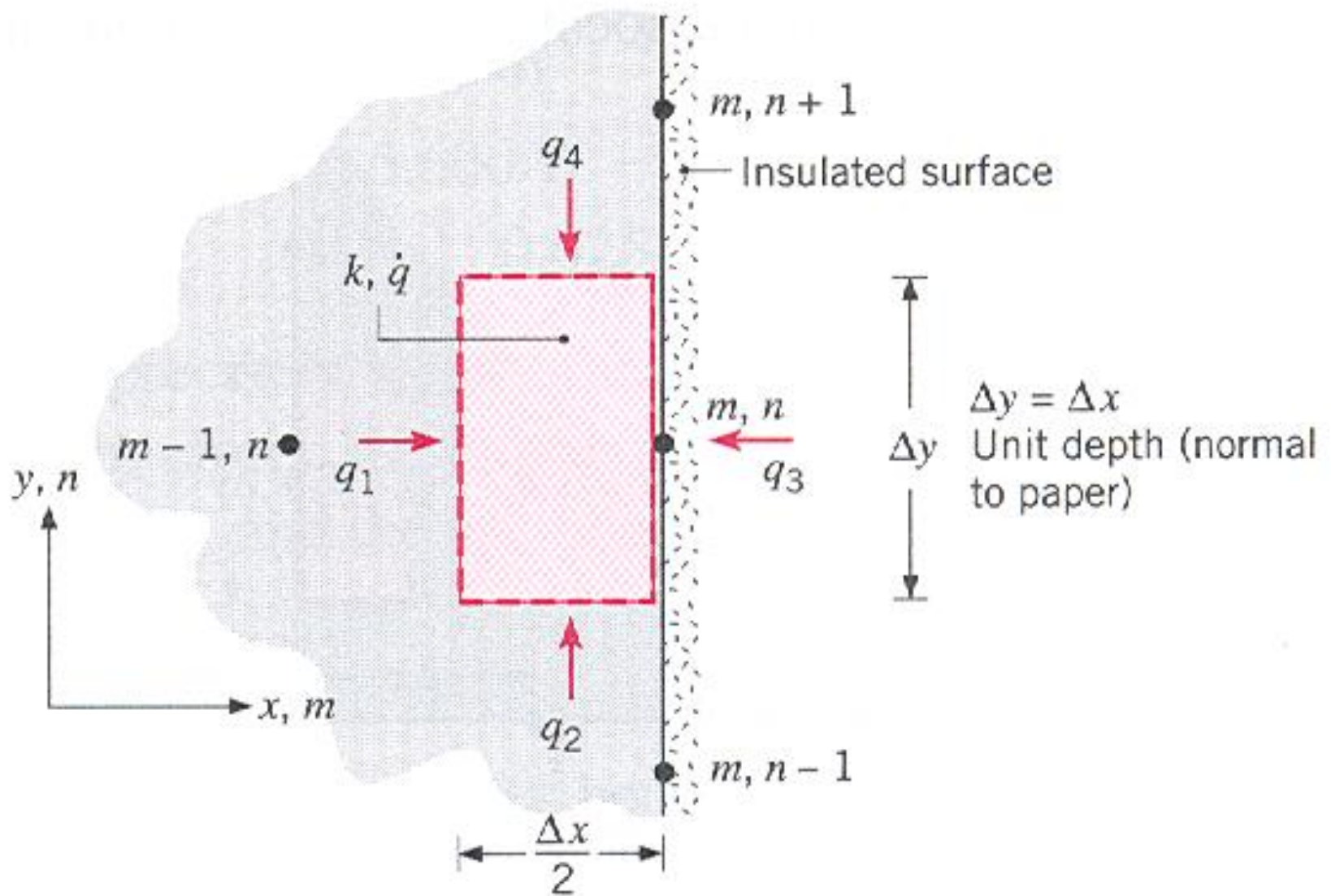
$$T_{m-1,n} + T_{m,n+1} + \frac{1}{2}(T_{m+1,n} + T_{m,n-1}) + \frac{h\Delta x}{k}T_{\infty} - \left(3 + \frac{h\Delta x}{k}\right)T_{m,n} = 0$$

Nodal energy balance equations for several common geometries is given in [Table 4.2](#).

Example 4.2

Using the energy balance method, derive the finite difference equation for the m, n nodal point located on a plane, insulated surface of a medium with

uniform heat generation.



Solution

Using the energy conservation for node m,n

$$q_1 + q_2 + q_3 + q_4 + \dot{q}''' \left(\frac{\Delta x}{2} \cdot \Delta y \cdot 1 \right) = 0$$

where

$$q_1 = k(\Delta y \cdot 1) \frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

$$q_2 = k\left(\frac{\Delta x}{2} \cdot 1\right) \frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

$$q_3 = 0$$

$$q_4 = k\left(\frac{\Delta x}{2} \cdot 1\right) \frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

Substituting in the energy balance equation and simplifying gives

$$2T_{m-1,n} + T_{m,n-1} + T_{m,n+1} + -4T_{m,n} + \frac{\dot{q}''' (\Delta x \cdot \Delta y)}{k} = 0$$

The finite difference equations may also be formulated using the thermal resistance concept. For the example of the corner node (internal)

$$\dot{q}_{(m-1,n) \rightarrow (m,n)} = \frac{T_{m-1,n} - T_{m,n}}{R_{t,cond}} = \frac{T_{m-1,n} - T_{m,n}}{\Delta x / (\Delta y \cdot 1 \cdot k)}$$

$$\dot{q}_{(\infty) \rightarrow (m,n)} = \frac{T_{\infty} - T_{m,n}}{R_{t,cond}} = \frac{T_{\infty} - T_{m,n}}{\left\{ h \left[\left(\frac{\Delta x}{2} \cdot 1 \right) + \left(\frac{\Delta y}{2} \cdot 1 \right) \right] \right\}^{-1}}$$

The utility of the resistance can come when considering two nodes of dissimilar materials with a contact resistance between them [figchp-4\fig4.7.pptx](#).

$$\dot{q}_{(m,n) \rightarrow (m,n-1)} = \frac{T_{m,n} - T_{m,n-1}}{R_{tot}}$$

Where

$$R_{\text{tot}} = \frac{\Delta y / 2}{k_A (\Delta x.1)} + \frac{R''_{t,c}}{\Delta x.1} + \frac{\Delta y / 2}{k_B (\Delta x.1)}$$

4.5 SOLVING THE FINITE DIFFERENCE EQUATIONS

These equations are the result of the nodal equations. For N-nodal points there will be N equations. The result is N linear algebraic with N unknown temperatures.

Two types of solutions will be dealt with

- Direct and iterative

For small equations, direct is suitable while for larger ones, iterative is better.

4.5.1 The Matrix Inversion Method

This is a direct solution. The nodal equations are

$$a_{11}T_1 + a_{12}T_2 + a_{13}T_3 + \dots + a_{1N}T_N = C_1$$

$$a_{21}T_1 + a_{22}T_2 + a_{23}T_3 + \dots + a_{2N}T_N = C_2$$

$$\begin{array}{cccccc} \cdot & & \cdot & & \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot & & \cdot & & \cdot \end{array}$$

$$a_{N1}T_1 + a_{N2}T_2 + a_{N3}T_3 + \dots + a_{NN}T_N = C_N$$

Using matrix notation, the above can be written as

$$[A][T] = [C]$$

where

$$A \equiv \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{bmatrix}, \quad T \equiv \begin{bmatrix} T_1 \\ T_2 \\ \cdot \\ \cdot \\ \cdot \\ T_N \end{bmatrix}, \quad C \equiv \begin{bmatrix} C_1 \\ C_2 \\ \cdot \\ \cdot \\ \cdot \\ C_N \end{bmatrix}$$

Since

$$[A]^{-1}[A] = I, \text{ then } [A]^{-1}[A][T] = [A]^{-1}[C] \quad \text{or}$$

$$[T] = [A]^{-1}[C]$$

If the inverse of A is defined as

$$[A]^{-1} \equiv \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1N} \\ b_{21} & b_{22} & \dots & b_{2N} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ b_{N1} & b_{N2} & \dots & b_{NN} \end{bmatrix}$$

The solution then becomes

$$T_1 = b_{11}C_1 + b_{12}C_2 + \dots + b_{1N}C_N$$

$$T_1 = b_{11}C_1 + b_{12}C_2 + \dots + b_{1N}C_N$$

$$\cdot \quad \cdot \quad \cdot \quad \dots \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \dots \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \dots \quad \cdot$$

$$T_1 = b_{11}C_1 + b_{12}C_2 + \dots + b_{1N}C_N$$

Matrix inversion can easily be done by calculators and computers. For large equation, this procedure is time consuming.

4.5.2 Gauss-Seidel Iteration

This is an iteration procedure and is good for large equations. The following procedure is followed.

1. As much as possible, the equations should be reordered to have the largest coefficient at the diagonal for quick convergence.
2. After reordering, each of the N equations should be written in explicit form as

$$T_i^{(k)} = \frac{C_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} T_j^{(k)} - \sum_{j=i+1}^N \frac{a_{ij}}{a_{ii}} T_j^{(k-1)}$$

where $i = 1, 2, \dots, N$. The superscript k refers to the level of iteration.

3. An initial ($k=0$) value is assumed for each temperature T_i .
4. Setting $k=1$ in the equation, values of $T_i^{(1)}$ are then calculated by substituting assumed values in the second summation or new values in the first summation. This step is the first ($k=1$) iteration. New values of $T_i^{(k)}$ are determined from $T_j^{(k)}$ values of the current iteration, where $1 \leq j \leq i-1$, and the $T_j^{(k-1)}$ values of the previous iteration, where $i+1 \leq j \leq N$

5. The iteration is terminated when a prescribed criterion is satisfied.

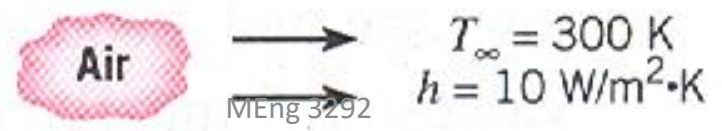
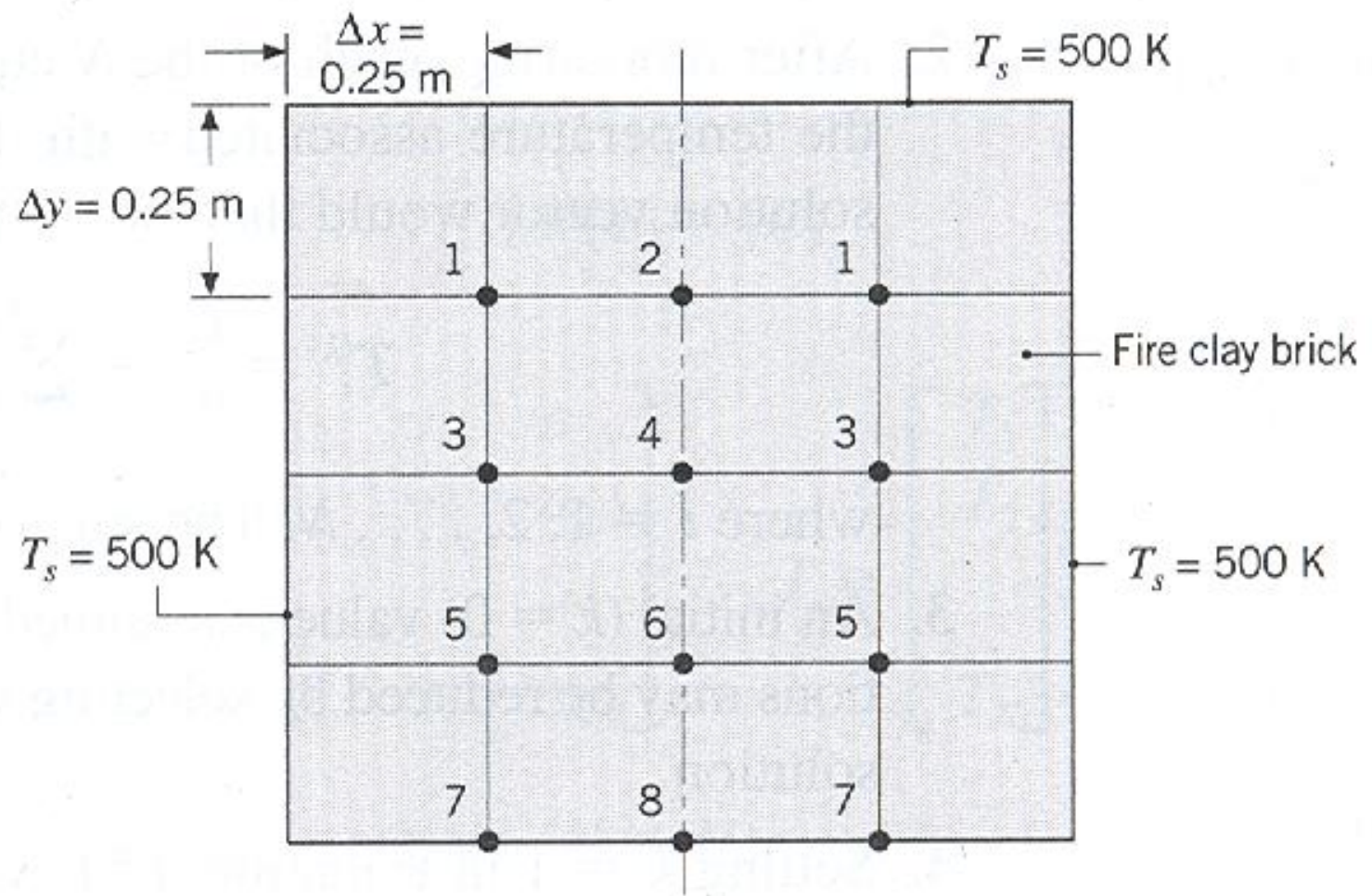
$$\left| T_i^{(k)} - T_i^{(k-1)} \right| \leq \varepsilon$$

where ε represents an error margin.

Example 4.3

A large industrial furnace is supported on a long column fire clay brick, which is 1m by 1m on a side. During steady-state operation, installation is such that three surfaces of the column are maintained at 500 K while the remaining surface is exposed to an airstream for which $T_\infty = 300$ K and $h = 10$ W/m².K. Using a grid of $\Delta x = \Delta y = 0.25$ m, determine the two dimensional

temperature distribution in the column and the heat rate to the airstream per unit length of column.



Solution

There are 12 nodes. By symmetry the unknowns will be reduced to 8.

For interior nodes 1, 2, 3, 4, 5, and 6, the finite difference equations are

$$\text{Node 1: } T_2 + T_3 + 1000 - 4T_1 = 0$$

$$\text{Node 2: } 2T_1 + T_4 + 500 - 4T_2 = 0$$

$$\text{Node 3: } T_1 + T_4 + T_5 + 500 - 4T_3 = 0$$

$$\text{Node 4: } T_2 + 2T_3 + T_6 - 4T_4 = 0$$

$$\text{Node 5: } T_3 + T_6 + T_7 + 500 - 4T_5 = 0$$

$$\text{Node 6: } T_4 + 2T_5 + T_8 - 4T_6 = 0$$

For the surface nodes 7 and 8, use equation 4.42 using

$$h \Delta x/k=2.5$$

$$\text{Node 7: } 2T_5 + T_8 + 2000 - 9T_7 = 0$$

$$\text{Node 8: } 2T_6 + 2T_7 + 1500 - 9T_8 = 0$$

$$-4\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 + 0 + 0 + 0 + 0 + 0 = -1000$$

$$2\mathbf{T}_1 - 4\mathbf{T}_2 + 0 + \mathbf{T}_4 + 0 + 0 + 0 + 0 = -500$$

$$\mathbf{T}_1 + 0 - 4\mathbf{T}_3 + \mathbf{T}_4 + \mathbf{T}_5 + 0 + 0 + 0 = -500$$

$$0 + \mathbf{T}_2 + 2\mathbf{T}_3 - 4\mathbf{T}_4 + 0 + \mathbf{T}_6 + 0 + 0 = 0$$

$$0 + 0 + \mathbf{T}_3 + 0 - 4\mathbf{T}_5 + \mathbf{T}_6 + \mathbf{T}_7 + 0 = -500$$

$$0 + 0 + 0 + \mathbf{T}_4 + 2\mathbf{T}_5 - 4\mathbf{T}_6 + 0 + \mathbf{T}_8 = 0$$

$$0 + 0 + 0 + 0 + 2\mathbf{T}_5 + 0 - 9\mathbf{T}_7 + \mathbf{T}_8 = -2000$$

$$0 + 0 + 0 + 0 + 0 + 2\mathbf{T}_6 + 2\mathbf{T}_7 - 9\mathbf{T}_8 = -1500$$

In matrix form $[A][T]=[C]$

$$A = \begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & -9 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & -9 \end{bmatrix} \quad [C] = \begin{bmatrix} -1000 \\ -500 \\ -500 \\ 0 \\ -500 \\ 0 \\ -2000 \\ -1500 \end{bmatrix}$$

The solution given by

$[T]=[A]^{-1}[C]$ is the column vector

$$[T] = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{bmatrix} = \begin{bmatrix} 489.30 \\ 485.15 \\ 472.07 \\ 462.01 \\ 436.95 \\ 418.74 \\ 356.99 \\ 339.05 \end{bmatrix}$$

The heat transfer from the column to the airstream per unit length will be

$$\begin{aligned}\left(\frac{\dot{q}}{L}\right) &= 2h \frac{\Delta x}{2} (T_s - T_\infty) + 2h \Delta x (T_7 - T_\infty) + h \Delta x (T_8 - T_\infty) \\ &= 2h \left[\left(\frac{\Delta x}{2}\right) (T_s - T_\infty) + \Delta x (T_7 - T_\infty) + \frac{\Delta x}{2} (T_8 - T_\infty) \right] \\ &= 2 \times 10 [0.125 \times 200 + 0.25 \times 56.99 + 0.125 \times 39.05] = 883 \text{ W / m}\end{aligned}$$

To check the solution, the equality of the conduction and convection heat transfers can be checked.

The temperature distribution of the above example can also be determine using the Gauss-Seidel iteration method.

As the equations give a diagonal dominant coefficient matrix, it does not need any reordering. This is a typical behavior of finite difference solutions to conduction problems.

The explicit form of the equation according to step 2 is

$$T_1^{(k)} = 0.25T_2^{(k-1)} + 0.25T_3^{(k-1)} + 250$$

$$T_2^{(k)} = 0.50T_1^{(k)} + 0.25T_4^{(k-1)} + 125$$

$$T_3^{(k)} = 0.25T_1^{(k)} + 0.25T_4^{(k-1)} + 0.25T_5^{(k-1)} + 250$$

$$T_4^{(k)} = 0.25T_2^{(k)} + 0.50T_3^{(k)} + 0.25T_6^{(k-1)}$$

$$T_5^{(k)} = 0.25T_3^{(k)} + 0.25T_6^{(k-1)} + 0.25T_7^{(k-1)} + 125$$

$$T_6^{(k)} = 0.25T_4^{(k)} + 0.50T_5^{(k)} + 0.25T_8^{(k-1)}$$

$$T_7^{(k)} = 0.2222T_5^{(k)} + 0.1111T_8^{(k-1)}$$

$$T_8^{(k)} = 0.2222T_6^{(k)} + 0.2222T_7^{(k)} + 166.67$$

With initial assumption at $k=0$ and convergence criterion of $\varepsilon=0.2$ K, the following table is generated.

k	T1	T2	T3	T4	T5	T6	T7	T8
0	480.0	470.0	440.5	430.0	400.0	390.0	370.0	350.0
1	477.5	471.3	451.9	441.3	428.0	411.8	356.2	337.3
2	480.8	475.7	462.5	453.1	432.6	413.9	355.8	337.7
3	484.6	480.6	467.6	457.4	434.3	415.9	356.2	338.3
4	487.0	482.9	469.7	459.6	435.5	417.2	356.6	338.6
5	488.4	484.0	470.8	460.7	436.1	417.9	356.7	338.8
6	488.7	484.5	471.4	461.3	436.5	418.3	356.9	338.9
7	489.0	484.8	471.7	461.6	436.7	418.5	356.9	339.0
8	489.1	485.0	471.9	461.8	436.8	418.6	356.9	339.0

The results given in row 8 are in excellent agreement with those obtained by matrix inversion