Chapter3
Mathematical Modeling of Dynamic Systems

3.1. Introduction

- **Causality:** the current output of the system (the output at time $t = 0$) depends on the past input (the input for $t < 0$) but does not depend on the future input (the input for $t > 0$). No future anticipation.
- **Linear Time-Invariant Systems:** dynamic systems that are composed of constant coefficients or functions of independent variable of differential equations.
- **Linear Time-Variant systems:** systems that are represented by differential equations whose coefficients are functions of time.

Cont. …

- **Mathematical model of a dynamic system:** is defined as a set of equations that represents the dynamics of the system accurately or, at least, fairly well, whether they are mechanical, electrical, thermal, economic, biological, and so on. In other words Mathematical model of a dynamic system is a mathematical relation which relates the excitation (input) of the system, the system itself and the output. It may be linear or non-linear.
- Mathematical model are more to analysis control system.
- Mathematical model is not unique for a given system (mathematical models may assume many different forms).

Cont. …

Example:

- In optimal control problems, it is advantageous to use state-space representations.
- In transient-response or frequency-response analysis of single-input-single-output, linear, time-invariant systems, the transfer function representation may be more convenient than any other.
- In mathematical modeling of linear lumped-parameter system, a certain non-linearity and distributed parameter ignored if their effects is small on the response.

Cont. …

- The most popular mathematical models are:

  i. Differential equation
  ii. Transfer function
  iii. Impulse response
  iv. State space representation

- Rather than the mathematical models, there are also other ways of describing a system that have an aim of giving a schematic overview of a system. The two popular schematic system descriptions are:

  i. Block diagrams
  ii. Signal flow graphs

3.2. Classical Mathematical Modeling Techniques of a Dynamic System

3.2.1. Differential equation

- The oldest way of a time domain system description.
- It includes linearly independent equations of a system as well as the appropriate initial conditions.
- Applications of differential equations

i. LTI electric Network

- To develop the over all differential equation of the electric network, KVL and KCL will be applied.
- The basic elements of an electrical system are resistor, inductor and capacitor.

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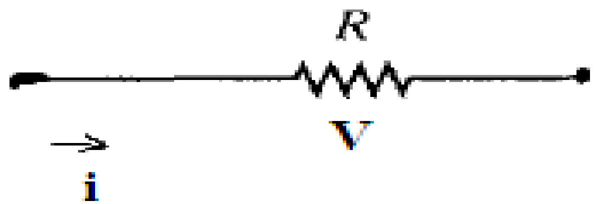
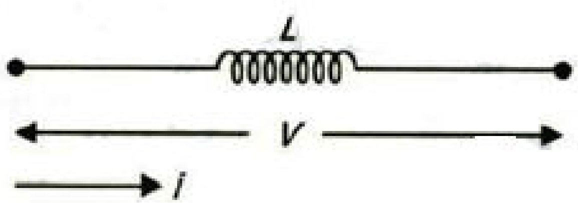
❖ **Resistor**



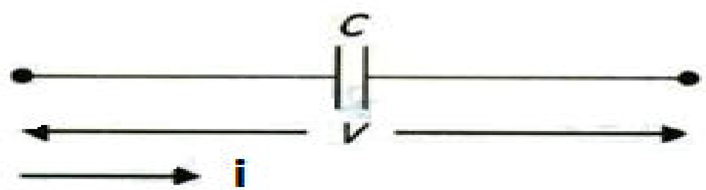
$V = Ri$

❖ **Inductor**



$$\mathbf{v = L\frac{di}{dt} \quad or \quad i = \frac{1}{L}\int_0^t vdt}$$

❖ **Capacitor**



$$v = \frac{1}{C}\int_0^t idt \quad or \quad i = C\frac{dV}{dt}$$

where V is voltage and i is current.

Cont. …

ii. LTI mechanical system

- Mechanical systems are classified into two types:

    i. Translational   ii. Rotational

- **Translational motion:** during this the motion of the body is along a straight line or a curved path.

- There are three elements which are dominantly involved in the analysis of translational motion. These are mass, spring and damper.

- **Mass:** in the physical model of a mass, it is assumed that its mass is concentrated at the centre of the mass. Consider the mass shown below with zero friction.

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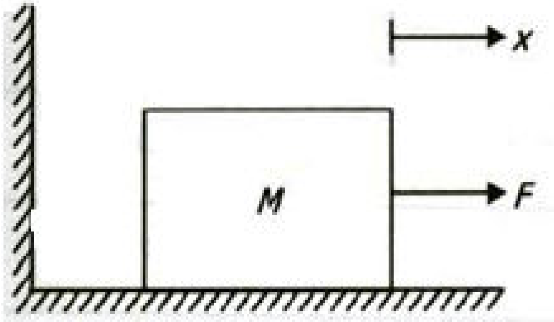

The relation between force F and mass M is given by

$$F = M\frac{d^2x}{dt^2} = M\frac{du}{dt} = Ma$$

Where **a** is the acceleration, **u** is the velocity, and **x** is the displacement of the mass.

❖ **Spring:** consider a spring having negligible mass as shown below. Let the linear spring constant for the spring be K. the spring subjected to force and it undergoes elastic deformation. Where **x** is the displacement.

$$F = Kx$$

Cont. …

Here x_2 is the displacement of end A, whereas x_1 is the displacement of end B.

If force is applied at end B,

$$F = K(x_1 - x_2)$$

If force is applied at end A,

$$F = K(x_2 - x_1)$$

- **Damper:** motion is opposed by friction. Types of frictional forces are given below.

i. **Coulomb frictional force:** the sliding friction between dry surfaces.

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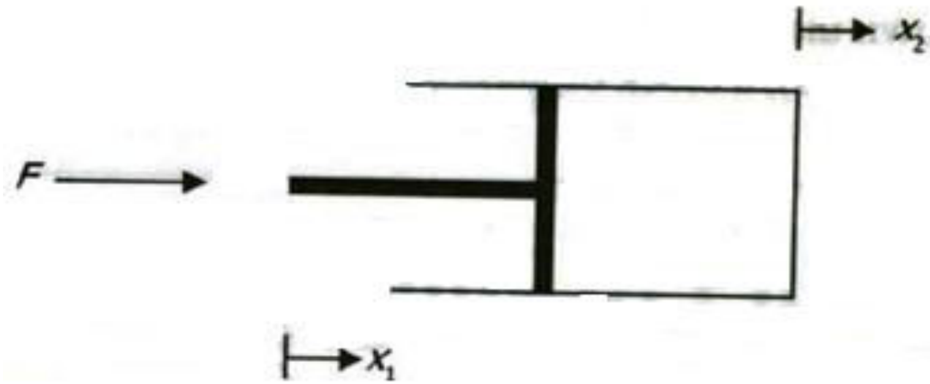
ii. **Viscous friction:** the friction between moving surfaces by a viscous fluid or the friction between a solid body and a fluid medium. Viscous friction is proportional to velocity.



Dashpot

$$f = D\frac{dx}{dt} = D\frac{d(x_1 - x_2)}{dt}$$

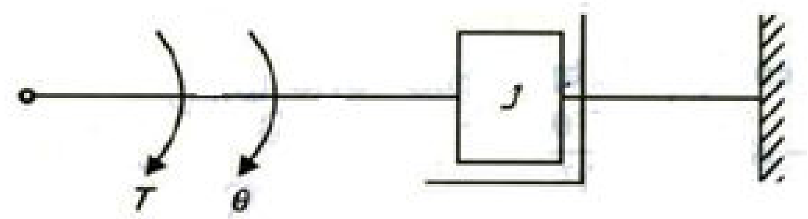
Where D is the damping coefficient

Cont. …

iii. **Stiction:** the friction force that required to initiate motion between two contact surfaces.

- **Rotational motion:** during this the motion of the body is about its own axis. It is similar to translational motion except torque, inertia and angular displacement are considered instead of force, mass and linear displacement. The three basic elements of rotational motion are inertia (J), damper (D) and spring (K).

- **Inertia**



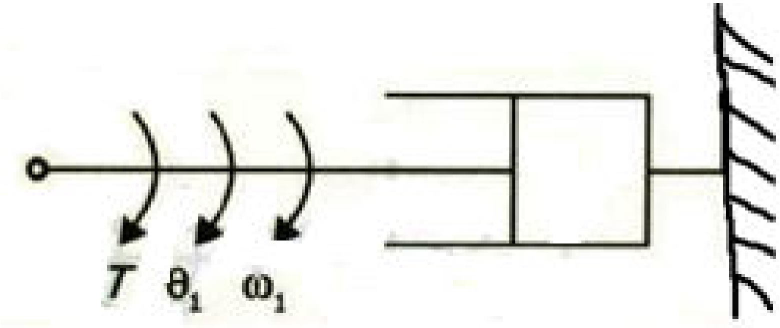
$$T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt} = J\alpha$$

T- is torque

Where α is the acceleration, ω is the velocity, and θ is the displacement of the inertia.

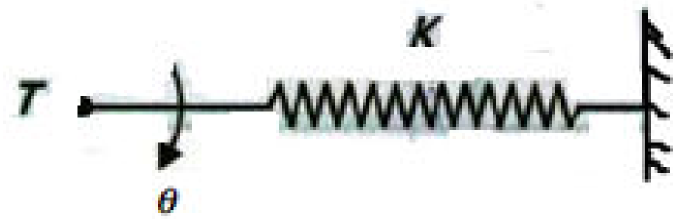
Cont. …

❖ **Damper:**



$$T = D\frac{d\theta}{dt} = D\omega$$

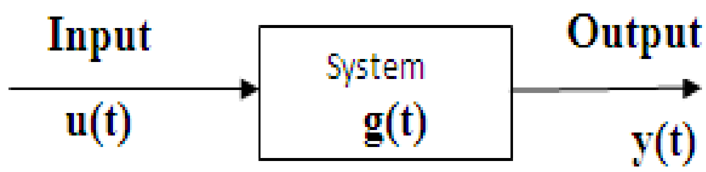
❖ **Spring:**



$$T = K\theta$$

3.2.2. Transfer function

- It is a description of frequency domain.
- It holds only for LTI systems having zero initial conditions.
- Consider the following



Let u(t) be the input signal y(t) be the output signal and g(t)be the representation of the system.

- Then, the transfer function of a LTI system with zero initial conditions is the ratio of the Laplace transform of the output y(t) to the Laplace transform of the input u(t).

Cont. …

- Mathematically

$$\text{Transfer function} = \frac{\text{laplace transform of output}}{\text{laplace transform of input}}$$

$$G(s) = \frac{Y(s)}{U(s)}\Big|\text{zero initial conditions}$$

- It is mathematical model expressing differential equation that relates output variable to input variable.
- The order of system is the highest power of s in the denominator of the transfer function.
- Poles and zeros of a transfer function is obtained from denominator and numerator of the transfer function respectively.
- Property of system itself, independent of magnitude and nature of input.

Cont. …

- Does not provide any information concerning physical structure of system.
- Used for the study of output for various input if transfer function is known.
- Stability is determined based on the location of poles of transfer function.

Exercise 3.1:

1) Consider the differential equation given by

$$a_n\frac{d^n y(t)}{dt^n}+a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}}+\cdots+a_0 y(t)=b_m\frac{d^m u(t)}{dt^m}+b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}}+\cdots+b_0 u(t)$$

Cont. …

Proof its transfer function is given by

$$\frac{Y(s)}{U(s)} = G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

2) The relation between input and output of the system can be expressed mathematically by using convolution integral as

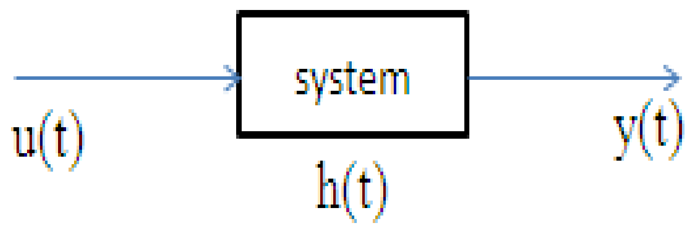
$$y(t) = u(t) * g(t)$$

$$y(t) = \int_0^t g(\tau) u(t-\tau) d\tau$$

Then proof $G(s) = \frac{Y(s)}{U(s)}$

3.2.3. Impulse response function

- Consider the following



- Impulse response h(t) of a linear system with zero initial conditions is the systems output when its input is the unit impulse function .

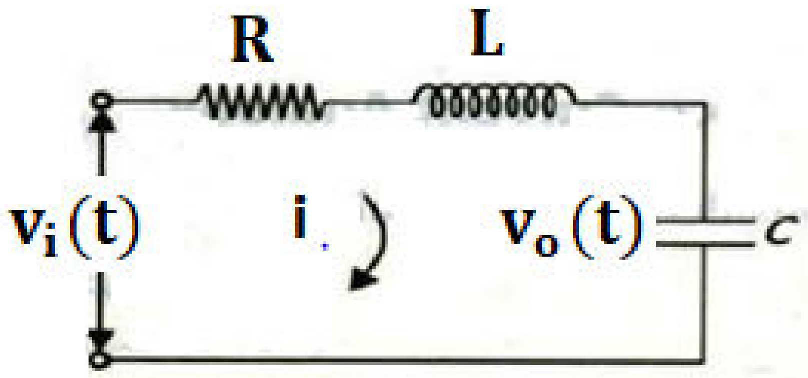
Then $Y(s) = H(s)U(s) = H(s)$ Since $\delta(s) = U(s) = 1$

- The impulse response of the above system is

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\{H(s)\} = h(t)$$

3.3. Mathematical models of electrical systems

Example 3.1: Consider RLC circuit shown below. Find the differential equation and transfer function of the electrical network. Assume the Vi(t) as an input and Vo(t) as an output of the given system.



solution

To get the differential equation of the system, let us apply KVL into the given network.

Cont. …

$$v_i(t) = Ri(t) + L\frac{di(t)}{dt} + v_o(t) \qquad (3.1)$$

$$v_o(t) = \frac{1}{C}\int i(t)dt$$

$$i(t) = C\frac{dv_o(t)}{dt} \qquad (3.2$$

- In order to get transfer function, let us apply Laplace transform into equation (3.1) and (3.2).

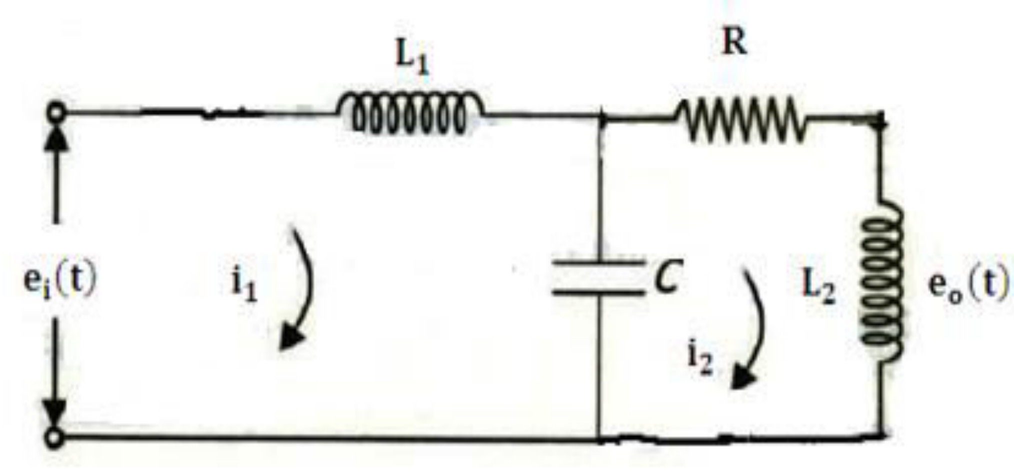
For eqn.(3.1), $v_i(s) = RI(s) + LsI(s) + v_o(s)$ (3.3)

For eqn.(3.2), $I(s) = CSv_o(s)$ (3.4)

- Substituting eqn.(3.4) into eqn.(3.3) and rearranging will result the following transfer function.

$$\frac{v_o(s)}{v_i(s)} = \frac{1/LC}{s^2+\frac{R}{L}s+1/LC}$$

Exercise 3.2: 1) Find the transfer function $\frac{E_0(s)}{E_i(s)}$ of the given network below.



Cont. …

2) Find the transfer function for the following **operational amplifier** circuits

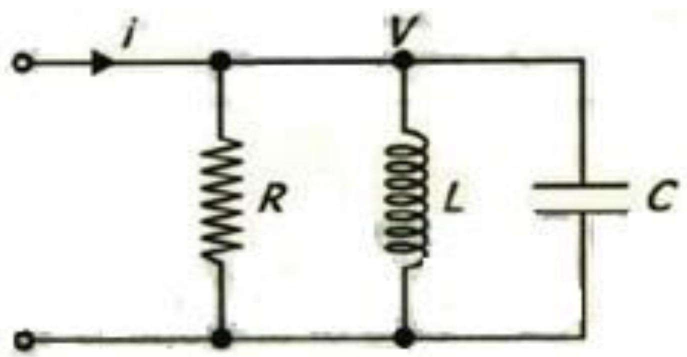
i. Inverting amplifier

ii. Non-inverting amplifier

iii. Differential amplifier

iv. Integral amplifier

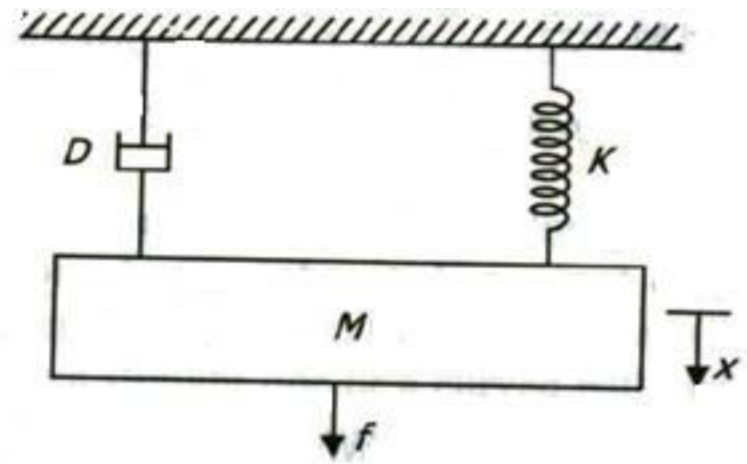
3) Find the differential equation that describe the network given below.



3.4. Mathematical models of mechanical systems

Example 3.2:

1) Determine the mathematical model for the system shown below.



Solution:

- The force f causes the displacement on mass M. This is oppose by spring as well as by damper.

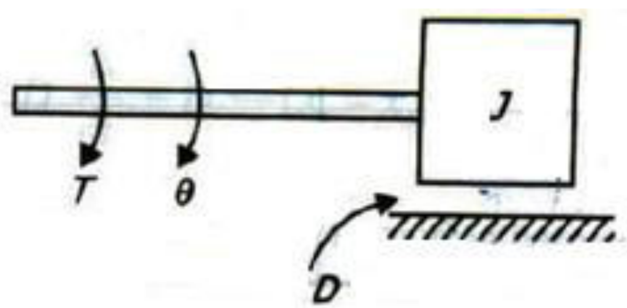
Cont. …

- So the mathematical model of the system is given by,

$$M\frac{dx^2}{dt} = f - D\frac{dx}{dt} - kx$$

$$f = M\frac{dx^2}{dt} + D\frac{dx}{dt} + Kx$$

2) Determine the torque equation of the system shown below.



Solution:

$$T - D\frac{d\theta}{dt} = J\frac{d^2\theta}{dt^2}$$

$$T = J\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt}$$

Exercise 3.3:

1. Determine the transfer function ${X_2(s)}/{F(s)}$ for the system shown in Figure 3.1. Both masses slide on a frictionless surface, and $k = 1\ {N}/{m}$.

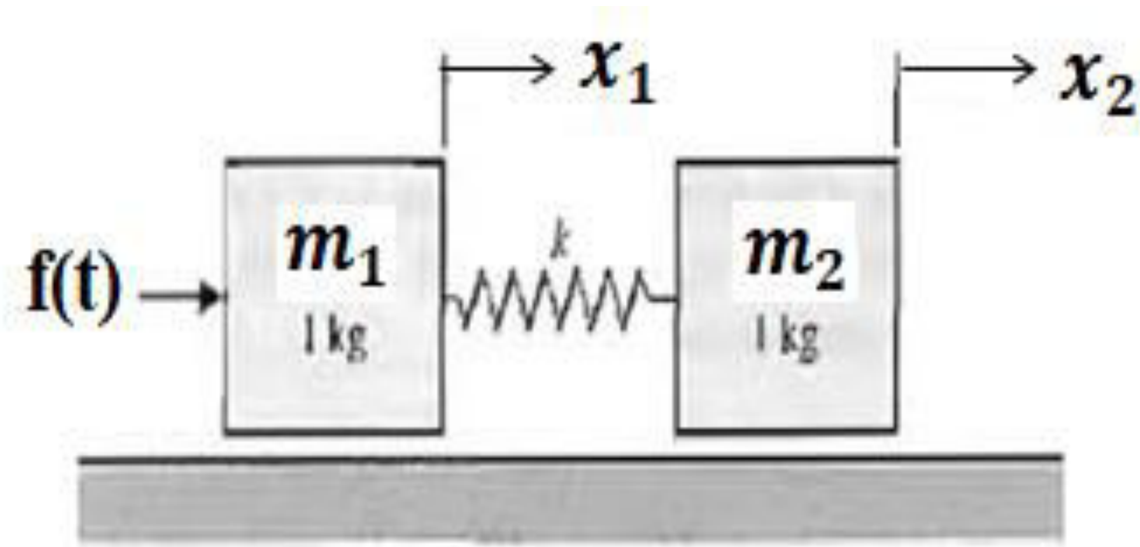


Fig.3.1: Two connected masses on a frictionless surface.

Cont. …

2. A two-mass model of the robot is shown in Figure 3.2. Find the transfer function ${Y(s)}/{F(s)}$.

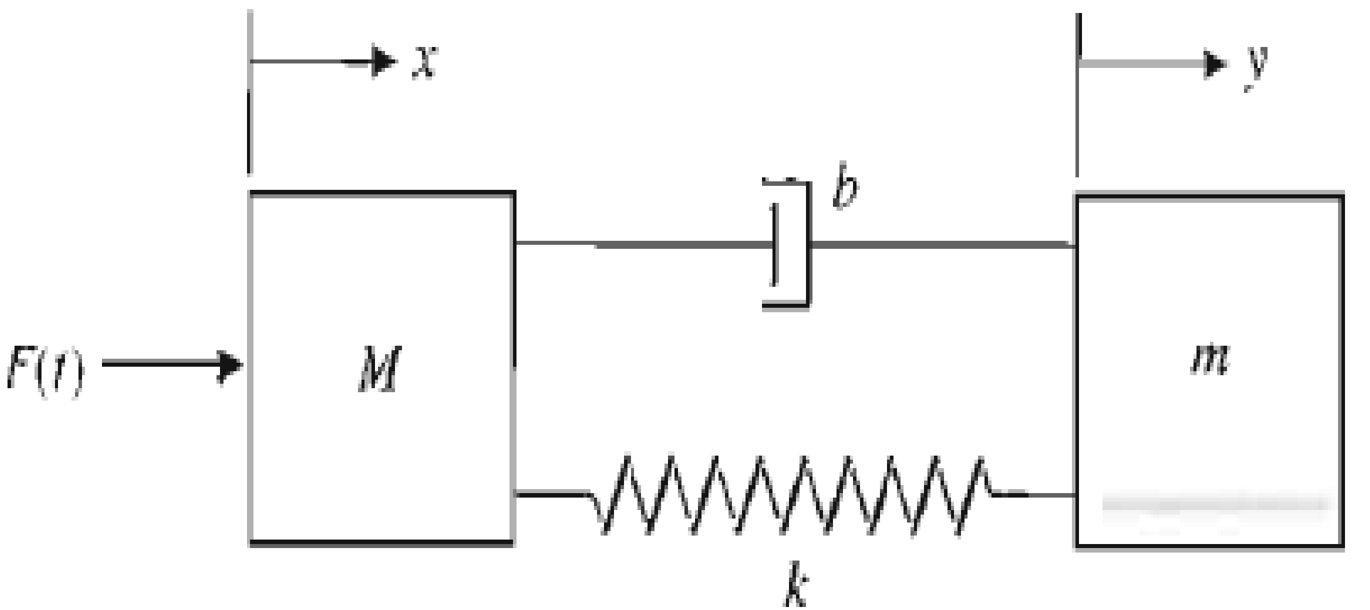


Fig.3.2: spring-mass-damper model of a robot arm.

Cont. …

3. For the three-cart system illustrated in Figure 3.3, obtain the equations of motion. The system has three inputs u_1, u_2, and u_3 and three outputs x_1, x_2, and x_3. Obtain three second-order ordinary differential equations with constant coefficients.

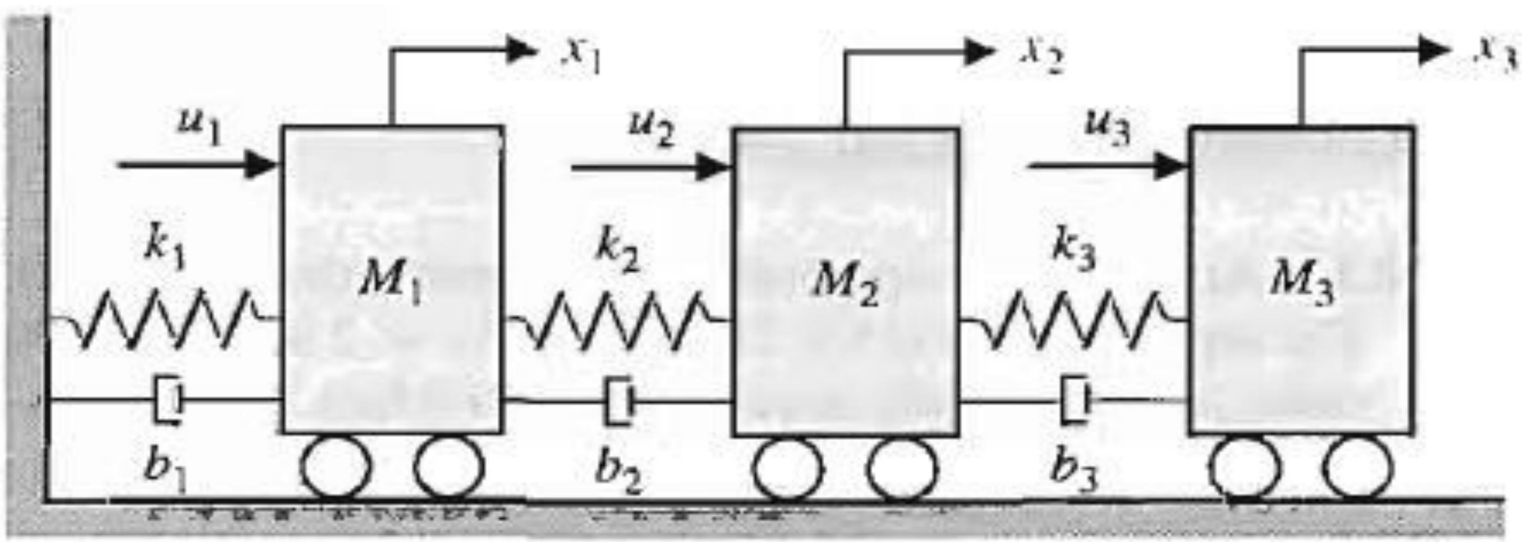


Fig.3.3: Three-cart system with Three inputs and three outputs.

3.5. Poles and zeros of a transfer function

- The n order transfer function of a system can be expressed as the following:

$$G(s) = \frac{b_0 S^m + b_1 S^{m-1} + b_2 S^{m-2} + b_3 S^{m-3} + \cdots \dots \dots + b_{m-1} S + b_m}{S^n + b_1 S^{n-1} + b_2 S^{n-2} + b_3 S^{n-3} + \cdots \dots \dots + b_{n-1} S + b_n}$$

- It is often convenient to factor the polynomials in the numerator and denominator, and to write the transfer function in terms of those factors:

$$G(s) = \frac{P(s)}{Q(s)} = \frac{K(S-z_1)(S-z_2)(S-z_3)\dots\dots(S-z_m)}{(S-p_1)(S-p_2)(S-p_3)\dots\dots(S-p_m)}$$

Where P(s) and Q(s) are the numerator and denominator polynomial equations of the transfer function respectively.

Cont. …

- The two polynomials, P(s) and Q(s), allow us to find the finite poles and zeros of the transfer function.

Definition 1: zeros

- The roots of $P(s) = 0$. $\{P(s)|_{S=z_i} = 0\}$ where $z_i, i = 1, 2, \ldots, m$ *are zero of the TF*
- The roots that will make the overall transfer function to zero. $\{\lim_{s \to z_i} G(s) = 0\}$

Definition 2: poles

- The roots of $Q(s) = 0$. $\{Q(s)|_{S=p_j} = 0\}$ where P_j, $j = 1, 2, \ldots, n$ *are pole of the TF*
- The roots that will make the overall transfer function to infinity. $\{\lim_{s \to p_j} G(s) = \infty\}$

Remark:

- The total number of finite and infinite poles and the total number of finite and infinite zeros of a transfer function are always equal.

Cont. …

- The total number of finite and infinite poles of a transfer function is always equal to the order of the transfer function.

Example 3.3: Find the zeros and poles of the following transfer functions

a) $G(s) = \frac{(S+2)}{(S+3)(S-2)}$

c) $G(s) = \frac{(S+2)}{(S+3)^2(S-2)}$

b) $G(s) = \frac{(S+2)}{(S+3)(S-3)(S-2)}$

d) $G(s) = \frac{(S+2)^2}{(S+3)^3(S-2)(S+1)}$

Solution **a)** **Poles :** $S = -3, 2$

Zeros:

- $\lim_{s\to\infty} G(s) = \lim_{s\to\infty} \frac{(S+2)}{(S+3)(S-2)} \approx \lim_{s\to\infty} \frac{1}{s} = 0$

Cont. …

- As we can see from this approximation, there will be one opportunity in which the zero of the system can be infinity.

In general the zeros of the given TF are $S = -2, \infty$

b) **Poles :** $S = -3, 3, 2$

Zeros:

- $\lim_{s\to\infty} G(s) = \lim_{s\to\infty} \frac{(S+2)}{(S+3)(S-3)(S-2)} \approx \lim_{s\to\infty} \frac{1}{S^2} = 0$

As we can see from this approximation, there will be two opportunities in which the zeros of the system can be infinity.

In general the zeros of the given TF are $S = -2, \infty, \infty$

c) **Poles :** $S = -3, -3, 2$

Zeros: $\lim_{s\to\infty} G(s) = \lim_{s\to\infty} \frac{(S+2)}{(S+3)^2(S-2)} \approx \lim_{s\to\infty} \frac{1}{S^2} = 0$

Cont. …

- As we can see from this approximation, there will be two opportunities in which the zeros of the system can be infinity.

In general the zeros of the given TF are $S = -2, \infty, \infty$

d) Poles : $S = -3, -3, -3, 2, -1$

Zeros:

- because $\lim_{s\to\infty} G(s) = \lim_{s\to\infty} \frac{(S+2)^2}{(S+3)^3(S-2)(S+1)}$

$$\approx \lim_{s\to\infty} \frac{1}{S^3} = 0$$

As we can see from this approximation, there will be three opportunities in which the zeros of the system can be infinity.

In general the zeros of the given TF are $S = -2, -2, \infty, \infty, \infty$

3.6. Diagrammatical representation mechanisms of a systems dynamic models

3.6.1. Block Diagram

- It is a pictorial representation of the functions performed by each component and of the flow of signals. Such a diagram depicts the interrelationships that exist among the various components.
- Differing from a purely abstract mathematical representation, *a block diagram has the advantage of indicating more realistically the signal flows of the actual system.*
- In a block diagram all system variables are linked to each other through functional blocks.

Cont. …

- The functional block or simply block is a symbol for the mathematical operation on the input signal to the block that produces the output.
- The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signals. Note that the signal can pass only in the direction of the arrows.
- The advantages of the block diagram representation of a system lie in the fact that it is easy to form the overall block diagram for the entire system by merely connecting the blocks of the components according to the signal flow and that it is possible to evaluate the contribution of each component to the overall performance of the system.

Cont. …

- In general, the functional operation of the system can be visualized more readily by examining the block diagram than by examining the physical system itself.
- A block diagram contains information concerning dynamic behavior, but it does not include any information on the physical construction of the system. Consequently, many dissimilar and unrelated systems can be represented by the same block diagram.
- It should be noted that a block diagram of a given system is not unique. A number of different block diagrams can be drawn for a system, depending on the point of view of the analysis.

Cont. …

Parts of a Block diagram

i. Summing Point

- As it is shown in figure 3.1, a circle with a cross is the symbol that indicates a summing operation. The plus or minus sign at each arrowhead indicates whether that signal is to be added or subtracted.
- Quantities being added or subtracted should be the same dimensions and the same units.

ii. Branch/ Takeoff Point

- It is a point from which the signal from a block goes concurrently to other blocks or summing points.

Block Diagram of a Closed-Loop System

- Figure 3.1 shows an example of a block diagram of a closed-loop system. The output Y(s) is fed back to the summing point, where it is compared with the reference input R(s) .

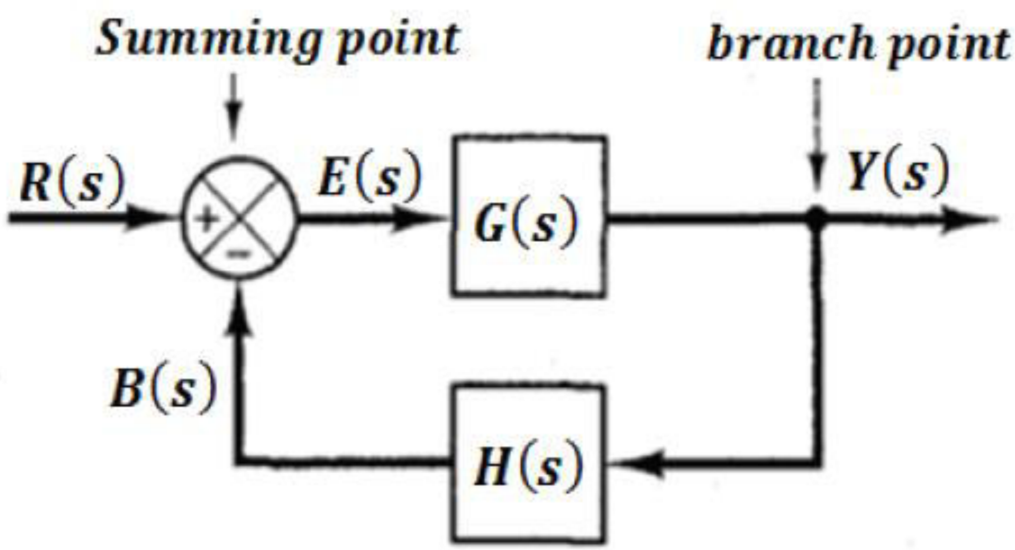


Figure 3.4: Block diagram of a closed loop system.

i. Open-Loop Transfer Function

- Referring to Figure 3.1, the ratio of the feedback signal B(s) to the actuating error signal E(s) is called the *open-loop transfer function. That is,*

$$\text{Open loop transfer function} = \frac{B(s)}{E(s)}$$

$$\Rightarrow B(s) = H(s)Y(s) \qquad \text{But } Y(s) = G(s)E(s)$$

$$\Rightarrow B(s) = H(s)G(s)E(s)$$

$$\Rightarrow \frac{B(s)}{E(s)} = H(s)G(s)$$

ii. Feed forward Transfer Function

➢ Referring to Figure 3.1, the ratio of the output signal Y(s) to the actuating error signal E(s) is called the *feed forward transfer function. That is,*

$$Feed\ forward\ transfer\ function = \frac{Y(s)}{E(s)}$$

$$\Rightarrow Y(s) = G(s)E(s)$$

$$\Rightarrow \frac{Y(s)}{E(s)} = G(s)$$

➢ If the feedback transfer function H(s) is unity, then the open-loop transfer function and the feed forward transfer function are the same.

iii. Closed-Loop Transfer Function

Referring to Figure 3.1, the ratio of the output signal Y(s) to the reference input signal R(s) is called the closed-loop transfer function. That is,

$$\textit{Closed loop transfer function} = \frac{Y(s)}{R(s)}$$

$$\Rightarrow Y(s) = G(s)E(s) \qquad \text{But } E(s) = R(s) - B(s) = R(s) - H(s)Y(s)$$

$$\Rightarrow Y(s) = G(s)\{R(s) - H(s)Y(s)\}$$

$$\Rightarrow Y(s) = G(s)R(s) - G(s)H(s)Y(s)$$

$$\Rightarrow Y(s) + G(s)H(s)Y(s) = G(s)R(s)$$

$$\Rightarrow Y(s)\{1 + G(s)H(s)\} = G(s)R(s)$$

$$\Rightarrow \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Cont. …

- In general, for a negative feedback system the closed loop transfer function can be easily obtained by using the following formula.

$$\text{Closed loop transfer function} = \frac{\text{Feed forward Transfer Function}}{1 + \text{Open loop transfer function}}$$

❖ Procedures for Drawing a Block Diagram

- To draw a block diagram for a system the following procedures will be followed.

i. Write the equations that describe the dynamic behavior of each component.

ii. Then take the Laplace transforms of these equations, assuming zero initial conditions.

iii. Represent each Laplace-transformed equation individually in block form.

iv. Assemble the elements into a complete block diagram.

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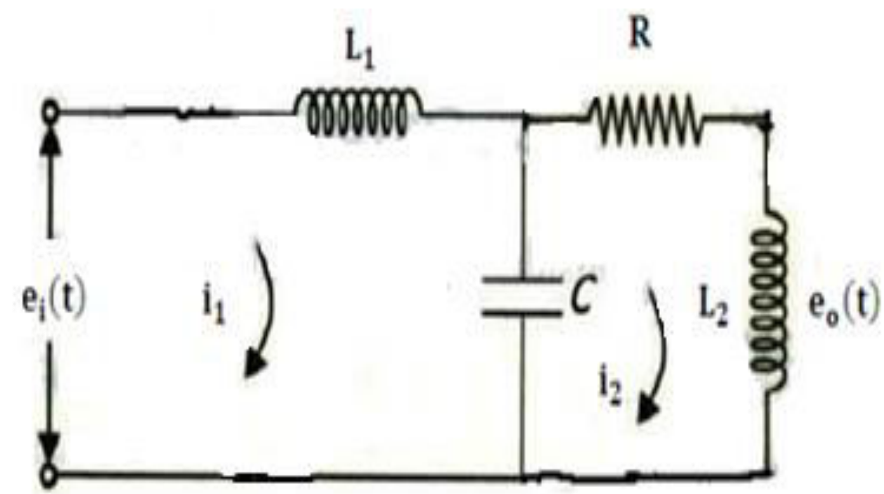
Example 3.4: Represent the following system in block form



Solution: The system differential equation (using KVL)

$$\textbf{Loop1:} \quad e_i = L_1 \frac{di_1}{dt} + \frac{1}{C}\int (i_1 - i_2)\, dt$$

$$\textbf{Loop2:} \quad Ri_2 + e_o + \frac{1}{C}\int (i_2 - i_1)\, dt$$

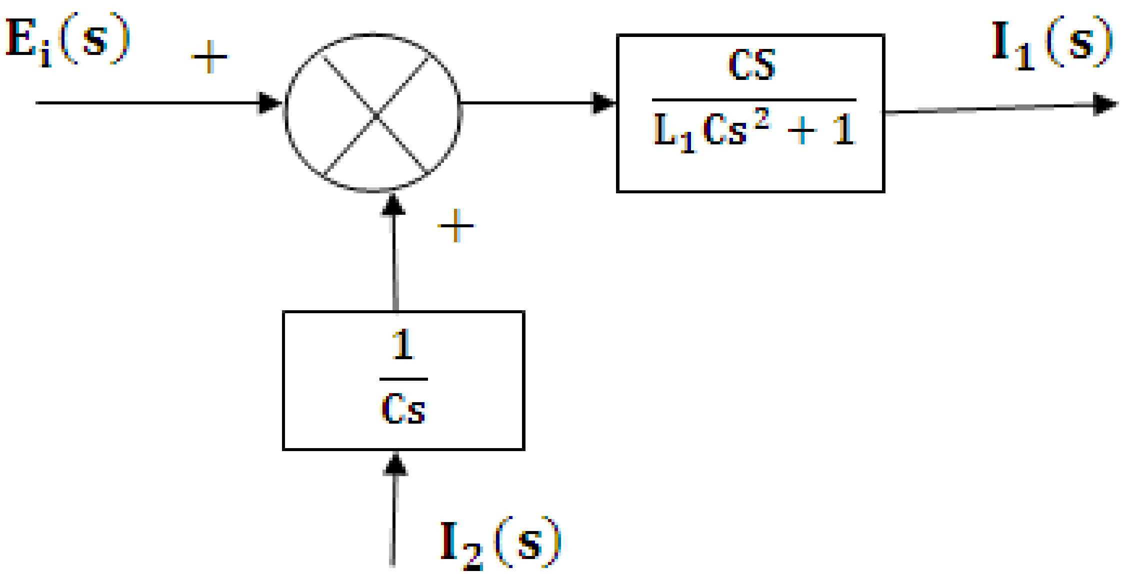
$$e_o = L_2 \frac{di_2}{dt}$$

Cont. …

Laplace transformed eqns.

loop1: $$E_i(s) = L_1 s I_1(s) + \frac{1}{Cs}\left(I_1(s) - I_2(S)\right)$$

$$\rightarrow \frac{Cs}{L_1 Cs^2 2+1}\left[E_i(s) + \frac{1}{Cs} I_2(s)\right] = I_1(s) \qquad (1)$$



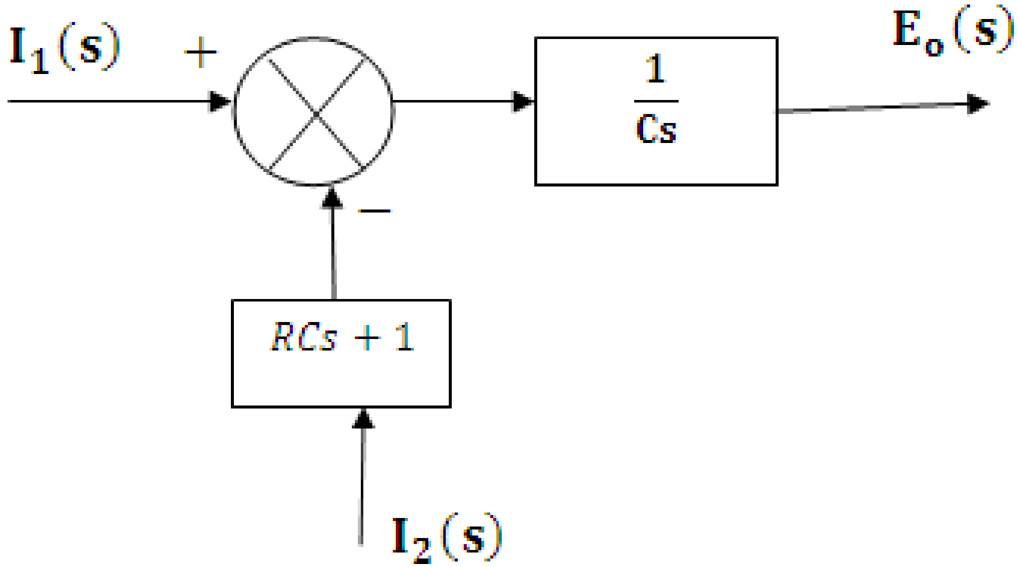
Block diagram representation of equation (1)

Cont. …

Loop2:

$$RI_2(s) + E_o(s) + \frac{1}{Cs}\left(I_2(s) - I_1(S)\right) = 0$$

$$\rightarrow \frac{1}{Cs} I_1(s) - \frac{RCs+1}{Cs} I_2(s) = E_o(s) \qquad (2)$$

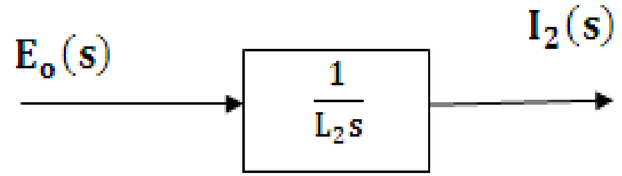


Block diagram representation of equation (2)

Cont. …

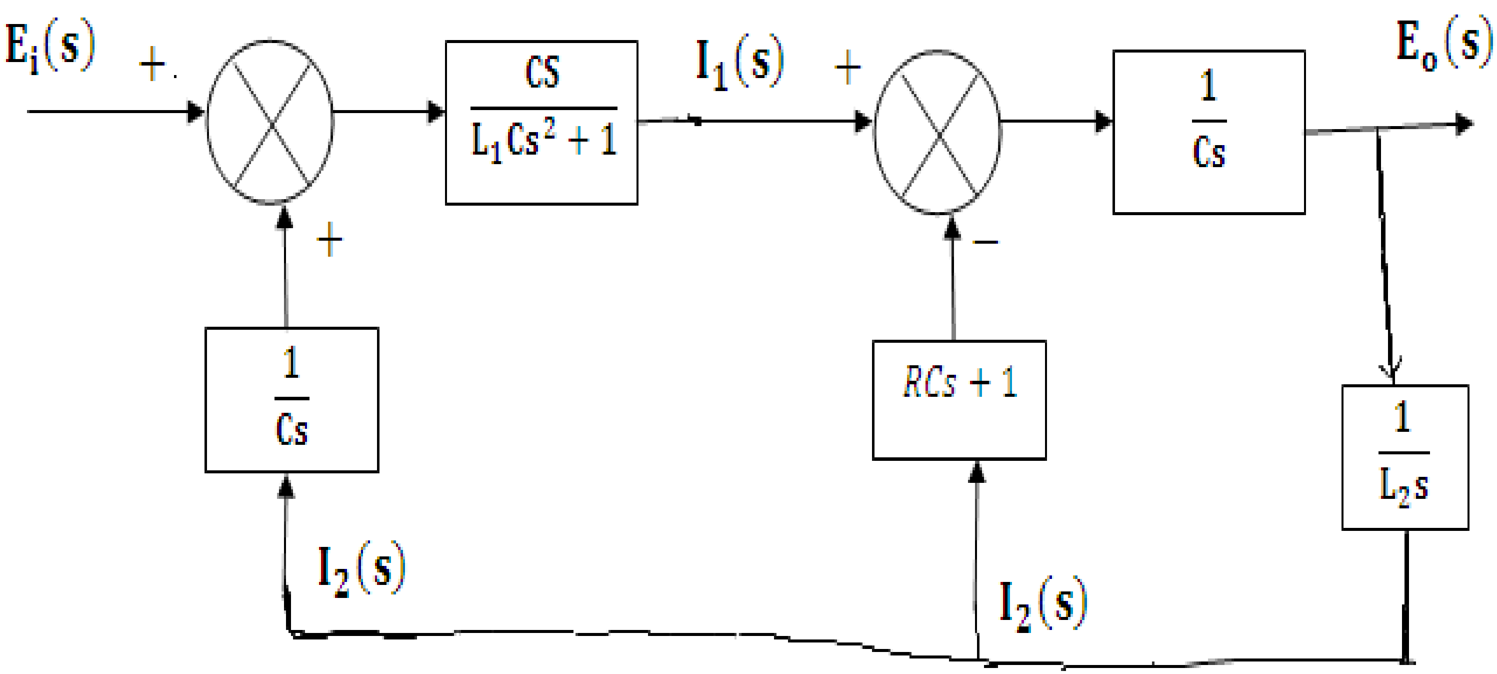
Also

$$E_o(s) = L_2 s I_2(s) \rightarrow I_2(s) = \frac{1}{L_2 s} E_o(s) \qquad (3$$



Block diagram representation for eqn. (3)

The complete block diagram of the system by combination of the above individual.

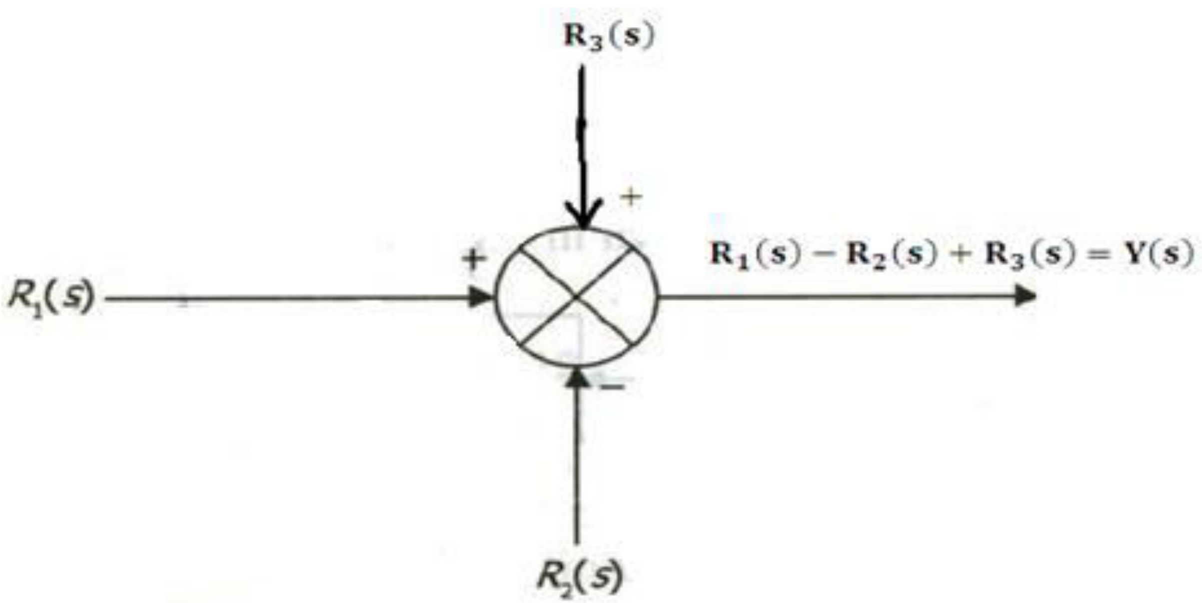


Cont. …

Rules For Block Diagram Reduction

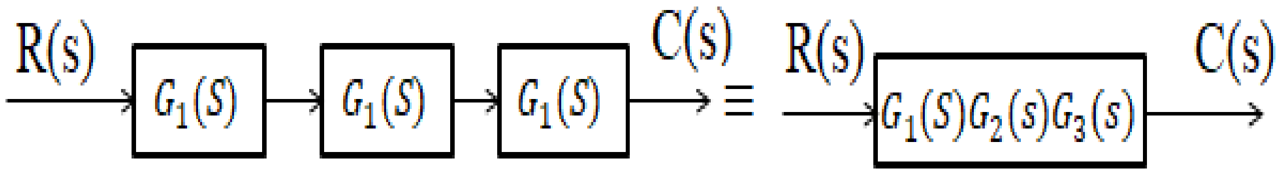
➢ The complex block diagram involving many feedback loops can be simplified by reduction of block diagram.

1) **Summing point:** it is a point where two or more signals may be added /subtracted.

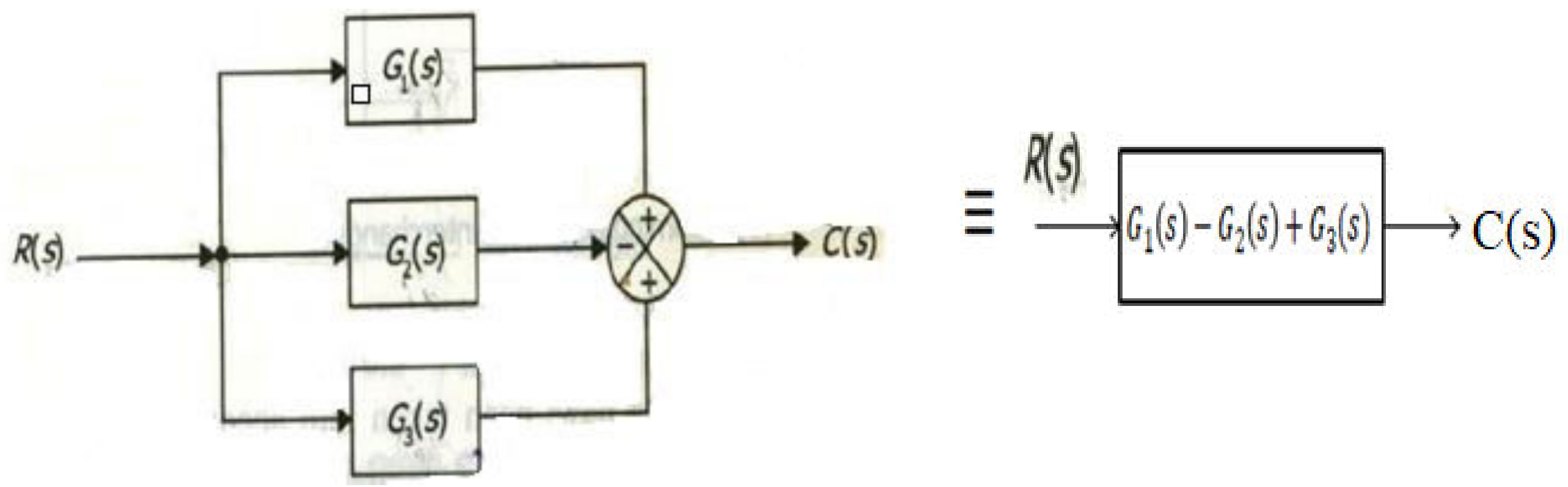


Cont. …

2) For blocks in cascade (series)

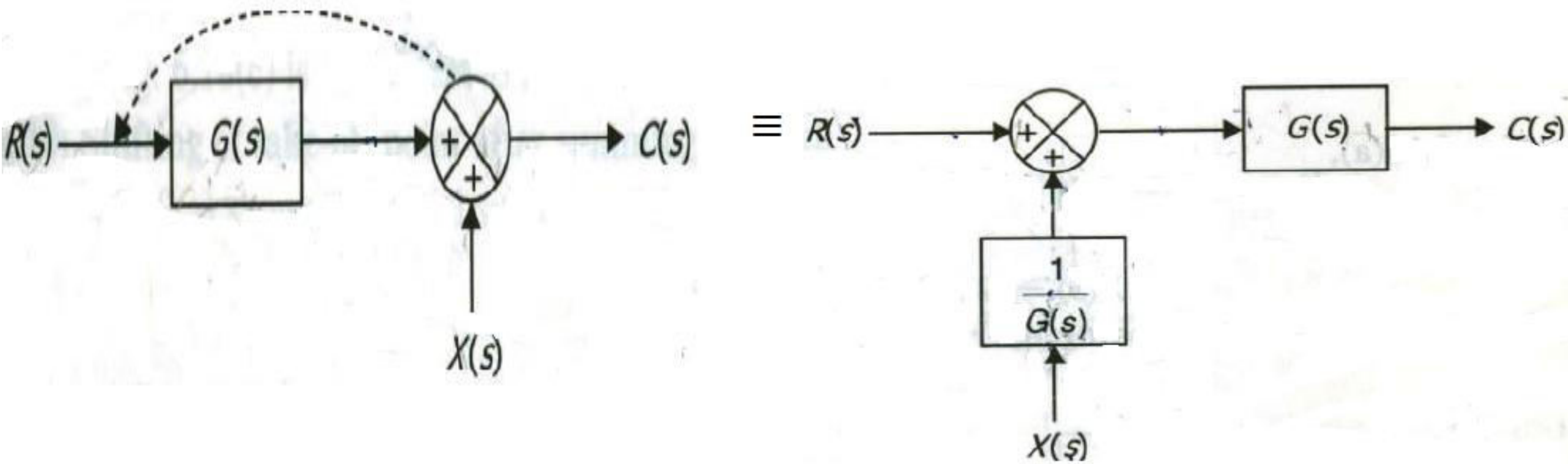


2) For blocks in parallel C(s)

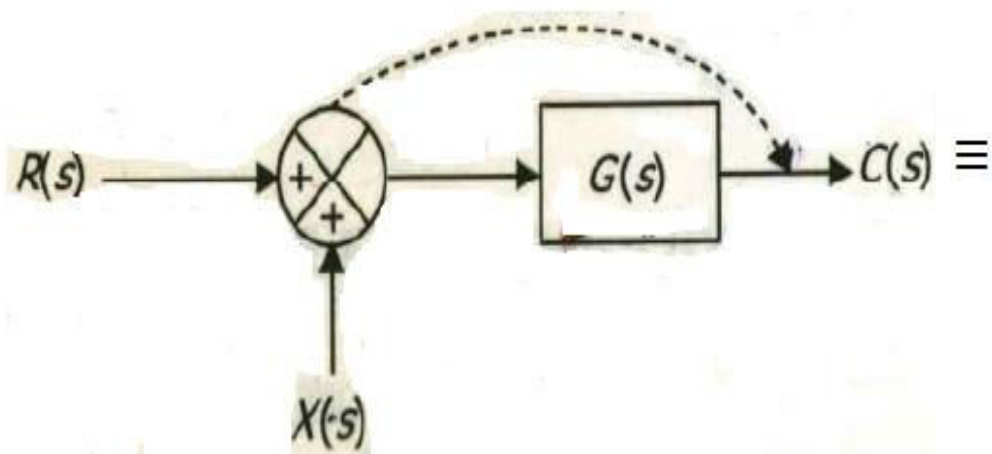


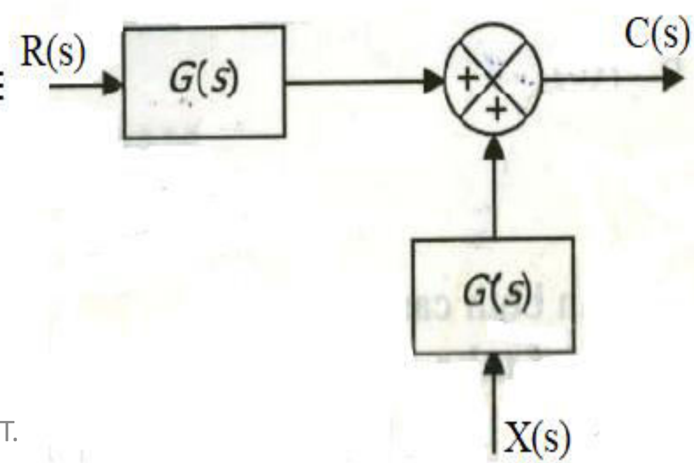
Cont. …

4) Shifting a summing point before a block



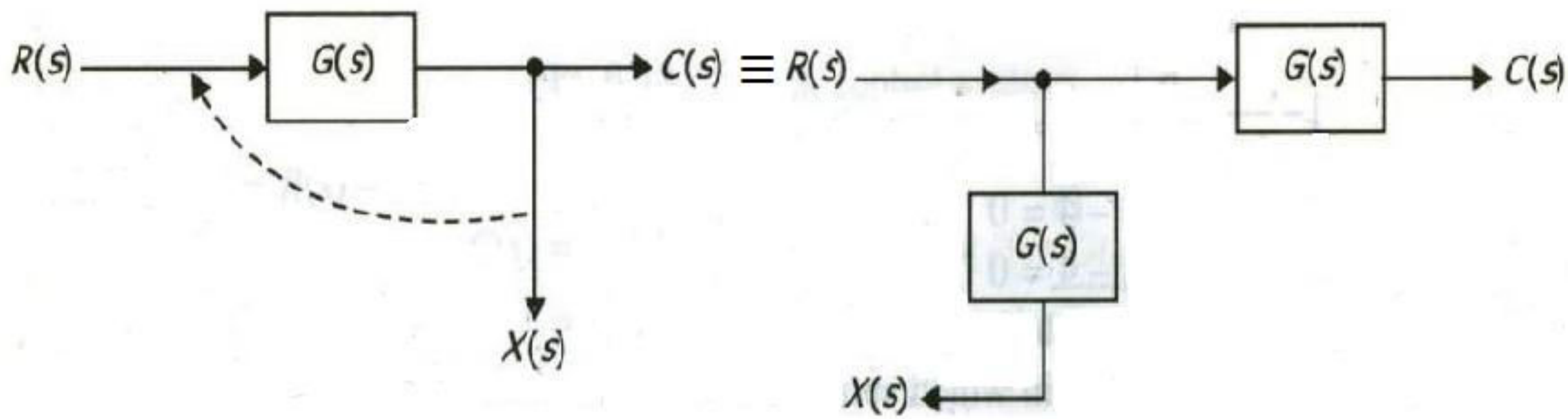
5) Shifting of summing point after a block



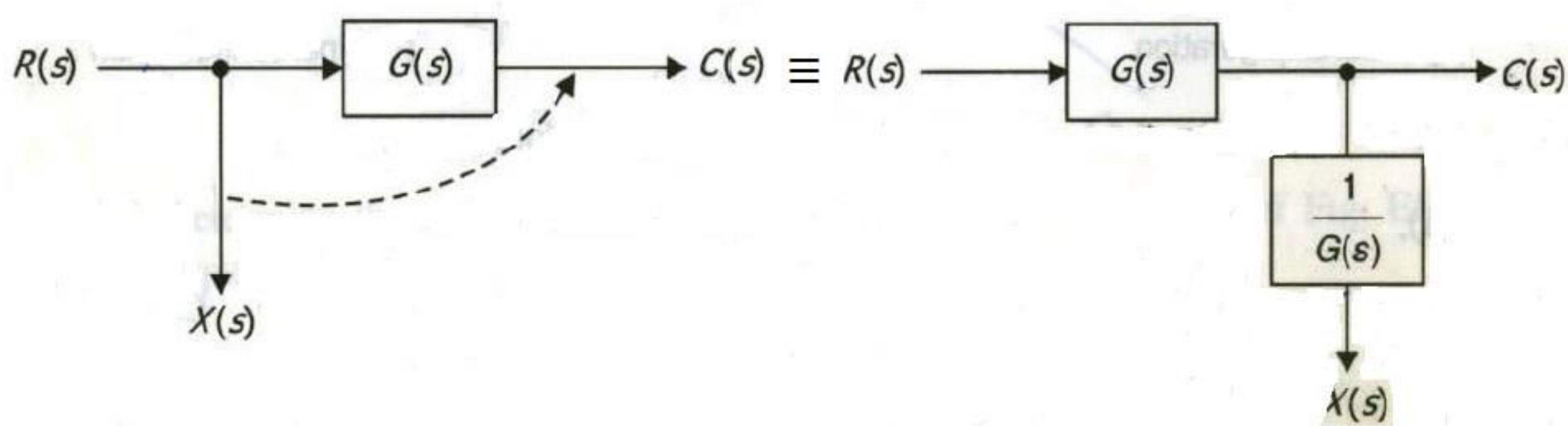


Cont. …

6) Shifting of take-off point before a block



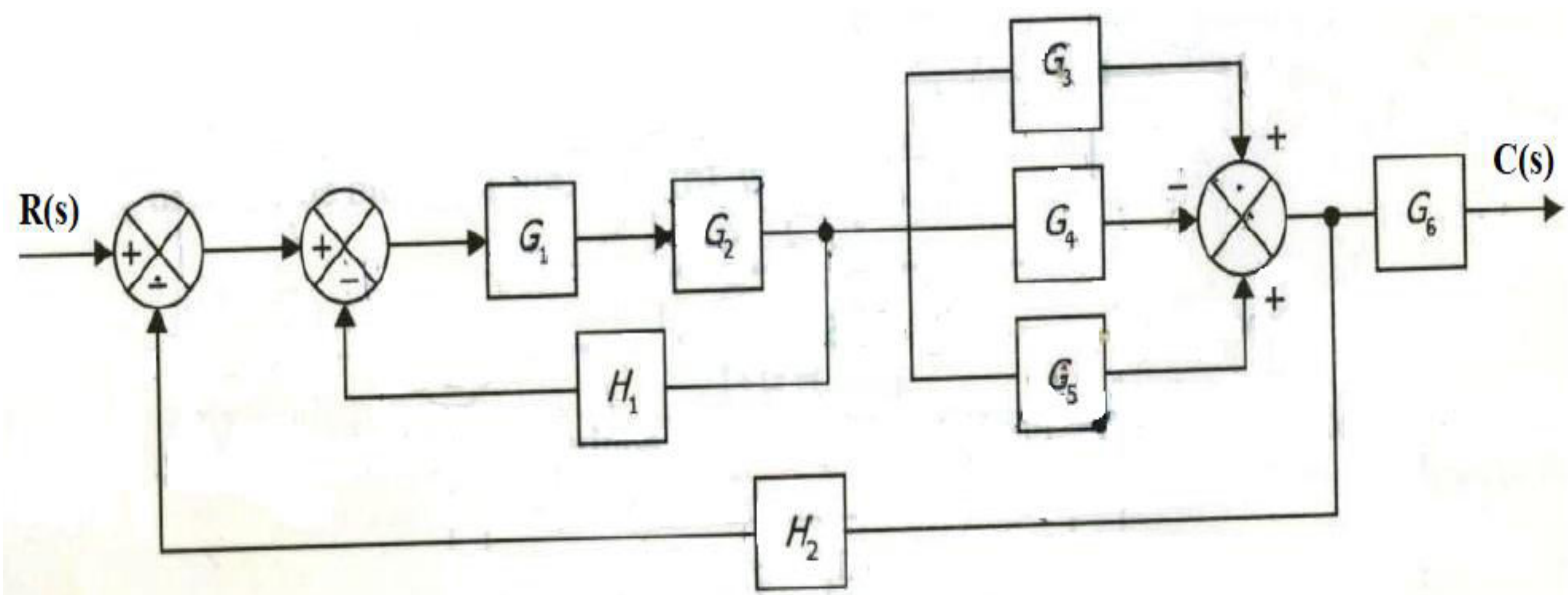
7) Shifting of take-off point after a block



❖ Procedure for block diagram reduction

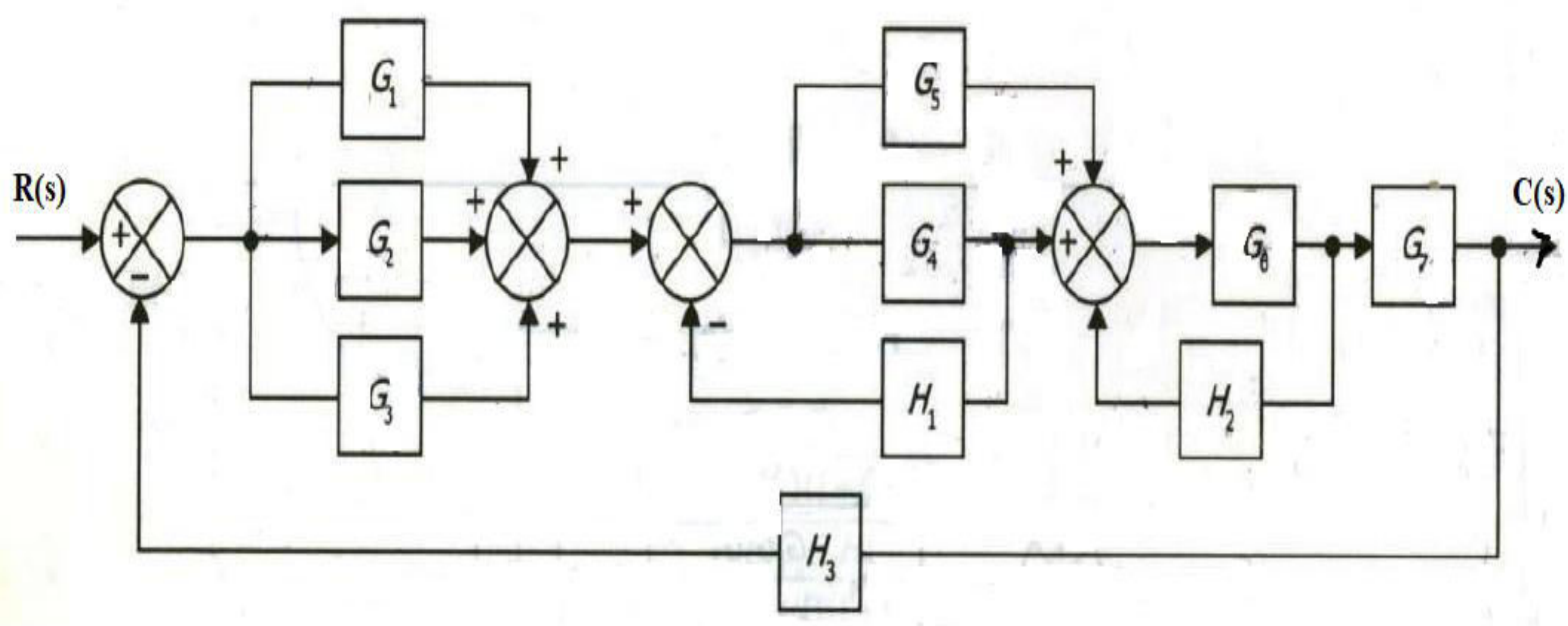
1) Reduce the cascade blocks.
2) Reduce the parallel blocks.
3) Reduce the internal feedback loops.
4) It is advisable to shift take-off points towards right and summing points towards left.
5) Repeat steps 1 to step 4 until the simple form is obtained.
6) Find transfer function of the overall system using the formula $C(s)/R(s)$.

Example 3.5: a) Find the single block equivalent and transfer function of the $C(s)/R(s)$.



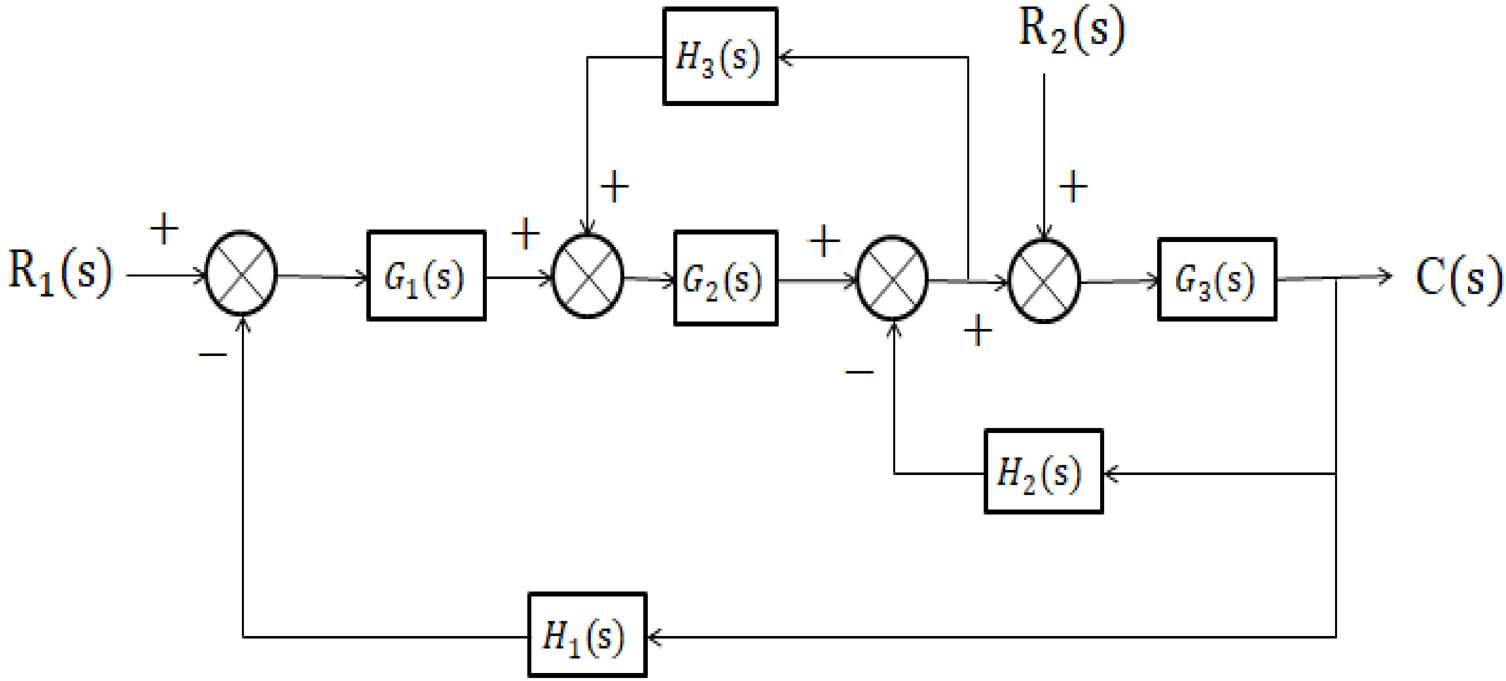
Cont. …

b) Obtain the simplified block and $C(s)/R(s)$.



Cont. …

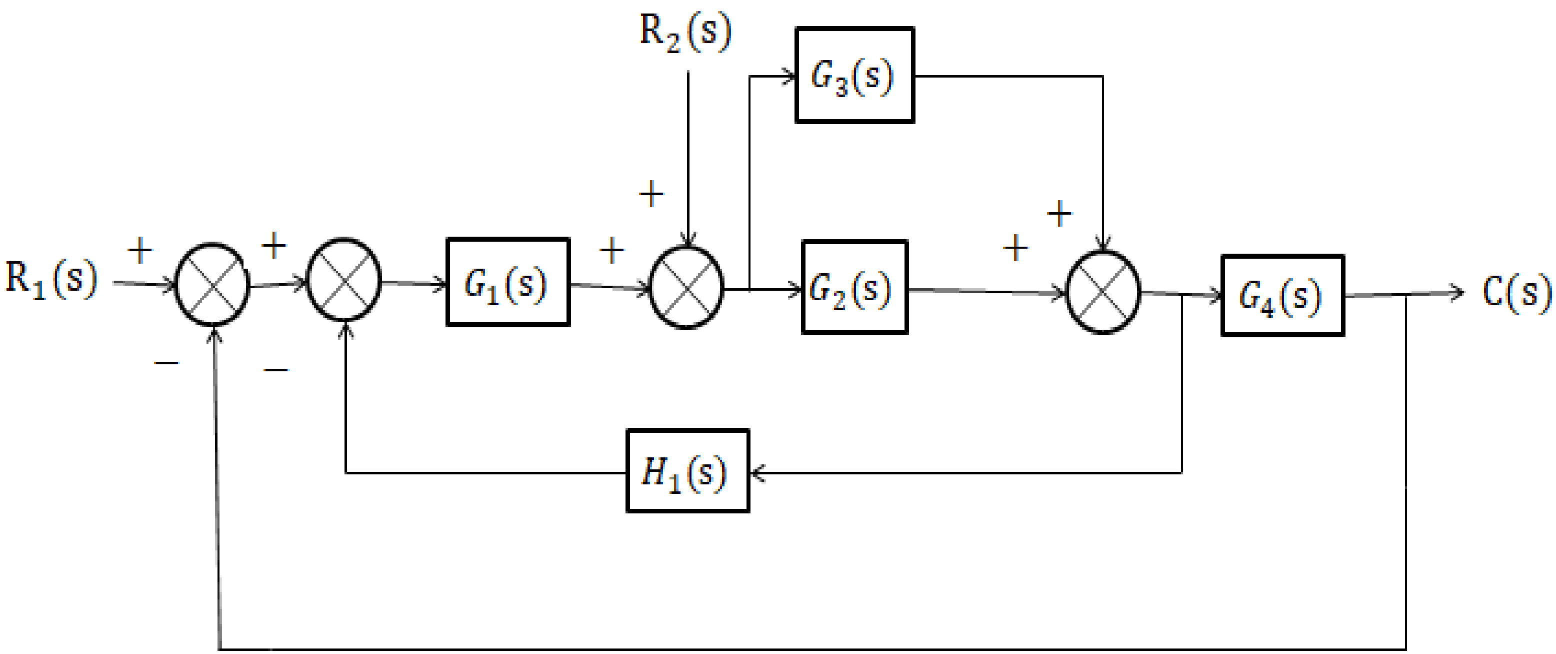
c) Obtain the expression for the output C(s) of the system.



Answer:

$$\boldsymbol{C}(s) = \frac{G_1(s)G_2(s)G_3(s)}{1 + H_2(s)G_3(s) - H_3(s)G_2(s)G_3(s) + H_1(s)G_1(s)G_2(s)G_3(s)} R_1(s)$$
$$+ \frac{(1 - H_2(s)G_2(s))G_3(s)}{1 - H_3(s)G_2(s) + G_3(s)[H_2(s) + H_1(s)G_1(s)G_2\ (s)]} R_2(s)$$

d) Obtain the expression for the output C(s) of the system



Answer:

$$C(s) = \frac{G_1 G_4 (G_2 + G_3)}{1 + H_1 G_1 (G_2 + G_3) + G_1 G_4 (G_2 + G_3)} R_1(s) + \frac{G_4 (G_2 + G_3)}{1 + H_1 G_1 (G_2 + G_3) + G_4 (G_2 + G_3)} R_2(s)$$

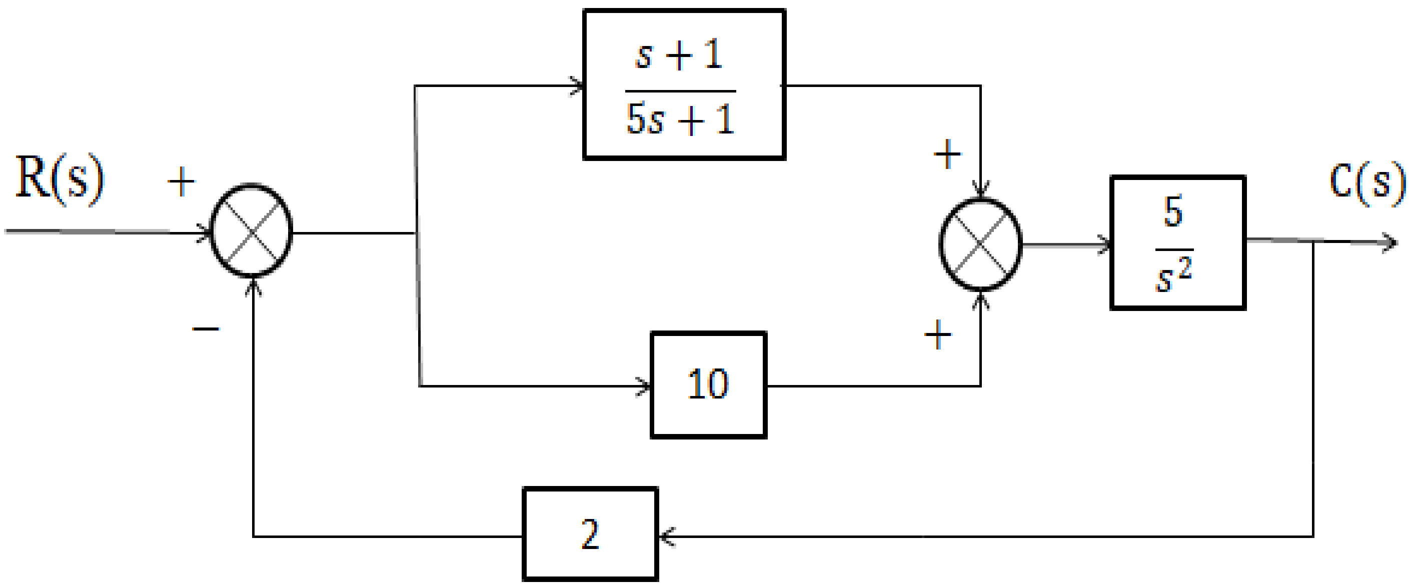
Cont. …

e) For the block diagram given below

i. Reduce the block diagram and find the closed loop transfer function.

ii. Find the poles and zeros of the closed loop system.

iii. Convert the system into a unity feedback.



Cont. …

Solution:

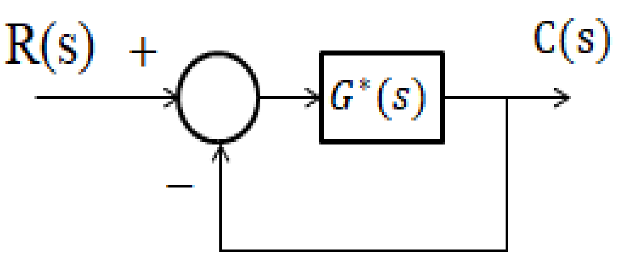
i. $\frac{C(s)}{R(s)} = \frac{255s+55}{5s^3+s^2+510s+110}$

ii. Characteristic equation: $5s^3 + s^2 + 510s + 110 = 0$

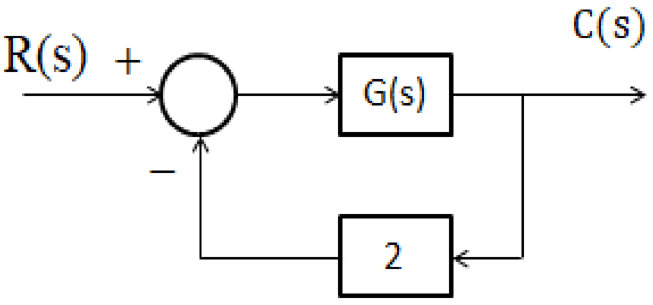
Zeros: $255s + 55 = 0 \rightarrow s = -\frac{55}{255} = -\frac{11}{51}$

poles: slove, $5s^3 + s^2 + 510s + 110 = 0$

iii. Unity feedback form



from given system



Cont. …

where $$G(s) = \left[\left(\frac{s+1}{5s+1}\right) + 10\right]\frac{5}{s^2}$$

$$\frac{\mathrm{C(s)}}{\mathrm{R(s)}} = \frac{\mathrm{G(s)}}{1 + \mathrm{H(s)G(s)}} \rightarrow \frac{G(s)}{1 + 2G(s)} = \frac{G^*(s)}{1 + G^*(s)}$$

Then solving for $G^*(s)$ gives,

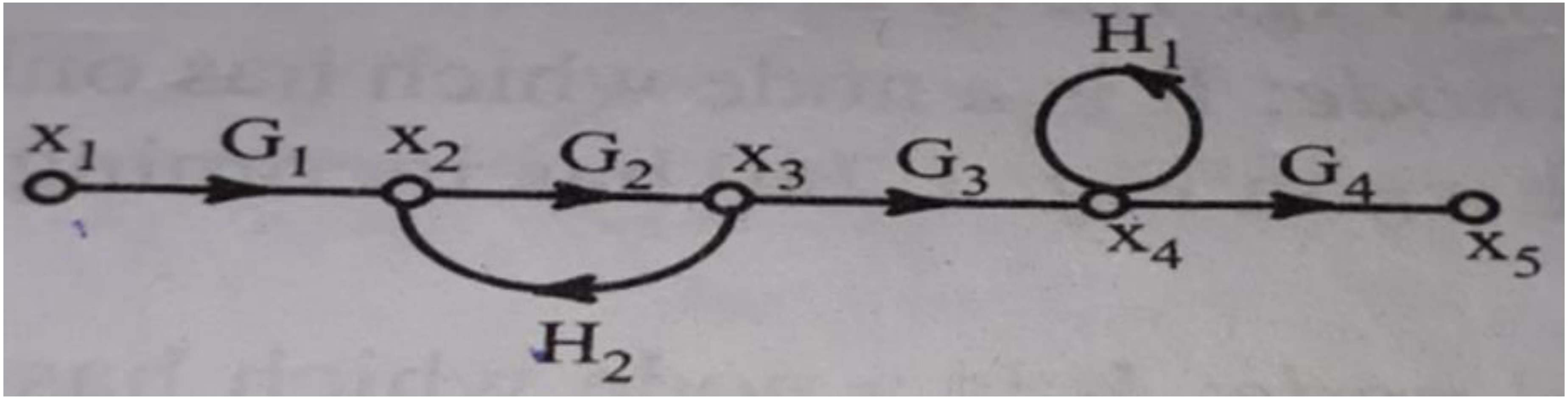
$$\rightarrow G^*(s) = \frac{G(s)}{1 + G(s)}$$

3.6.2. Signal Flow Graph (SFG)

- A signal flow graph is another graphical representation of control system. In which nodes represent system variable are connected by direct branches.
- It may be regarded as a simplified version of a block diagram.

❖ Signal Flow Graph Terminology

- Consider the signal flow graph shown below:



Cont. …

- **Node:** is system variable which is equal to the sum of all incoming signal at node, while outgoing signal from node do not affect the value of node.
- **Example** $x_2 = G_1x_1 + H_2x_3$ & $x_4 = G_3x_3 + H_1x_4$
- **Branch:** is direct line between nodes.
- **Transmittance:** is gain between nodes/branch gain.

  Example G_1, G_2, G_3 & G_4.
- **Input node/source node:** is node which has only outgoing branches.

  Example: x_1.
- **Output node/sink node:** is node which has only incoming branches.

  Example: x_5.

Cont. …

- **Chain node/mixed node:** is node which has both incoming and outgoing branches. Example: x_2, x_3 & x_4.
- **Forward path:** is path from input node to output node.

  Example: $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4 \rightarrow x_5$.
- **Closed loop/feedback loop:** is loop which start from particular node and ends to same node. Example: $x_2 \rightarrow x_3 \rightarrow x_2$.
- **Self loop:** start from a node and end at same node. Example: x_4.
- **Path gain:** the product of gains going through a forward path.

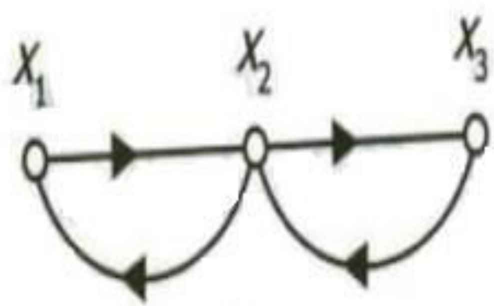
  Example: the path gain of the path $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4 \rightarrow x_5$ is $G_1G_2G_3G_4$.
- **Loop gain:** product of branch gains in a closed loop.

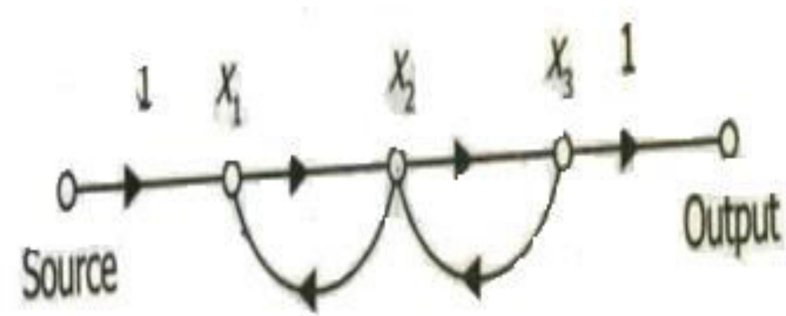
  Example: $x_2 \rightarrow x_3 \rightarrow x_2$ is G_2H_2.

Cont. …

❖ **Dummy Node:** If the incoming as well as outgoing branches exist at the first and the last node representing input and output variables, these nodes cannot be taken as input and output nodes. In such case, separate input and output nodes are created by adding branches with gain 1.

Example





Without input and output nodes. **Dummy nodes.**

Construction of signal flow graph from linear equations

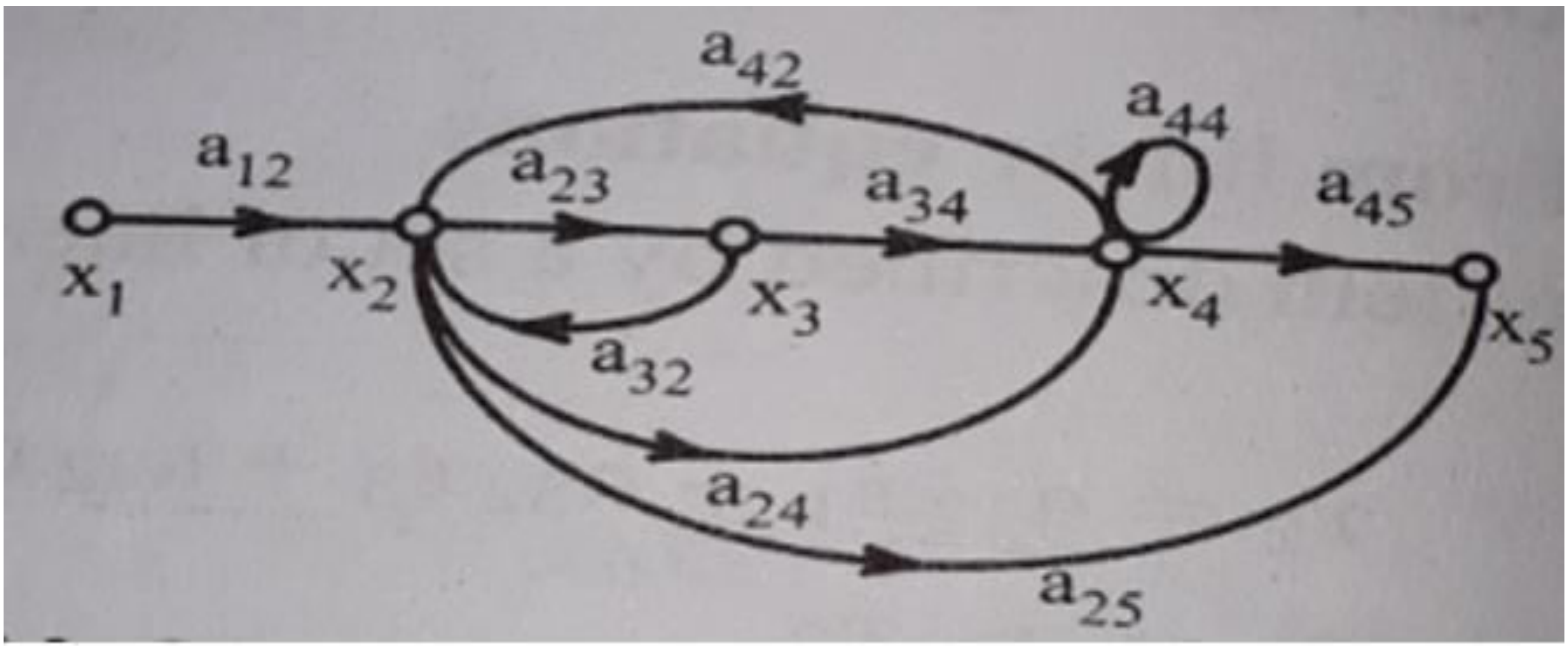
❖ Let consider the system described by linear equation shown below:

$x_2 = a_{12}x_1 + a_{32}x_3 + a_{42}x_4$

$x_3 = a_{23}x_2$

$x_4 = a_{24}x_2 + a_{34}x_3 + a_{44}x_4$

$x_5 = a_{25}x_2 + a_{45}x_4$



Procedure to draw signal flow graph from block diagram

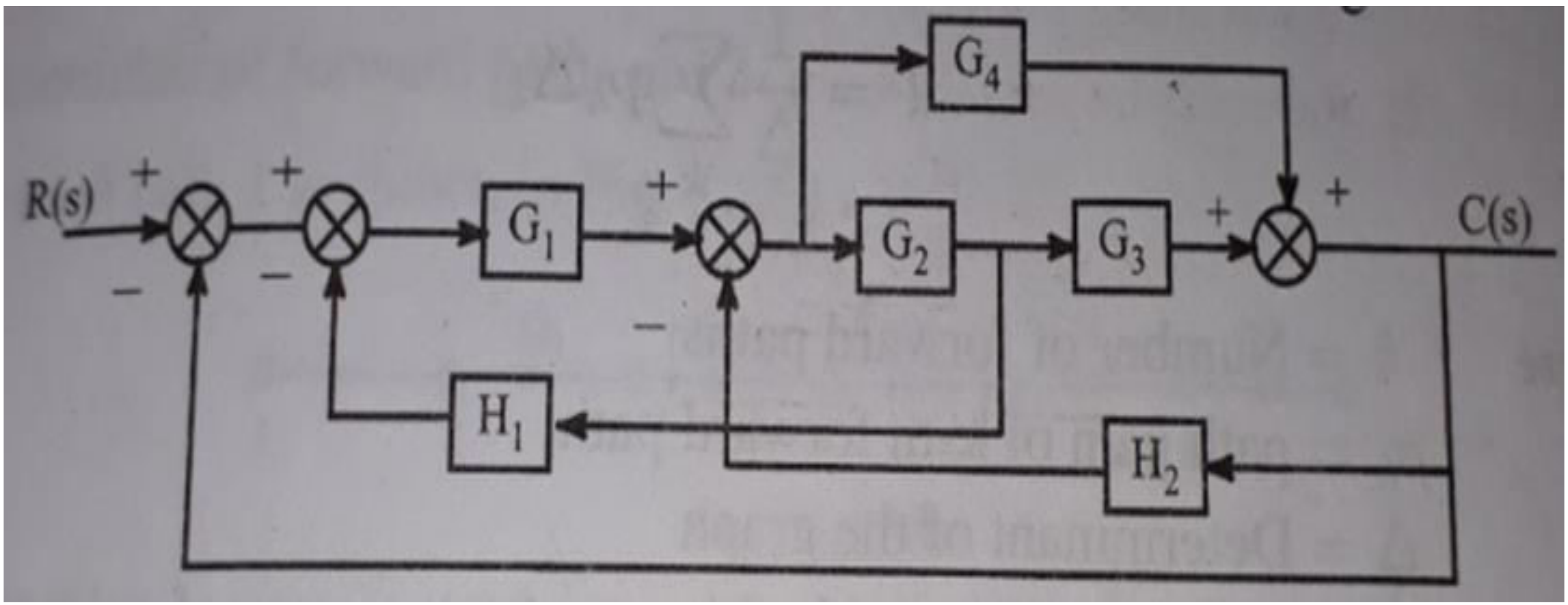
❖ Steps to draw signal flow graph from block diagram are:

1) Replace input and output signal by nodes.
2) Replace the summing point by nodes.
3) Replace all the take off points by nodes.
4) If the branch connecting summing point and take off point has unity gain, combined and represented by a single node.
5) If there are more take off points from the same signal, all take off points combined and represented by a single node.
6) If the gain of the link connecting two summing points is one, the two summing points combined and represented by a single node.

Cont. …

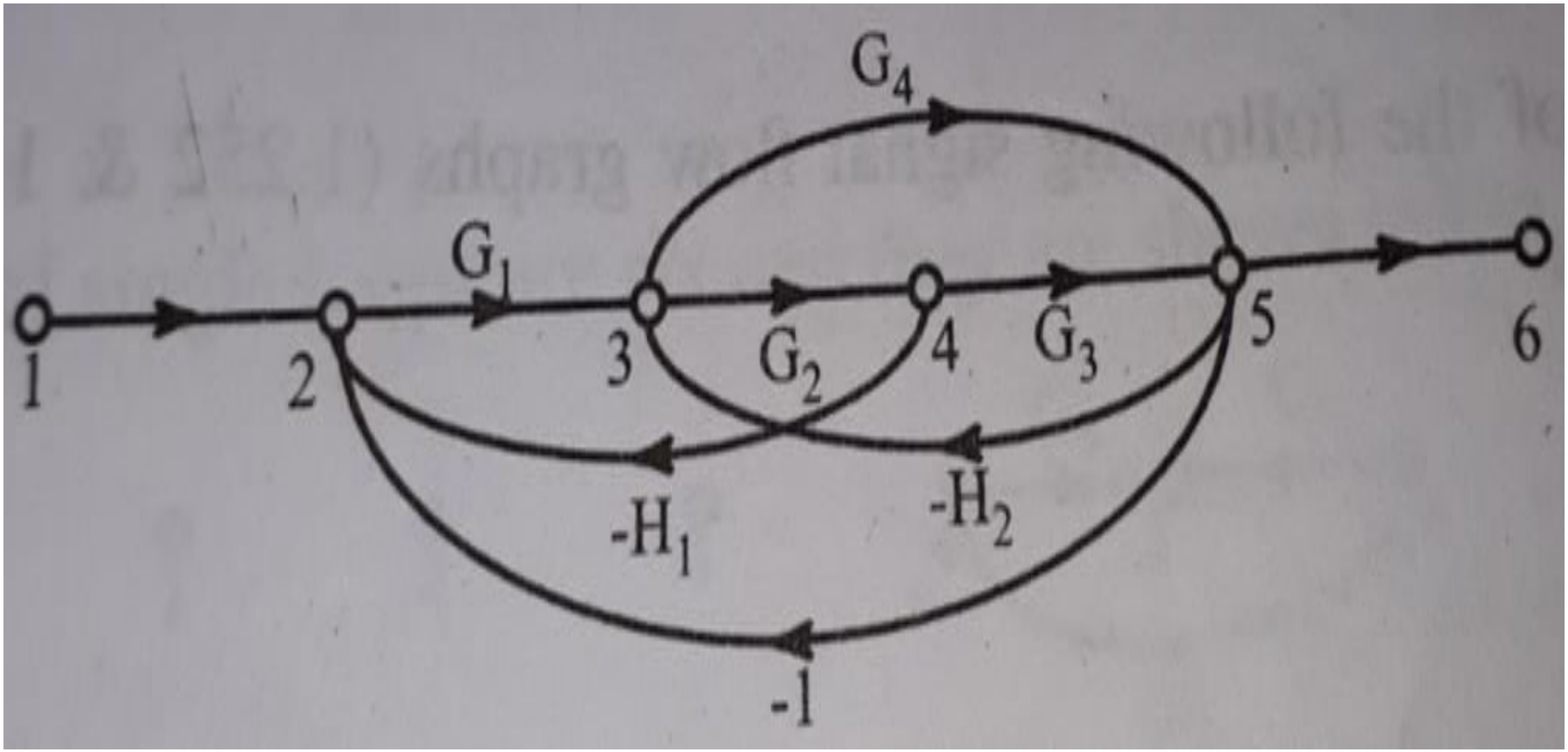
7) In summing point, if a signal is subtracted instead of addition then multiply the transmittance by -1 while representing in signal flow graph.

Example1: Draw the signal flow graph for the block diagram shown below.



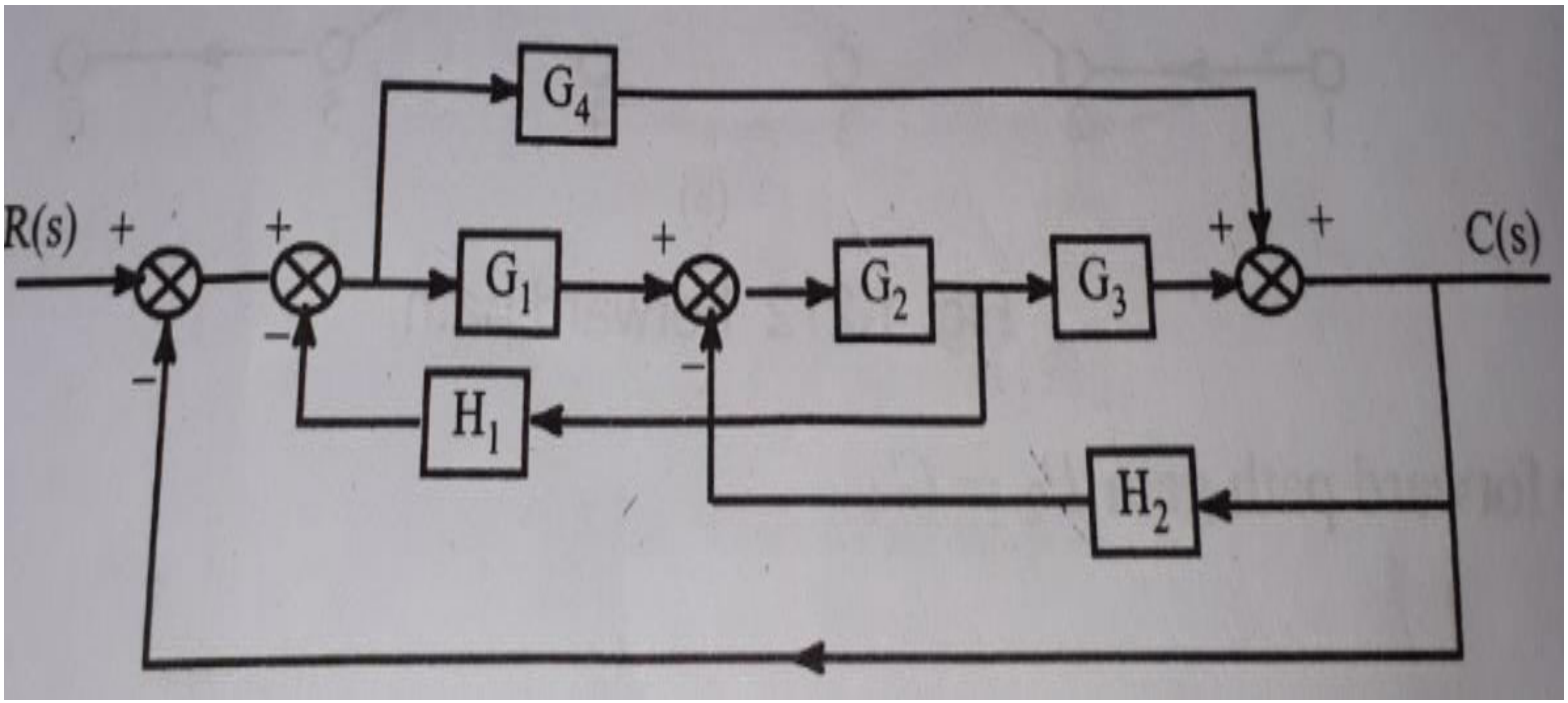
Cont. …

Solution



Cont. …

Exercise 3.3: Construct the equivalent signal flow graph for the block diagram shown below.



Mason's Gain formula

- Mason's gain formula is used for the determination of the overall transfer function of a system.
- It is convenient and easy way of finding the relation between input and output variables from the signal flow graph.
- Since high number of steps involved in block diagram reduction and time consuming, any complex block diagram can be converted into signal flow graph and transfer function can be easily obtained using Mason's gain formula.

Mason's gain formula is given by

$$T = \frac{1}{\Delta} \sum_{k} p_k \Delta_k$$

Cont. …

where $k = Number\ of\ forward\ paths$

$p_k = path\ gain\ of\ k^{th}$ forward path,

$\Delta = determinant\ of\ the\ graph$ given by

$\Delta = 1 - (sum\ of\ individual\ loop\ gains) +$

$(sum\ of\ gain\ product\ of\ all\ combinations\ of\ two\ non\ touching\ loops) -$

$(sum\ of\ gain\ products\ of\ all\ combinations\ of\ three\ non\ touching\ loops) + \cdots$

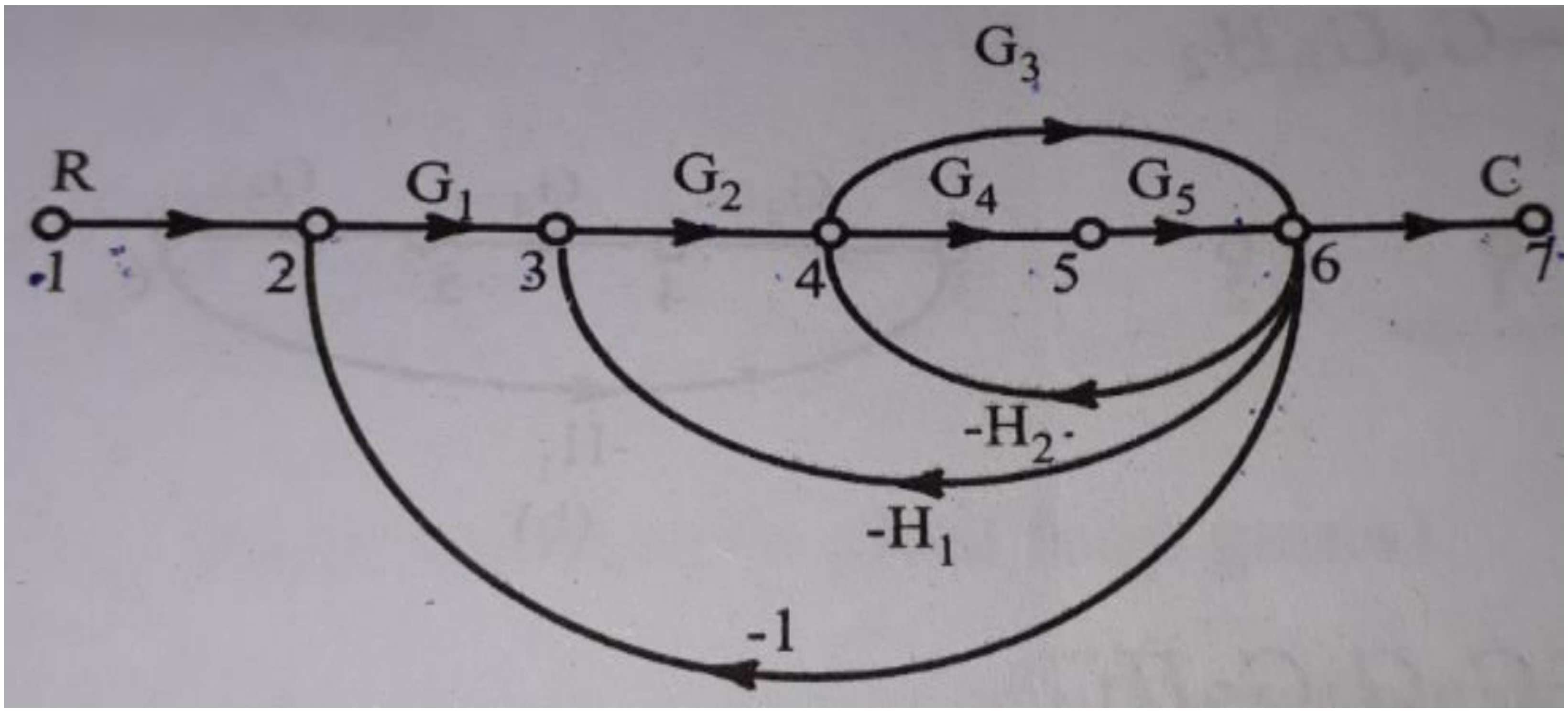
$\Delta_k = \Delta$ part of graph not touching the k^{th} forward path.

❖ **Non-touching loop:** it is a subgraph forming loop (closed path) but does not touch the forward path.

Cont. …

Example: Find transfer function ($\frac{C}{R}$) of the following signal flow graphs using Mason's gain formula.

a)

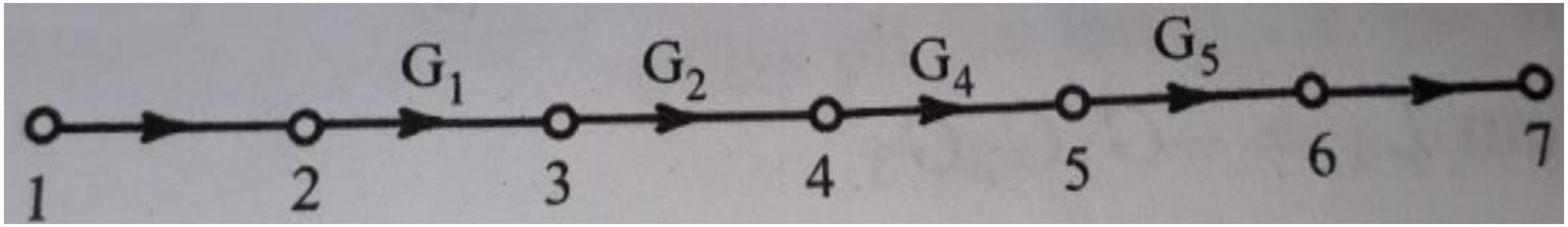


Cont. …

Solution:

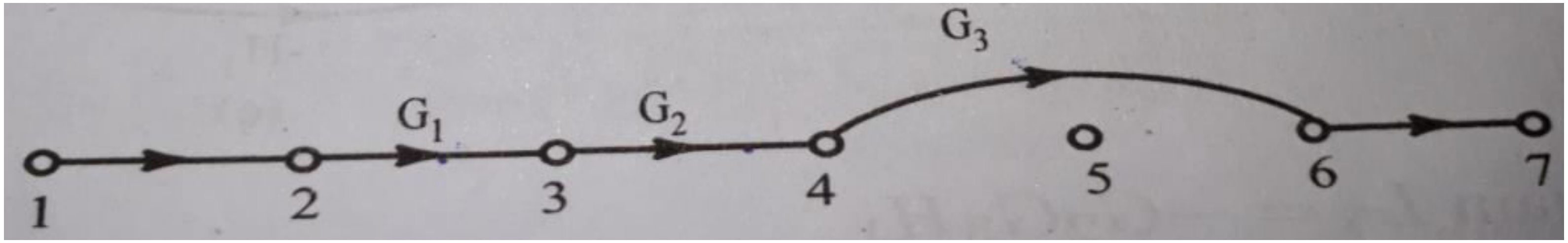
a) The number of forward paths are 2

Forward path 1:



$$\boldsymbol{pathgain\ p_1 = G_1G_2G_4G_5}$$

Forward path 2:

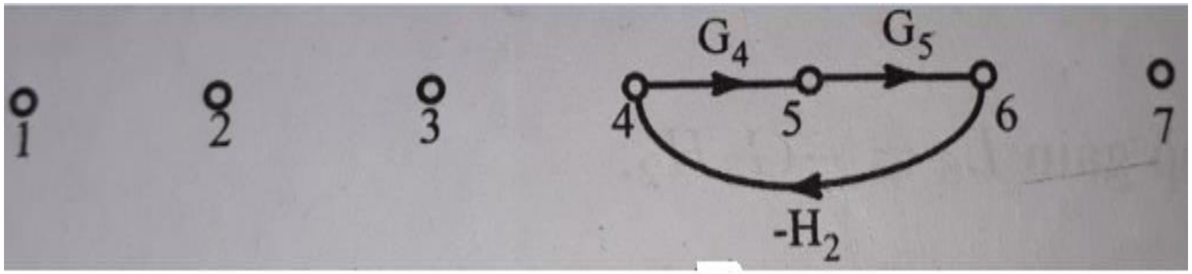


$$\boldsymbol{pathgain\ \ p_2 = G_1G_2G_3}$$

Cont. …

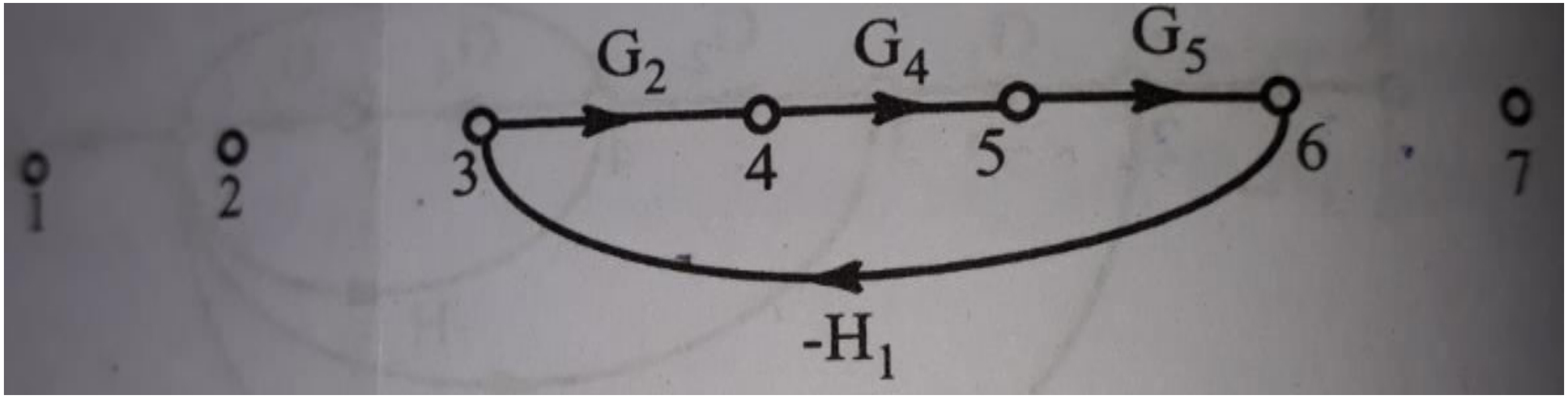
➢ The number of single loops are six.

Loop1:



$$Loop\ gain\ L_1 = -G_4G_5H_2$$

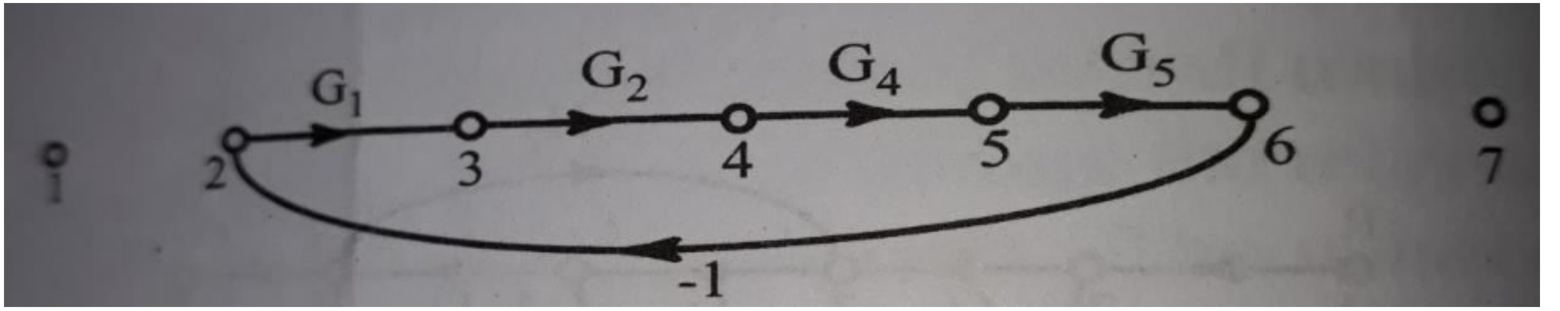
Loop2:



$$Loop\ gain\ L_2 = -G_2G_4G_5H_1$$

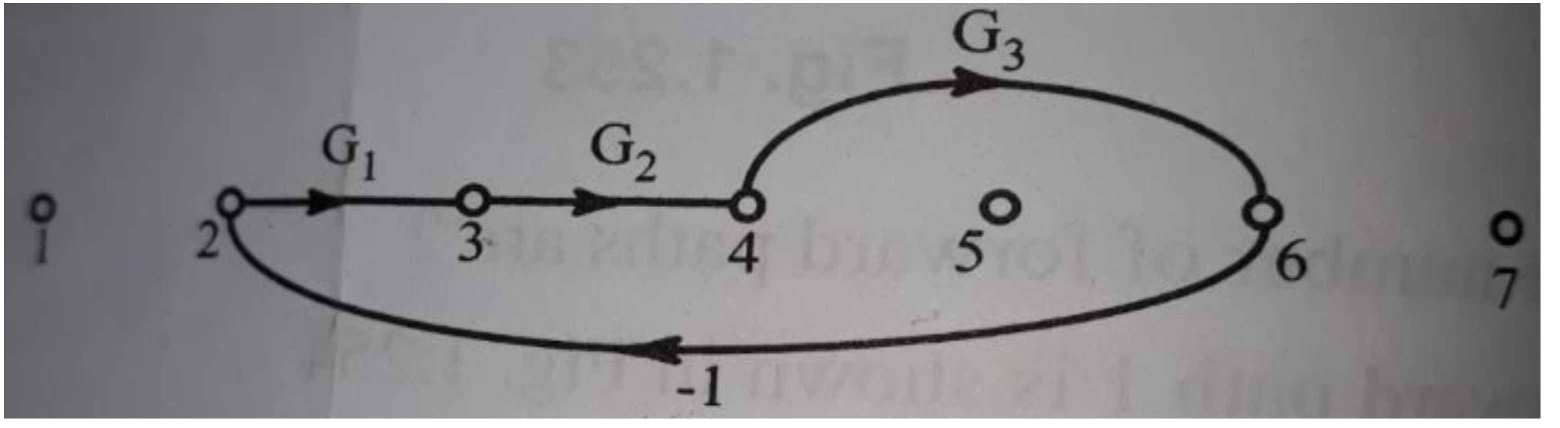
Cont. …

Loop3:



$Loop\ gain\ L_3 = -G_1G_2G_4G_5$

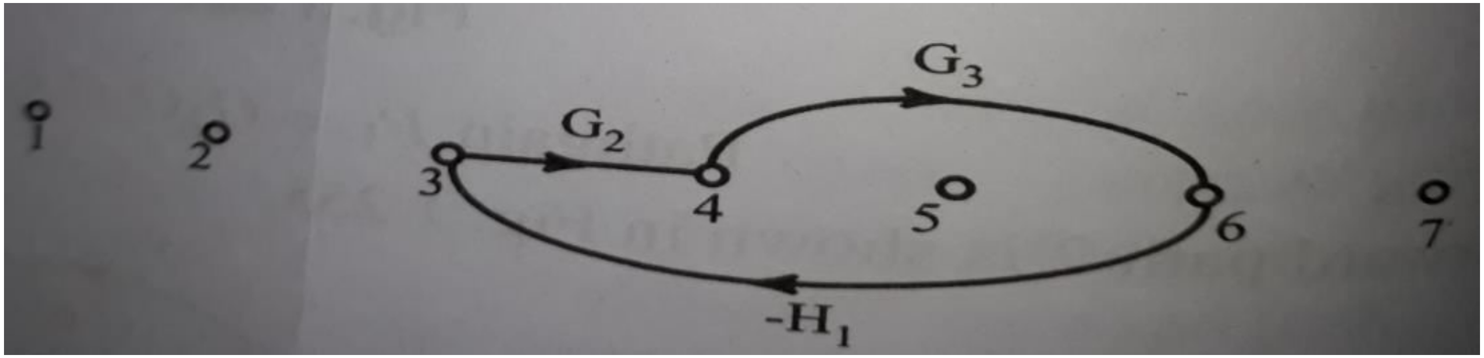
loop4:



$loop\ gain\ L_4 = -G_1G_2G_5$

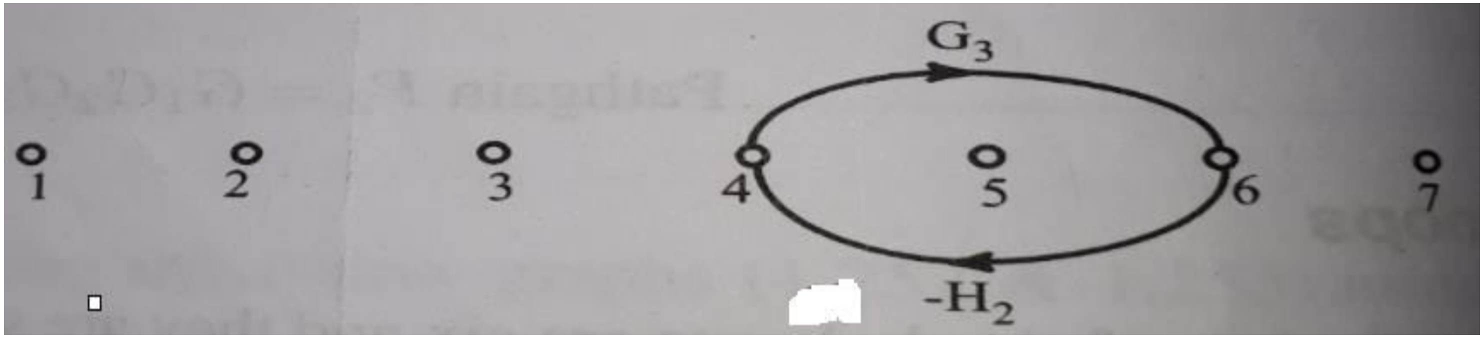
Cont. …

Loop5:



$$Loop\ gain\ L_5 = -G_2G_3H_1$$

Loop6:



$$Loop\ gain\ L_6 = -G_3H_2$$

- The number of two and more than two non-touching loops are zero.

Cont. …

Mason's gain formula

$$T = \frac{1}{\Delta}\sum_k p_k \Delta_k$$

$\Delta = 1 - (\textit{sum of individual loop gains}) +$

$(\textit{sum of gain product of all combinations of two non touching loops}) -$

$(\textit{sum of gain products of all combinations of three non touching loops}) + \cdots$

$$\Delta = 1 - (-H_2G_4G_5 - G_2G_4G_5H_1 - G_1G_2G_4G_5 - G_1G_2G_3 - G_2G_3H_1 - G_3H_2) + 0$$

$$= 1 + H_2G_4G_5 + G_2G_4G_5H_1 + G_1G_2G_4G_5 + G_1G_2G_3 + G_2G_3H_1 + G_3H_2$$

Cont. …

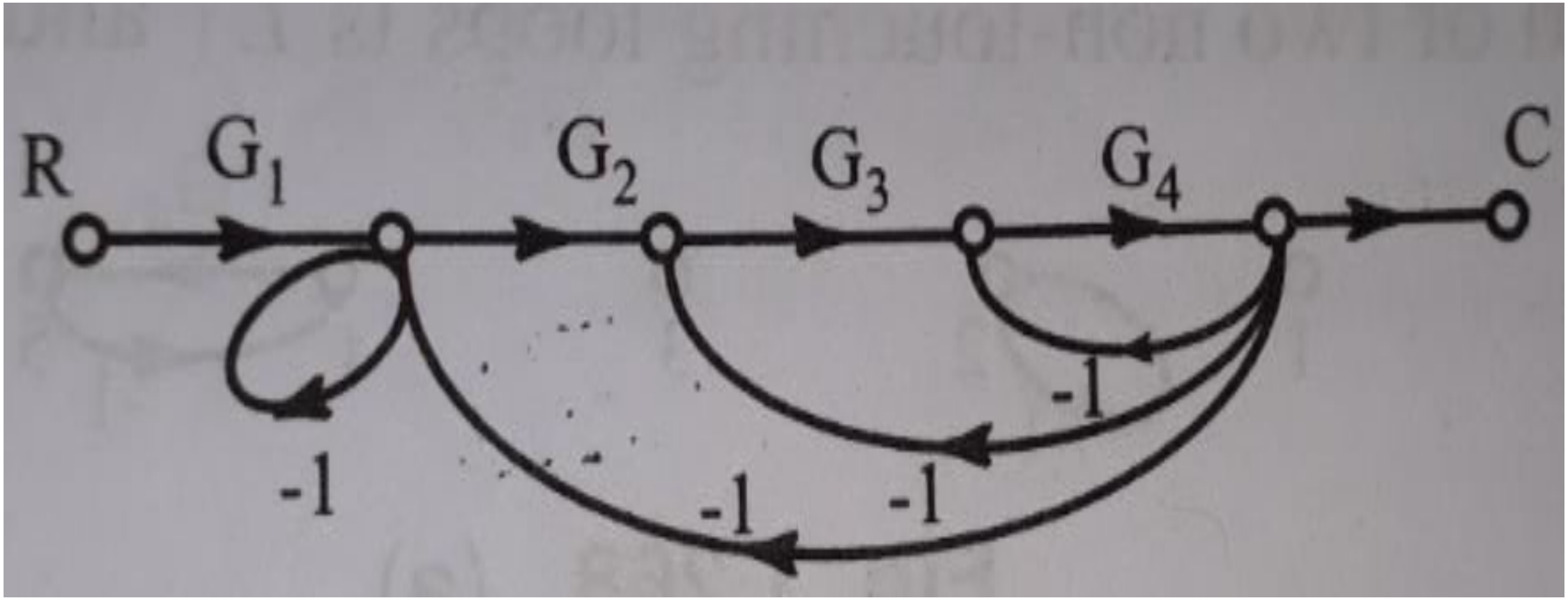
$\Delta_k = \Delta$ for that part of graph which is not touching k^{th} forward path.

$\Delta_1 = 1 - 0 = 1;\ \Delta_2 = 1 - 0 = 1,$ since all the loops are touching the two forward paths.

$$\mathrm{T(s)} = \frac{1}{\Delta}(\mathrm{p}_1\Delta_1 + \mathrm{p}_2\Delta_2)$$

$$T(s) = \frac{G_1 G_2 G_4 G_5 + G_1 G_2 G_3}{1 + H_2 G_4 G_5 + G_2 G_4 G_5 H_1 + G_1 G_2 G_4 G_5 + G_1 G_2 G_3 + G_2 G_3 H_1 + G_3 H_2}$$

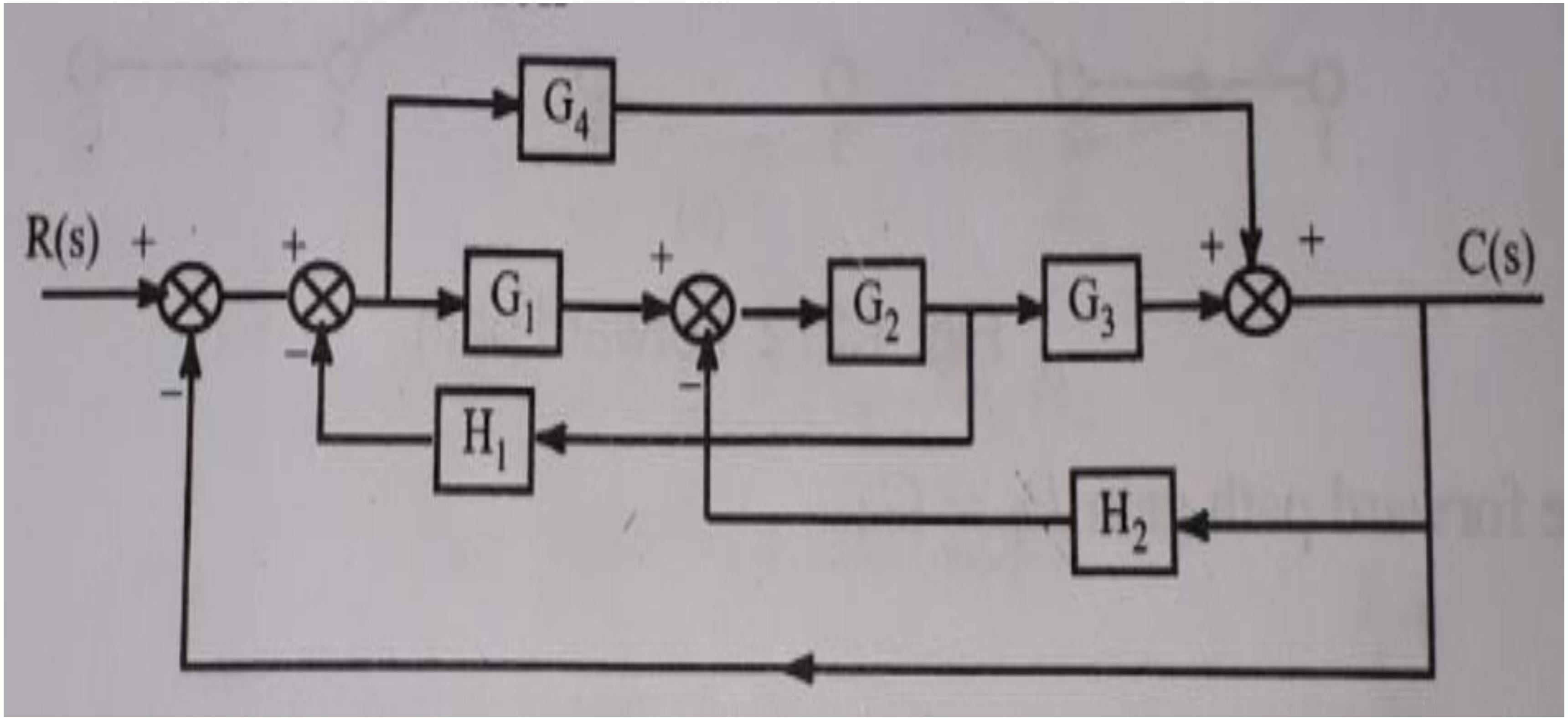
Cont. …

b) 

Answer: $T(s) = \frac{1}{\Delta} p_1 \Delta_1 = \frac{G_1 G_2 G_3 G_4}{2+2G_4+2G_3G_4+G_2G_3G_4}$

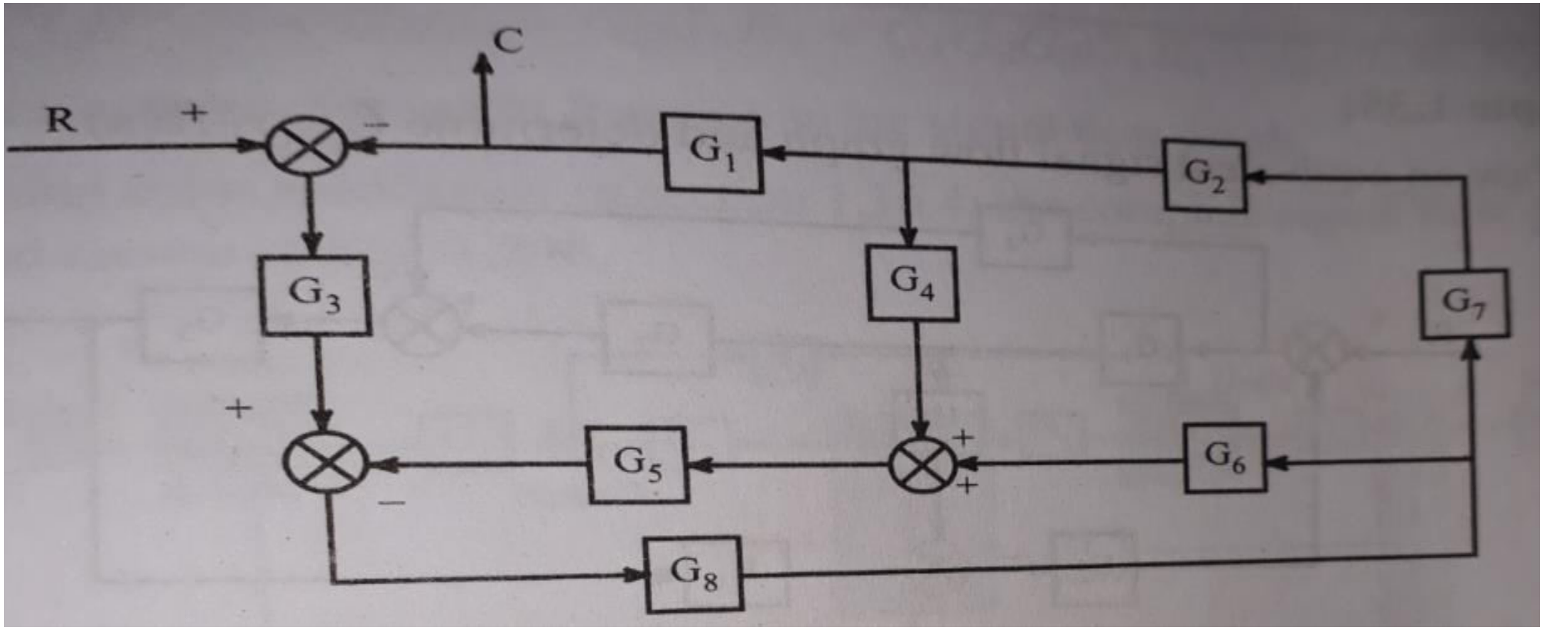
Cont. …

Exercise3.4: Construct an equivalent signal flow graph for the block diagram shown below and evaluate the transfer function $\frac{C(s)}{R(s)}$.



Cont. …

Exercise3.5: Construct an equivalent signal flow graph for the block diagram shown below and evaluate the transfer function $\frac{C(s)}{R(s)}$.



Answer: $T(s) = \frac{C(s)}{R(s)} = \frac{1+G_5G_6G_8+G_2G_4G_5G_7G_8}{1+G_5G_6G_8+G_2G_4G_5G_7G_8+G_1G_2G_3G_7G_8}$

3.7. Introduction about Automatic industrial controllers

❖ **Automatic controllers**

- Automatic controller compares the actual value of the plant output with the reference input (desired value), determines the deviation, and produces a control signal that will reduce the deviation to zero or to a small value.
- The manner in which the automatic controller produces the control signal is called the ***control action***.
- The controller detects the actuating error signal, which is usually at a very low power level, and amplifies it to a sufficiently high level and fed to an actuator such as electric motor or valve.

Cont. …

➢ The actuator is a power device that produces the input to the plant according to the control signal.

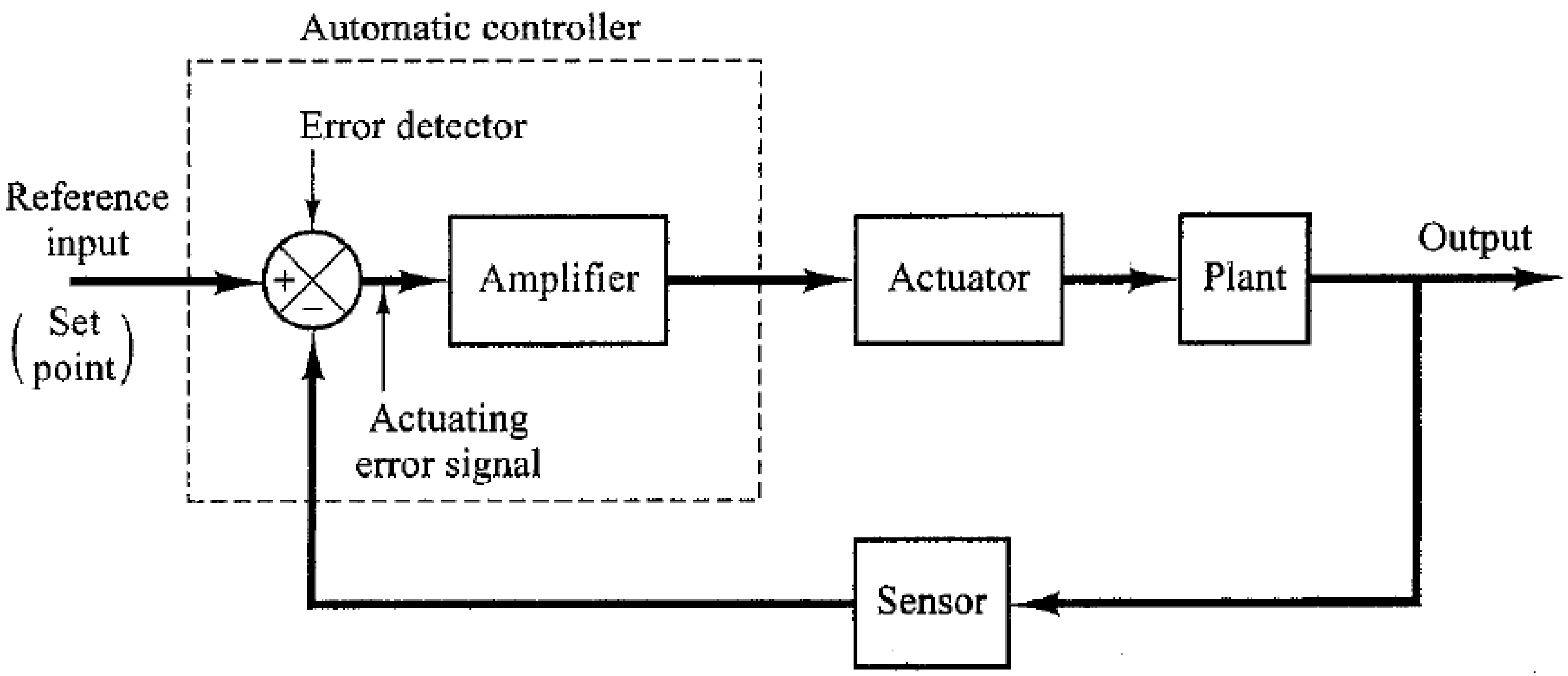


Fig.3.4: Block diagram of industrial control system consisting (automatic controller, actuator, plant and sensor).

Cont. …

- The sensor (measuring element) is a device that converts the output variable into another suitable variable, such as a displacement, pressure, or voltage, that can be used to compare the output to the reference input signal.
- The set point of the controller must be converted to the same units as the feedback signal from the sensor.
- Industrial controllers may be classified according to their control actions as:
  1) Two-position/on-off controllers
  2) Proportional controllers
  3) Integral controllers

Cont. …

4) Proportional-plus-integral controllers

5) Proportional-plus-derivative controllers

6) Proportional-plus-Integral-plus-derivative controllers

- *Controllers may also be classified according to the kind of power employed in the operation, such as pneumatic controllers, hydraulic controllers, or electronic controllers.*

- **Two-Position or On-Off Controller**

- In two-position control, the control signal *u(t)* remains at either a maximum or minimum value, depending on whether the actuating error signal is positive or negative, so that

$$u(t) = \begin{cases} u_1 \; for \; e(t) > 0 \\ u_2 \; for \; e(t) < 0 \end{cases}$$

Cont. …

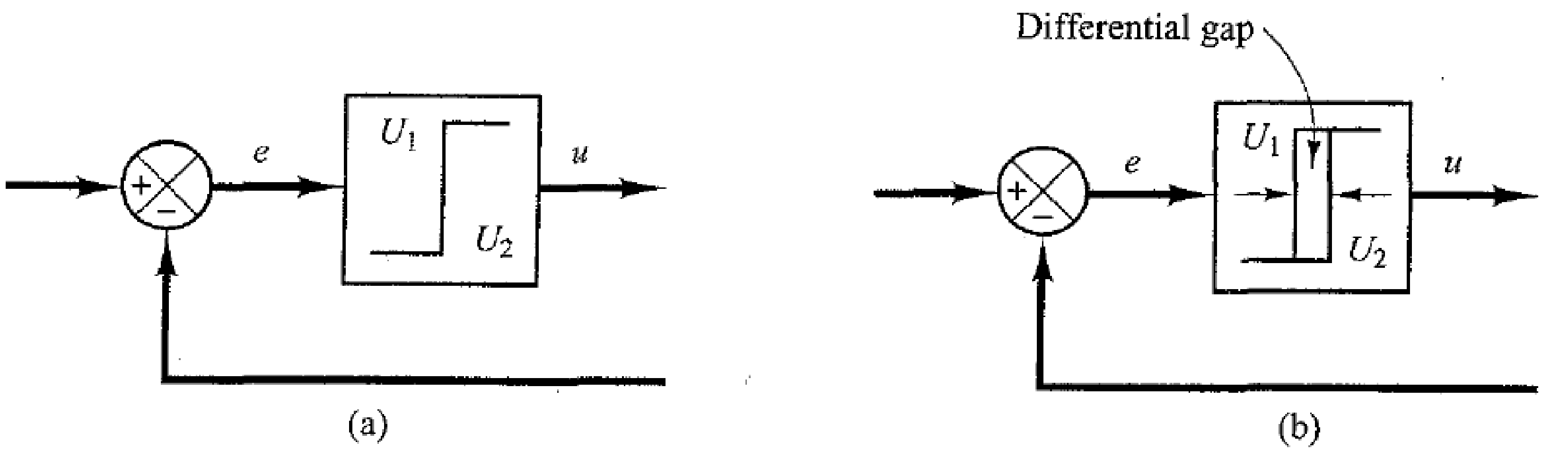


Fig.3.5: (a) Block diagram of an on-off controller; (b) block diagram of an on-off controller with differential gap.

- The range through which the actuating error signal must move before the switching occurs is called the differential gap.
- Such controllers are relatively simple and inexpensive and, for this reason, widely used in both industrial and domestic control systems.

Cont. …

- However, reducing output oscillation by decreasing the differential gap, increases the number of on-off switching per minute and reduces the useful life of the component.

Example:

- Liquid-level control system using electromagnetic valve to control the inflow rate.

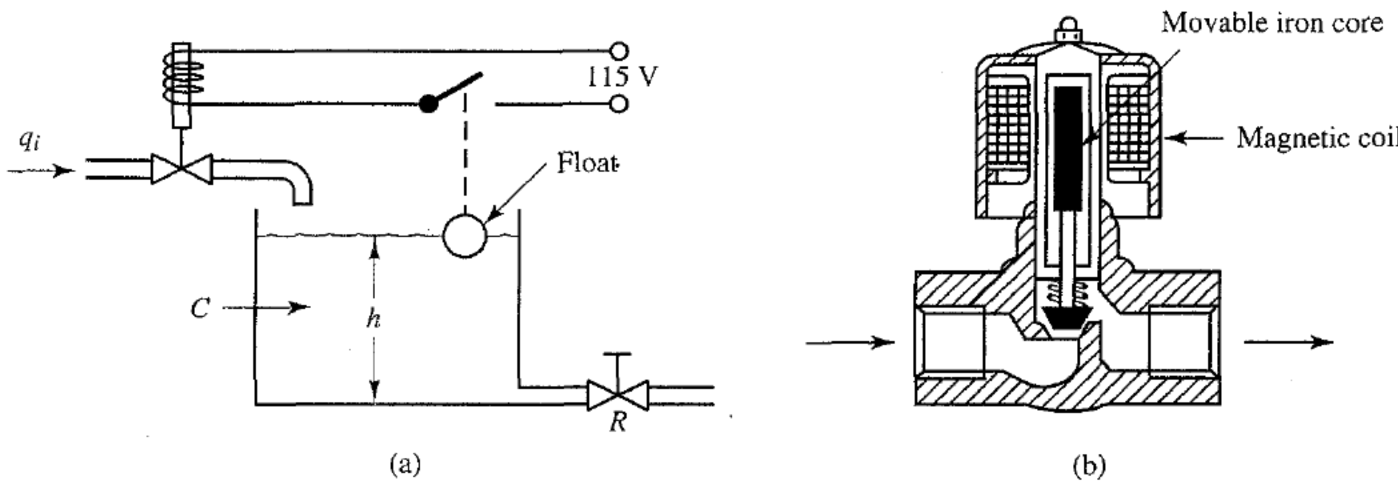


Fig.3.6: (a) Liquid-level control system (b) electromagnetic valve.

Cont. …

❖ Proportional controller (P-controller):

- In this controller, relationship between the output of the controller ***u(t)*** and the actuating error signal ***e(t)*** is

$$u(t) = k_p e(t)$$

- The transfer function of the proportional controller is

$$\frac{U(s)}{E(s)} = k_p$$

Where, k_p- is proportional gain.

- Proportional controller is essentially an amplifier with an adjustable gain.

Cont. …

❖ Integral Controller (I-Controller):

➢ In this controller, the value of the controller output **u(t)** is changed at a rate proportional to the actuating error signal **e(t).** That is,

$$\frac{du(t)}{dt} = k_i e(t) \quad \text{or} \quad u(t) = k_i \int_0^t e(t)dt$$

Where k_i is an adjustable constant.

➢ The transfer function of the integral controller is

$$\frac{U(s)}{E(s)} = \frac{k_i}{s}$$

Cont. …

❖ **Proportional-Plus-Integral Controller:**

➢ Its control action is defined by

$$u(t) = k_p e(t) + \frac{k_p}{T_i}\int_0^t e(t)dt$$

➢ The transfer function of the controller is

$$\frac{U(s)}{E(s)} = k_p(1 + \frac{1}{T_i s})$$

where, T_i is called the integral time.

Cont. …

❖ Proportional-Plus-Derivative Controller(PD controller):

- Its control action is defined by

$$u(t) = k_p e(t) + k_p T_d \frac{de(t)}{dt}$$

- The transfer function of the controller is

$$\frac{U(s)}{E(s)} = k_p(1 + T_d s)$$

where, T_d is called the derivative time.

Cont. …

❖ **Proportional-Plus-Integral-Plus-Derivative (PID) Controller:**

➢ It is the combination of proportional control action, integral control action, and derivative control action.

➢ This combined action has the advantages of each of the three individual control actions. The equation of a controller is given by

$$u(t) = k_p e(t) + \frac{k_p}{T_i}\int_0^t e(t)dt + k_p T_d \frac{de(t)}{dt}$$

➢ The transfer function of the controller is

$$\frac{U(s)}{E(s)} = k_p(1 + \frac{1}{T_i s} + T_d s)$$

Cont. …

Where, k_p- is the proportional gain, T_i- is the integral time, and T_D- is the derivative time.

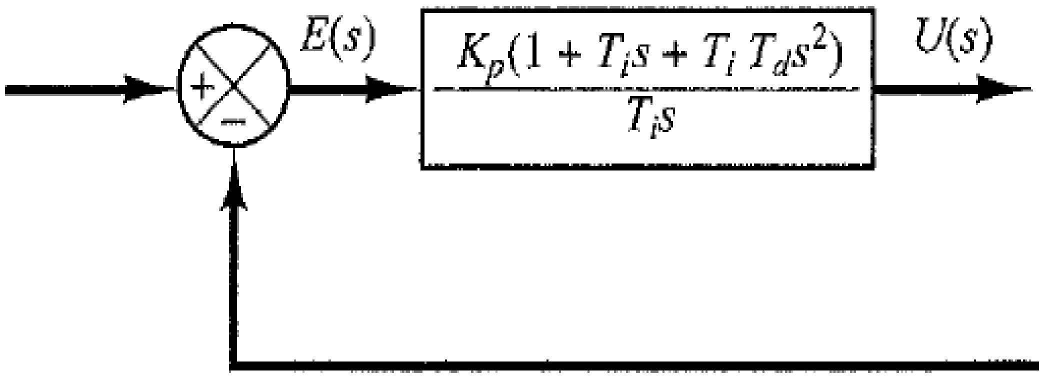


Fig.3.7: Block diagram of a proportional-plus-integral-plus-derivative controller.

3.8. Mathematical models of thermal and fluid systems (reading assignment)