Chapter4

Time Domain Analysis of Control Systems

4.1. Introduction

- The first step in analysing a control system was to derive a mathematical mode of the system. Once such a model is obtained, various methods are available for the analysis of system performance.
- Performance of a system is its response to an input signal. In designing a control system we have to choose a system with better performance, i.e., systems are to be compared by their response to particular test signal.

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- The commonly used test input signals are those of step functions, ramp functions, impulse functions, sinusoidal functions and the like.

❖ **Transient response**

- The transient response of a system is a particular part of the response of the system which tends to zero as time increases. It goes from initial state to the final state.

❖ **Steady state response**

- The steady state response of a system is a particular part of the response of the system which remains after the transient part has reached zero.

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- The total solution of a system y(t) is the sum of the transient response (y_{tr}) and the steady state response (y_{ss}).

$$y(t) = y_{tr}(t) + y_{ss}(t)$$

$$\lim_{t \to \infty} y_{tr}(t) = 0$$

$$\lim_{t \to \infty} y(t) = y_{ss}(t)$$

4.2. Time Domain Transient Response Specifications

- In many practical cases, the desired performance characteristics of control systems are specified in terms of time domain quantities. Systems with energy storage can not respond instantaneously and will exhibit responses whenever they are subjected to inputs or disturbances.

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- Frequently, the performance characteristics of a control system are specified in terms of the transient response of a unit step input since it is easy to generate and is sufficiently drastic.
- In specifying the transient response characteristics of a control system to a unit step input, It is common to specify the following.

i. **Delay time (t_d):** it is the time required for the response to reach half of its final value.

$$y(t_d) = 50\%.r(t) = {}^{1}/_{2}.1 = 0.5$$

i. **Peak time (t_p):** it is the time required for the response to reach the first peak of the shoot. $\frac{dy(t)}{dt}\Big|t = t_p = 0$

Cont. …

iii. **Rise time (t_r):** it is the time require for the response to rise from 10% to 90%, 5% to 95% or 0% to 100% of its final value. For under damped second order systems, the 0% to 100% rise time is normally used. For over damped systems, the 10% to 90% rise time is commonly used.

For 0 to 100% rising

$$y(t_r) = 100\%.r(t) = 1.1 = 1 \text{ since } r(t) = 1$$

iv. **Settling time (t_s):** this is the time required by the response y(t) to reach and remain within a certain range of its final value. This range is usually from 2 to 5% of amplitude of the final value.

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v. **Maximum overshoot (M_p):** it is the maximum peak value of the response curve measured from unity. It is the difference between the maximum value of the response and the final value (the steady state response). The amount of maximum par cent overshoot directly indicates the relative stability of the system.

$$M_p = \frac{y(t_p) - y(\infty)}{y(\infty)} * 100\%$$

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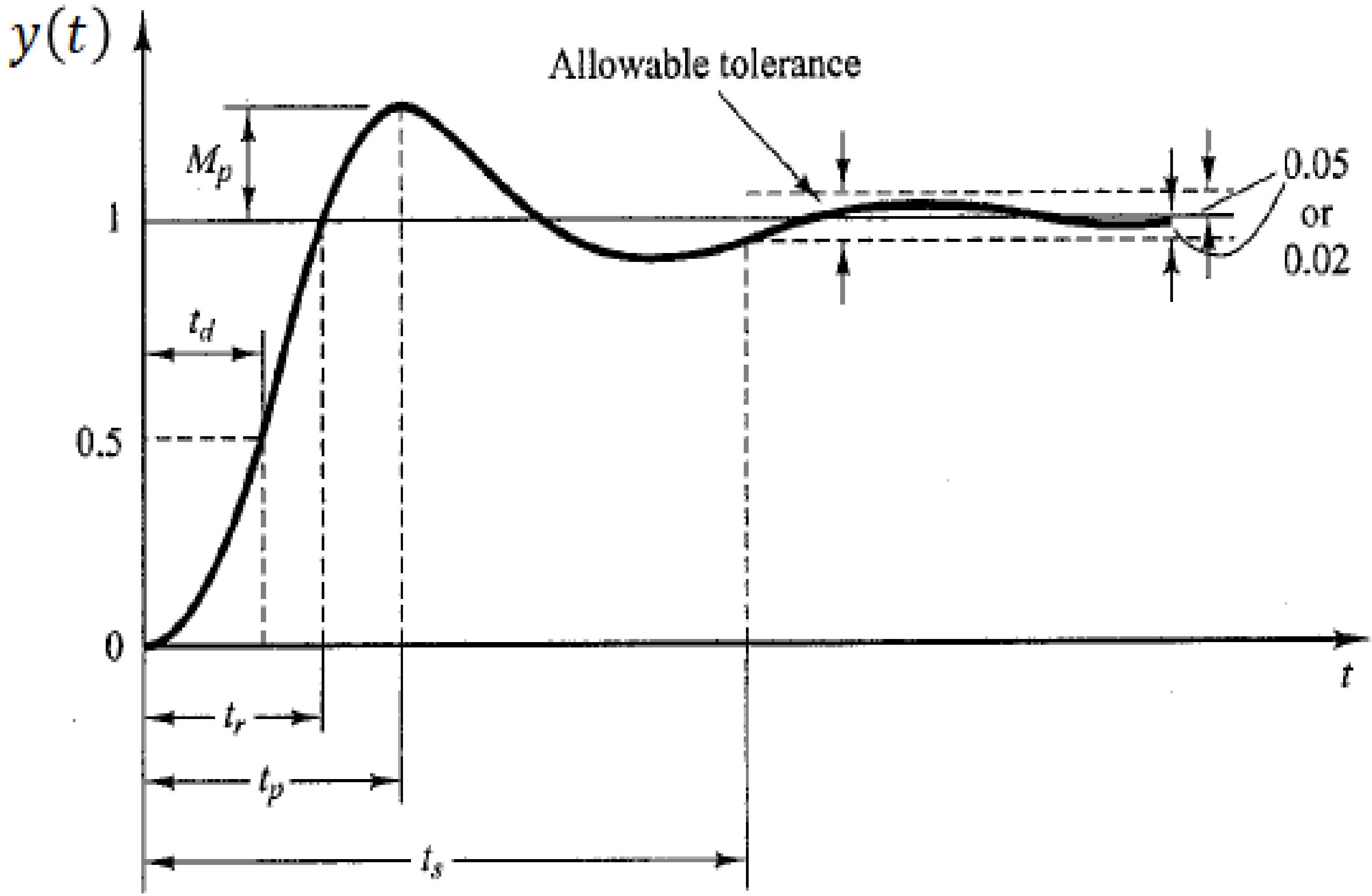

Fig.4.1: Unit-step response curve showing t_d, t_r, t_p, t_s, and M_p.

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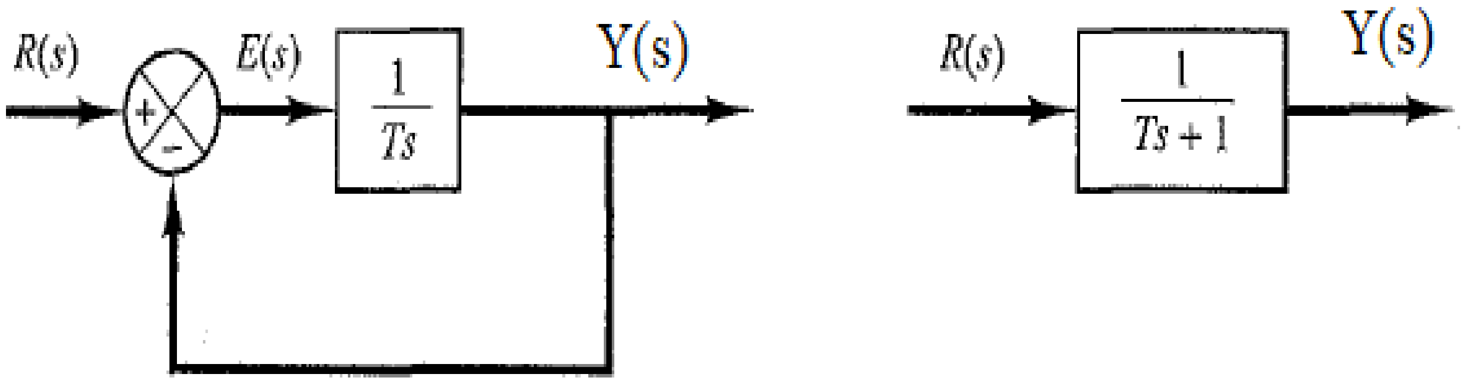
4.2.1. Analysis of first order systems

- Consider a first order and its equivalent circuit diagram as shown below



The input output relationship is given by

$$\frac{Y(s)}{R(s)} = \frac{1}{Ts + 1}$$

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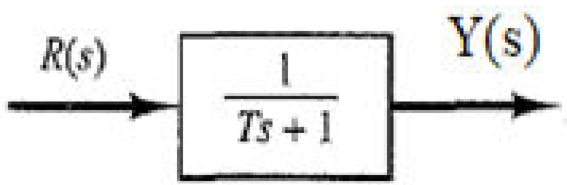
i. Unit-Step Response of First-Order Systems.



$$\frac{Y(s)}{R(s)} = \frac{1}{Ts+1} \rightarrow Y(s) = \frac{1}{Ts+1} R(s)$$

But $R(s) = \frac{1}{s}$ then $Y(s) = \frac{1}{(Ts+1)s}$

Expanding Y(s) into partial fractions gives

$$Y(s) = \frac{1}{s} - \frac{1}{Ts+1} = \frac{1}{s} - \frac{1}{s + {}^{1}/_{T}}$$

Taking the inverse Laplace transform

$$y(t) = 1 - e^{-\left({}^{1}/_{T}\right)t} \quad \text{for } t \geq 0$$

- From this equation, initially the output y(t) is zero and finally it becomes unity.

Cont. …

- One important characteristic of such an exponential response curve y(t) is that at t = T the value of y(t) is 0.632, or the response y(t) has reached 63.2% of its total change. This may be easily seen by substituting t = T in y(t). That is,

$$\text{At } t = T, \quad y(T) = 1 - e^{-(T/T)} = 1 - e^{-1} = 0.632$$

- For 2% of final value, the settling time can be derived as

$$y(t_s) = 0.98 * r(t) = 0.98 * 1 = 0.98$$

$$y(t_s) = 0.98 \quad \rightarrow 1 - e^{-(t_s/T)} = 0.98$$

$$\rightarrow t_s \approx 4T$$

- For $t \geq 4T$, the response reaches to its steady state value.

Cont. …

Steady state response: $y_{ss}(t) = \lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(1 - e^{-\left(t/T\right)}\right) = 1$

Transient response: $y_{tr}(t) = y(t) - y_{ss}(t) = \left(1 - e^{-t/T}\right) - 1 = e^{-t/T}$

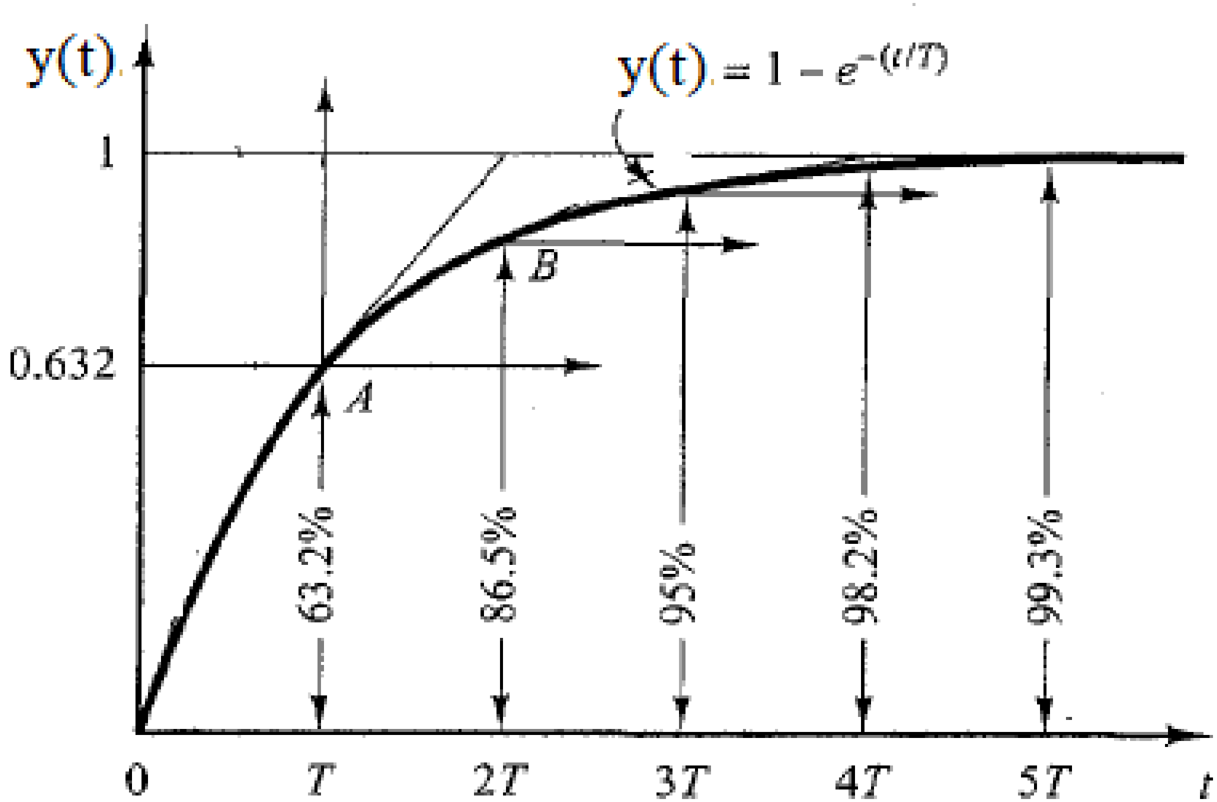


Fig.4.2: unit step response of first order system

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ii. Unit-Ramp Response of First-Order Systems.

$$Y(s) = \frac{1}{TS+1}R(s) = \frac{1}{s^2(Ts+1)}$$

Since the Laplace transform of the unit-ramp function is $R(s) = \frac{1}{s^2}$

Expanding Y(s) into partial fraction gives

$$Y(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

Taking the inverse Laplace transform gives

$$y(t) = t - T + Te^{-t/T}, \qquad \text{for } t \geq 0$$

The transient response of y(t) can be obtained as

$$y_{tr}(t) = \lim_{t\to\infty} y(t) = \lim_{t\to\infty}(t - T + Te^{-t/T}) = 0$$

Cont. …

- From this limit only $Te^{-t/T}$ will converges to zero as time tends to infinity. $\rightarrow y_{tr}(t) = Te^{-t/T}$
- Steady state response:

$$y_{ss}(t) = y(t) - y_{tr}(t) = t - T + Te^{-t/T} - Te^{-t/T} = t - T$$

- The error signal e(t) is then

$$e(t) = r(t) - y(t) = t - \left(t - T + Te^{-t/T}\right) = T(1 - e^{-t/T})$$

- At steady state the error signal will converges to T

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} T(1 - e^{-t/T}) = T$$

- The smaller the time constant T, the smaller the steady state error in ramp input.

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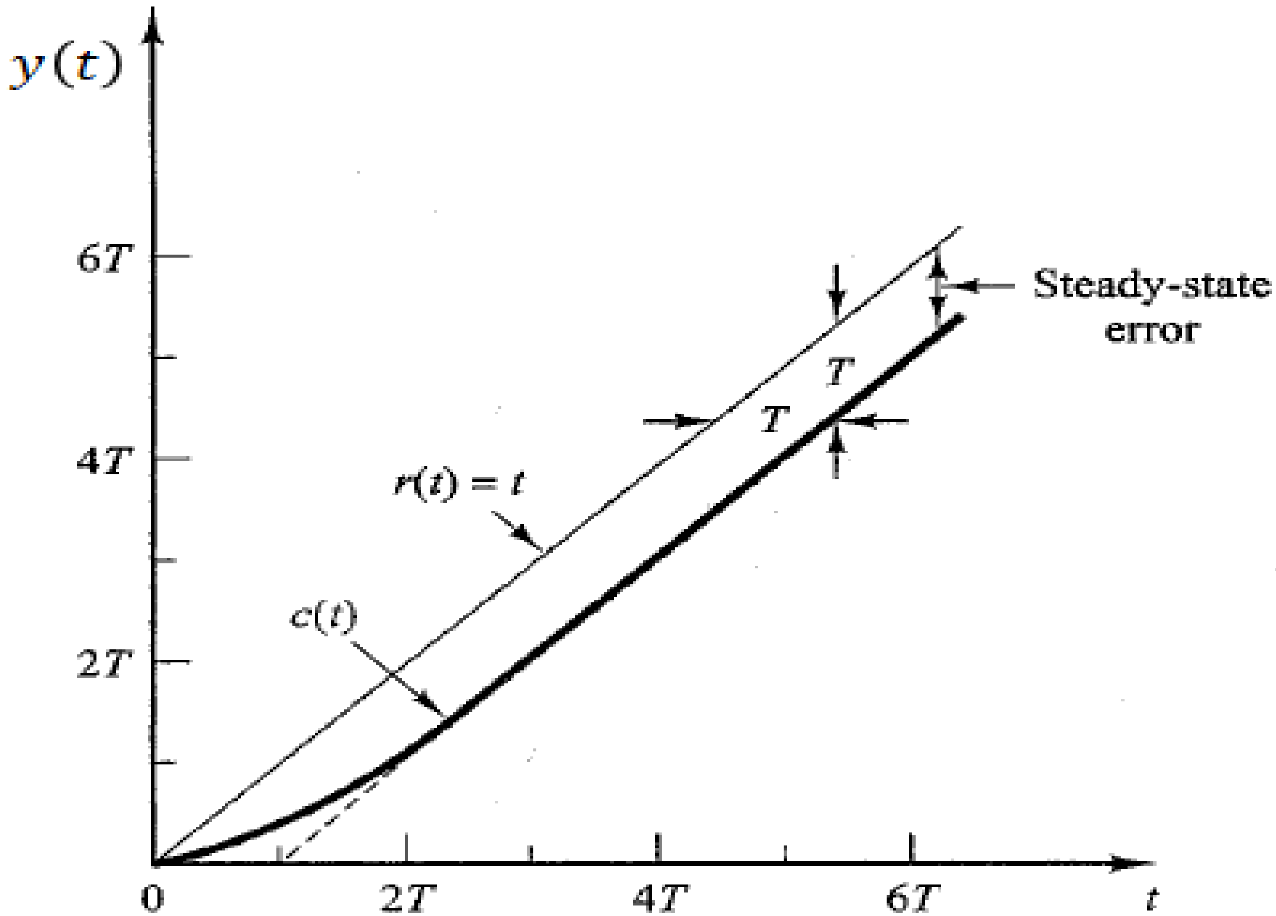

Fig.4.3: Ramp response of first order system.

Cont. …

iii. Unit-Impulse Response of First-Order Systems.

- For the unit-impulse input, R(s) = 1 and the output of first order system can be obtained as

$$Y(s) = \frac{1}{TS+1}R(s) = \frac{1}{(Ts+1)} = \frac{1/T}{s+1/T}$$

$$\rightarrow y(t) = \frac{1}{T}e^{-t/T} \qquad \text{for } t \geq 0$$

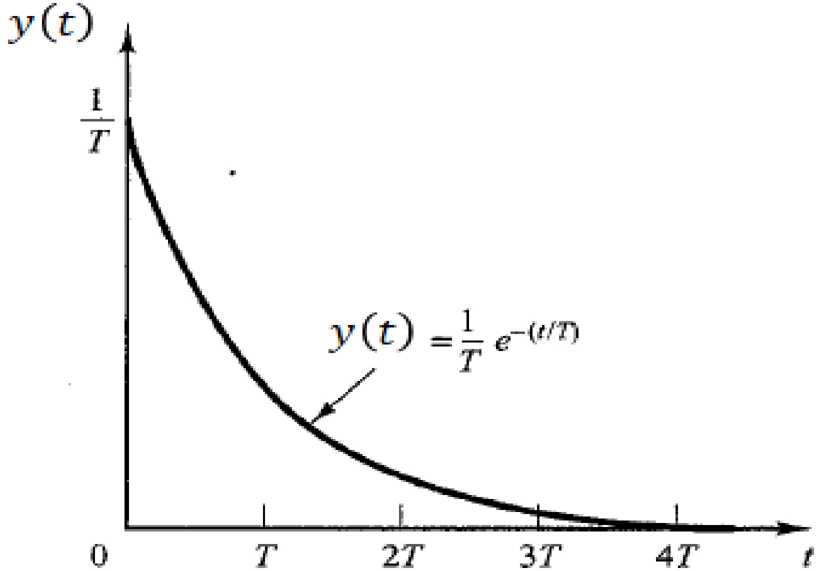

Fig.4.4:impulse response of first order system

Exercise 4.1: find steady state and transient response.

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❖ An Important Property of LTI Systems.

- The response to the derivative or integral of the input signal can be obtained by differentiating or integrating the response of the signal respectively.
- For unit ramp response of first order systems

$$\text{If} \quad u(t) \xrightarrow{\frac{1}{Ts+1}} 1 - e^{-t/T}$$

$$\int_0^t u(t)dt = r(t) \xrightarrow{\frac{1}{Ts+1}} \int_0^t \left(1 - e^{-\tau/T}\right)d\tau = t - T + Te^{-t/T}$$

- For unit impulse response of first order systems

$$\text{If} \;\; u(t) \xrightarrow{\frac{1}{Ts+1}} 1 - e^{-t/T}$$

$$\frac{du(t)}{dt} = \delta(t) \xrightarrow{\frac{1}{Ts+1}} \frac{d}{dt}\left(1 - e^{-t/T}\right) = \frac{1}{T}e^{-t/T}$$

Cont. …

4.2.2. Analysis of second order systems

- Consider a second order system with transfer function

$$G(s) = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

Where the constants ω_n and ζ are called the natural undamped frequency and the damping ratio of the system respectively

- G(s) can be also written in the form of the following

$$G(s) = \frac{{\omega_n}^2}{(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})}$$

Cont. …

- The poles of G(s) are

$$s_{1,\,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)}$$

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{-1}\sqrt{\left(1-\zeta^2\right)} = -\zeta\omega_n \pm j\omega_n\sqrt{\left(1-\zeta^2\right)}$$

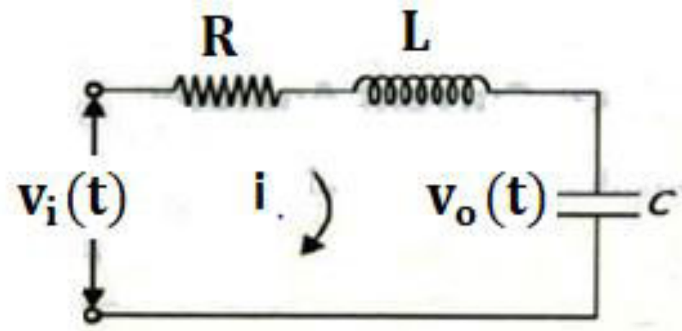
$$s_{1,2} = -\sigma \pm j\omega_d$$

where $\sigma = \zeta\omega_n$ is damping constant or attenuation

$\omega_d = \omega_n\sqrt{\left(1-\zeta^2\right)}$ is damped natural frequency of the system

Cont. …

Example4.1: RLC circuit



$$v_i(t) = Ri(t) + L\frac{di(t)}{dt} + v_o(t) \qquad (4.1)$$

$$v_o(t) = \frac{1}{C}\int i(t)dt$$

$$i(t) = C\frac{dv_o(t)}{dt} \qquad (4.2)$$

Taking Laplace transform gives

$$v_i(s) = RI(s) + LsI(s) + v_o(s) \qquad (4.3)$$

$$I(s) = CSv_o(s) \qquad (4.4)$$

Cont. …

➢ Substituting eqn.(4.4) into eqn.(4.3) and rearranging will result the following transfer function.

$$\frac{\mathbf{v_o(s)}}{\mathbf{v_i(s)}} = \frac{\mathbf{1/_{LC}}}{\mathbf{s^2+\frac{R}{L}s+1/_{LC}}}$$

➢ When we comparing this result to the general second order system, we will got the following results.

$$\omega_n{}^2 = \frac{1}{LC} \ \rightarrow \omega_n = \sqrt{\frac{1}{LC}} = \frac{1}{\sqrt{LC}}$$

$$2\zeta\omega_n = \frac{R}{L} \ \rightarrow \zeta = \frac{R}{2L\omega_n} = \frac{R}{2L} * \sqrt{LC} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

Cont. …

- The dynamic behaviour of the second order systems can then be described in terms of two parameters ζ and ω_n.

- **case1:** If $0 < \zeta < 1$, the closed loop poles are complex conjugates and lie in the left half s-plane. The system is called under damped and the transient response is oscillatory.
- **Case2:** If $\zeta = 0$, the transient response does not die out.
- **Case3:** If $\zeta = 1$, the two poles are equal and the system is called critically damped.
- Case4: If $\zeta > 1$, the closed loop poles are negative real and unequal and the system is called over damped.

4.2.2.1. Unit step response of second order systems

$$C(s) = \frac{\omega_n{}^2}{s(s^2+2\zeta\omega_n s+\omega_n{}^2)} \quad \text{where} \quad R(s) = \frac{1}{s}$$

$$C(s) = \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n{}^2}$$

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n{}^2} - \frac{\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n{}^2}$$

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+(\omega_d)^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+(\omega_d)^2}$$

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n{}^2(1-\zeta^2)} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n{}^2(1-\zeta^2)}$$

Cont. …

❖ Case1: undamped system ($\zeta = 0$)

$$C(s) = \frac{1}{s} - \frac{s}{s^2+\omega_n{}^2}$$

$$C(s) = \frac{s^2+\omega_n{}^2-s^2}{s(s^2+\omega_n{}^2)} = \frac{\omega_n{}^2}{s(s^2+\omega_n{}^2)}$$

➢ In this case the poles of C(s) are 0 and imaginary $s = \pm j\omega_n$. If we expand C(s) in partial fractions, we have

$$C(s) = \frac{1}{s} - \frac{s}{s^2+\omega_n{}^2}$$

$$\therefore \quad c(t) = L^{-1}\{C(s)\} = L^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+\omega_n{}^2}\right\} = 1 - \cos\omega_n t$$

Cont. …

- As we can observe from the above equation, the response c(t) is a sustained oscillation with constant frequency ω_n and constant amplitude equal to 1. in this case the system is called undamped.

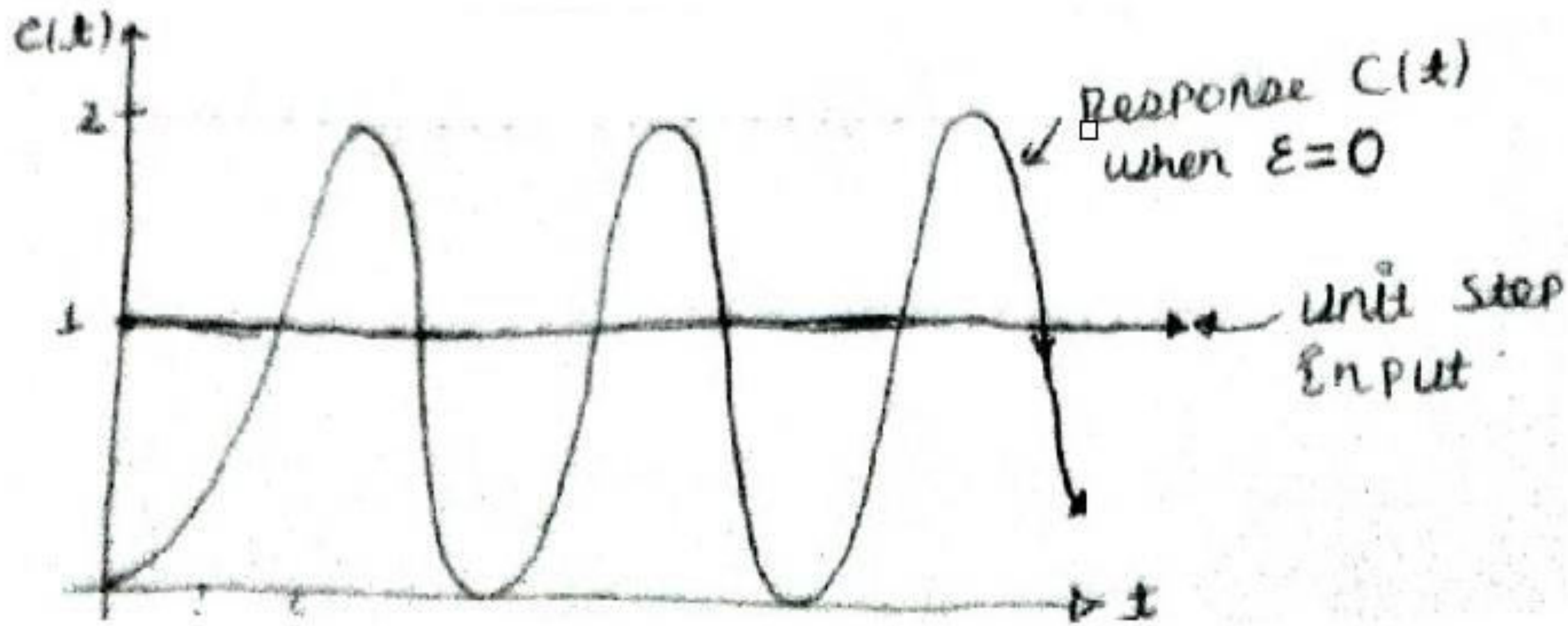


Fig.4.5: unit step response of undamped second order system.

Cont. …

❖ **Case2: Under damped system ($0 < \zeta < 1$):**

- In this case the poles of G(s)=C(s)/R(s) are complex conjugate pair since $s_{1,2} = -\sigma \pm j\omega_d$.
- For a unit step input, the response C(s) can be written as

$$C(s) = \frac{{\omega_n}^2}{s(s^2+2\zeta\omega_n s+{\omega_n}^2)}$$

$$C(s) = \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s^2+2\zeta\omega_n s+{\omega_n}^2)}$$

$$C(s) = \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s^2+2\zeta\omega_n s+\zeta^2{\omega_n}^2-\zeta^2{\omega_n}^2+{\omega_n}^2)}$$

$$C(s) = \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s^2+2\zeta\omega_n s+\zeta^2{\omega_n}^2)+{\omega_n}^2(1-\zeta^2)}$$

Cont. …

But $s^2 + 2\zeta\omega_n s + \zeta^2{\omega_n}^2 = (s + \zeta\omega_n)^2$ and ${\omega_n}^2(1 - \zeta^2) = {\omega_d}^2$

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n+\zeta\omega_n}{(s+\zeta\omega_n)^2+{\omega_d}^2}$$

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+{\omega_d}^2} - \left\{\frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+{\omega_d}^2}\right\} * \frac{\omega_d}{\omega_d}$$

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+{\omega_d}^2} - \frac{\zeta\omega_n}{\omega_d}\left\{\frac{\omega_d}{(s+\zeta\omega_n)^2+{\omega_d}^2}\right\} * \frac{\omega_d}{\omega_d}$$

But $L^{-1}\left\{\frac{s}{s^2+{\omega_d}^2}\right\} = cos\omega_d t$ $\quad L^{-1}\left\{\frac{\omega_d}{s^2+{\omega_d}^2}\right\} = sin\omega_d t$

$L^{-1}\left\{\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+{\omega_d}^2}\right\} = e^{-\zeta\omega_n t}cos\omega_d t$ $\quad L^{-1}\left\{\frac{\omega_d}{(s+\zeta\omega_n)^2+{\omega_d}^2}\right\} = e^{-\zeta\omega_n t}sin\omega_d t$

Then $c(t) = L^{-1}\{C(s)\}$

Cont. …

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t$$

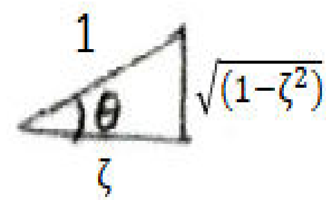
$$c(t) = 1 - e^{-\zeta\omega_n t}\left\{\cos\omega_d t + \frac{\zeta\omega_n}{\omega_n\sqrt{(1-\zeta^2)}}\sin\omega_d t\right\}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left\{\cos\omega_d t + \frac{\zeta}{\sqrt{(1-\zeta^2)}}\sin\omega_d t\right\}$$

$$c(t) = 1 - \frac{1}{\sqrt{(1-\zeta^2)}}\left\{\zeta sin\omega_d t + \sqrt{(1-\zeta^2)}\cos\omega_d t\right\}e^{-\zeta\omega_n t}$$

$$Let\ cos\theta = \zeta,\ \ sin\theta = \sqrt{(1-\zeta^2)} \quad \rightarrow tan\theta = \frac{\sqrt{(1-\zeta^2)}}{\zeta}$$

$$Then\ \theta = tan^{-1}\left\{\frac{\sqrt{(1-\zeta^2)}}{\zeta}\right\}$$



$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\{\cos\theta\sin\omega_d t + \sin\theta\cos\omega_d t\}$$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t + \theta)$$

Cont. …

$$\therefore \mathbf{c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}} \sin(\omega_d t + \tan^{-1}\frac{\sqrt{(1-\zeta^2)}}{\zeta})} \qquad \text{for } t \geq 0.$$

- Thus , when $0 < \zeta < 1$ we observe that the response c(t) is damped oscillation which tends to 1 as $t \rightarrow \infty$. In this case, we say that the system is under damped.

$$\lim_{t\to\infty} c(t) = 1 = C_{ss}(t)$$

The error signal for this system is

$$e(t) = r(t) - c(t) = 1 - (1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}} \sin(\omega_d t + \theta)$$

$$e(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}} \sin(\omega_d t + \theta)$$

Cont. …

- This error signal exhibits a damped sinusoidal oscillation. At steady state ($t \to \theta$), no error exists between the input and output.

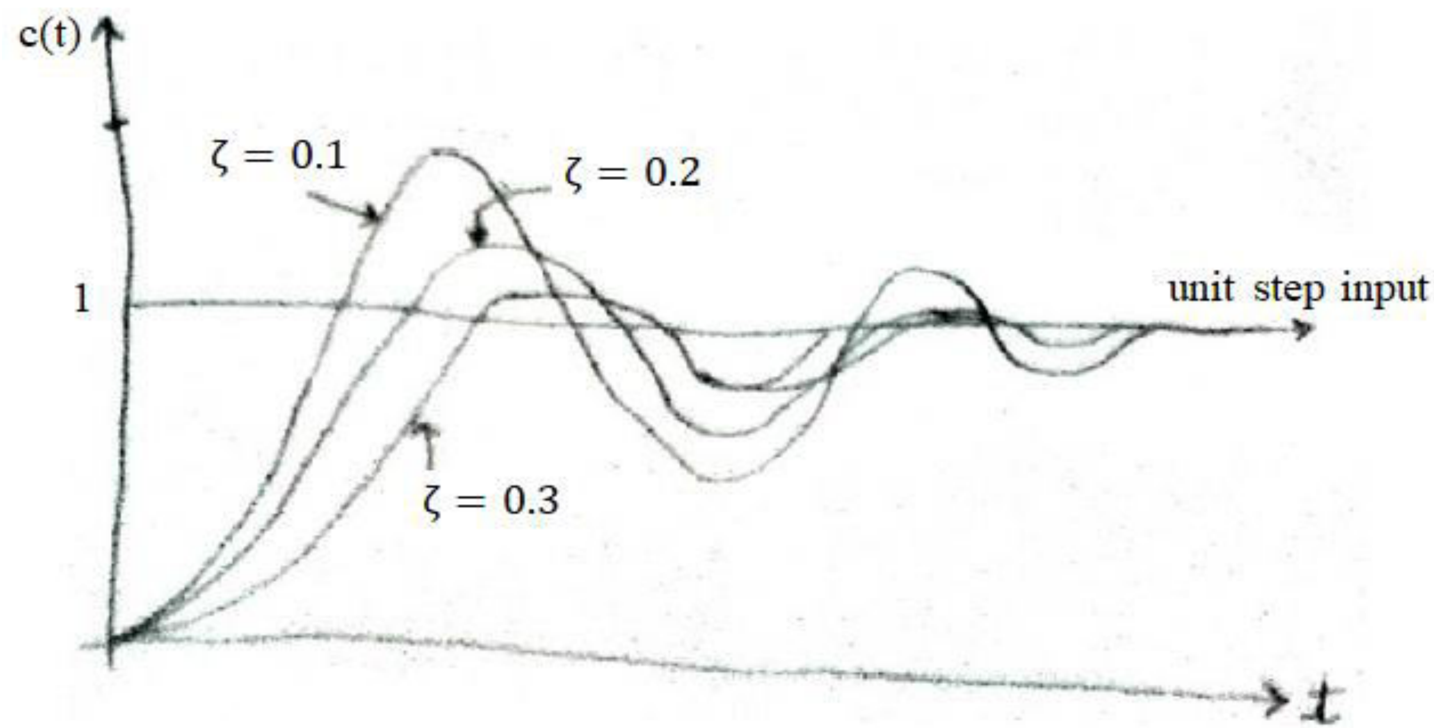


Fig.4.6: Unit step input response of under damped system for different damping ratio.

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❖ **Case3: Critically damped system ($\zeta = 1$)**

➢ In this case the poles of G(s)=C(s)/R(s) are real double pole $-\omega_n$. For unit step input $R(s) = \frac{1}{s}$, the response C(s) of critically damped case becomes

$$C(s) = \frac{{\omega_n}^2}{s(s^2+2\omega_n s+{\omega_n}^2)} = \frac{{\omega_n}^2}{s(s+\omega_n)^2}$$

If we expand C(s) in partial fractions we have

$$C(s) = \frac{1}{s} - \frac{1}{(s+\omega_n)} - \frac{\omega_n}{s(s+\omega_n)^2}$$

$$c(t) = L^{-1}\{C(s)\} = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t), \quad \text{for} \quad t \geq 0.$$

Cont. …

- Thus, when $\zeta = 1$ we observe that the wave form of the response c(t) involves no oscillations, asymptotically tends to 1 as $t \rightarrow \infty$.

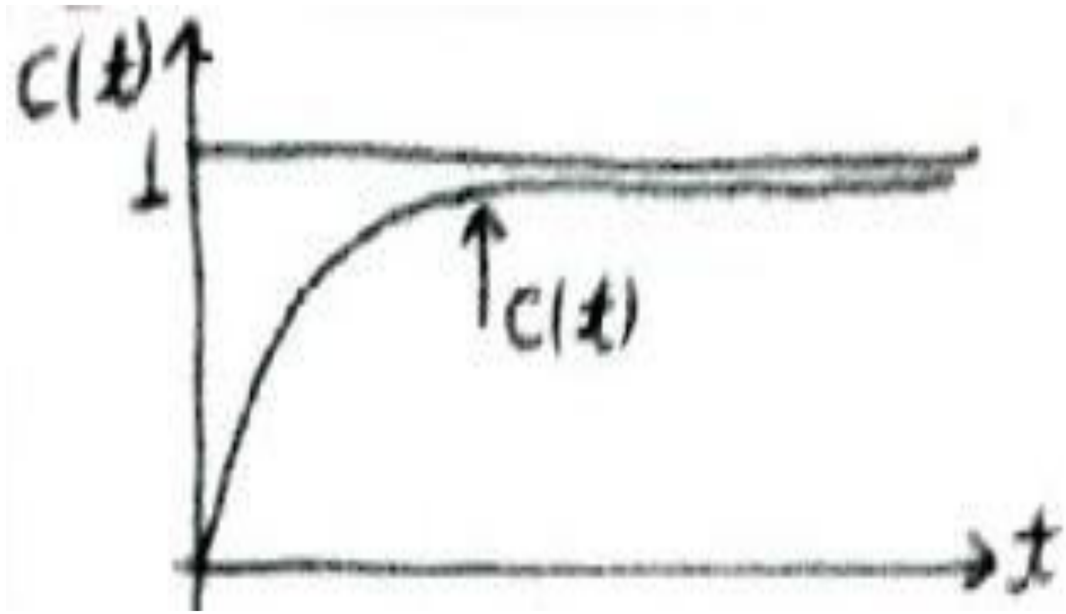

Fig.4.7: unit step response of second order critically damped system.

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Case4: overdamped system ($\zeta > 1$)

- In this case, the two poles of G(s)=C(s)/R(s) are negative real and unequal $s_{1,2} = -\sigma \pm \omega_n\sqrt{\zeta^2 - 1}$. For a unit-step input, $R(s) = \frac{1}{s}$ and C(s) can be written

$$C(s) = \frac{{\omega_n}^2}{s(s^2+2\zeta\omega_n s+{\omega_n}^2)}$$

proof after taking inverse Laplace transform it gives,

$$\mathbf{c(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right)}$$

Where $s_1 = \omega_n\left(\zeta + \sqrt{\zeta^2 - 1}\right)$ and $s_2 = \omega_n(\zeta - \sqrt{\zeta^2 - 1})$

Cont. …

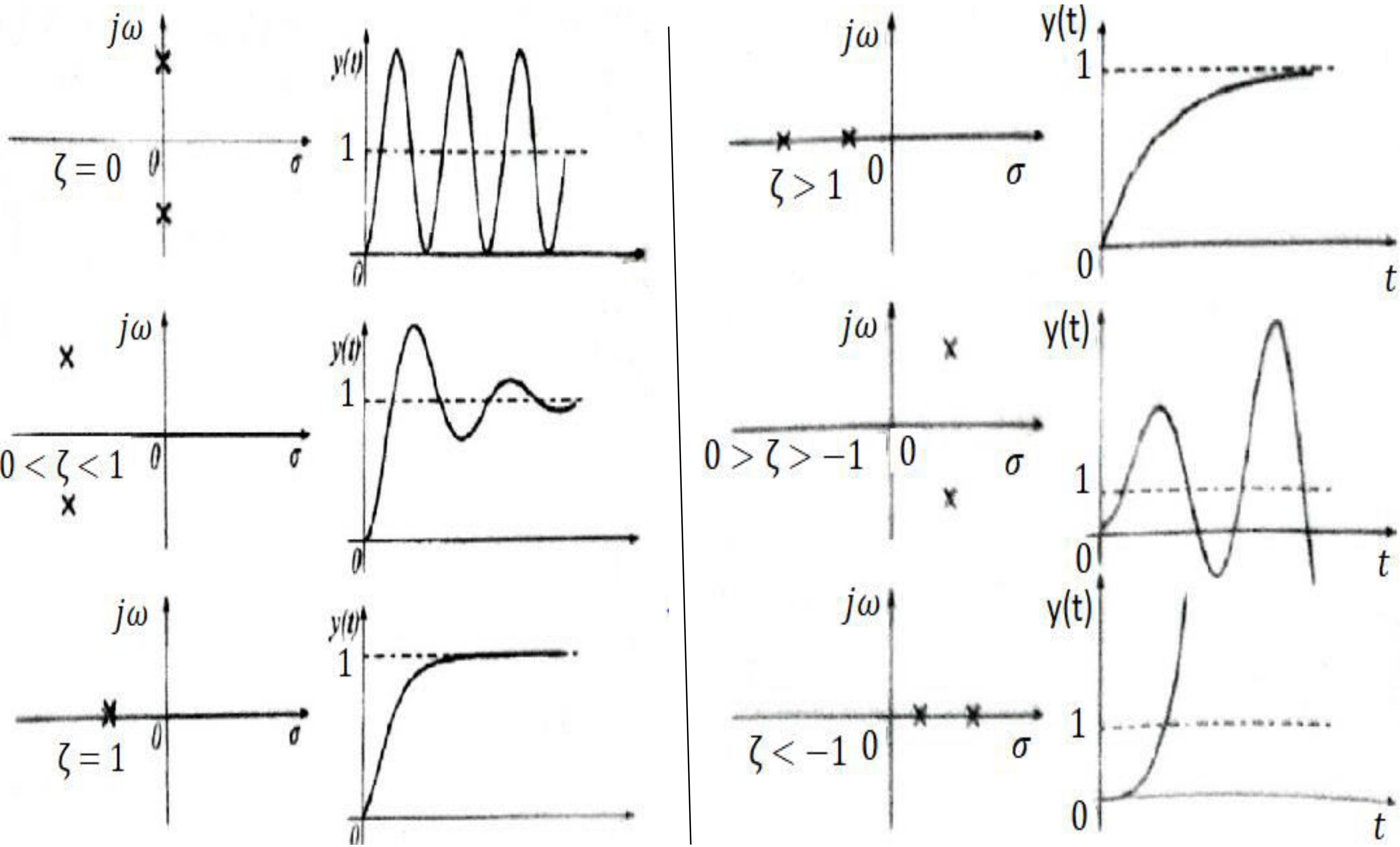


Fig.4.8: unit step response comparison for various pole locations in the s-plane of second order system.

Cont. …

- For a desirable transient response of a second order system, the damping ratio must be between 0.4 and 0.8. The Small value of ζ ($\zeta < 0.4$) yield excessive overshoot in the transient response, and a system with large value of ζ ($\zeta > 0.8$) responds sluggishly.

4.2.2.2. Transient response specification parameters of under-damped Second-Order Systems

i. **Rise time (t_r):** Based up on the time required for the response to rise from 0% to 100%, the rise time can be obtained from

$$c(t) = 100\%r(t), \quad \text{where } r(t) = u(t) = 1 \quad \text{for } t \geq 0$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left\{\cos\omega_d t + \frac{\zeta}{\sqrt{(1-\zeta^2)}}\sin\omega_d t\right\} \quad \text{for } t \geq 0$$

Then at t_r, $c(t) = 1$

$$\rightarrow c(t) = 1$$

$$\rightarrow 1 - e^{-\zeta\omega_n t_r}\left\{\cos\omega_d t_r + \frac{\zeta}{\sqrt{(1-\zeta^2)}}\sin\omega_d t_r\right\} = 1$$

$$\rightarrow e^{-\zeta\omega_n t_r}\left\{\cos\omega_d t_r + \frac{\zeta}{\sqrt{(1-\zeta^2)}}\sin\omega_d t_r\right\} = 0$$

Cont. …

Since $e^{-\zeta\omega_n t_r} \neq 0$, the above equation can be simplified to

$$\rightarrow \cos\omega_d t_r + \frac{\zeta}{\sqrt{(1-\zeta^2)}}\sin\omega_d t_r = 0$$

$$\rightarrow \sin\omega_d t_r = -\frac{\sqrt{(1-\zeta^2)}}{\zeta}\cos\omega_d t_r$$

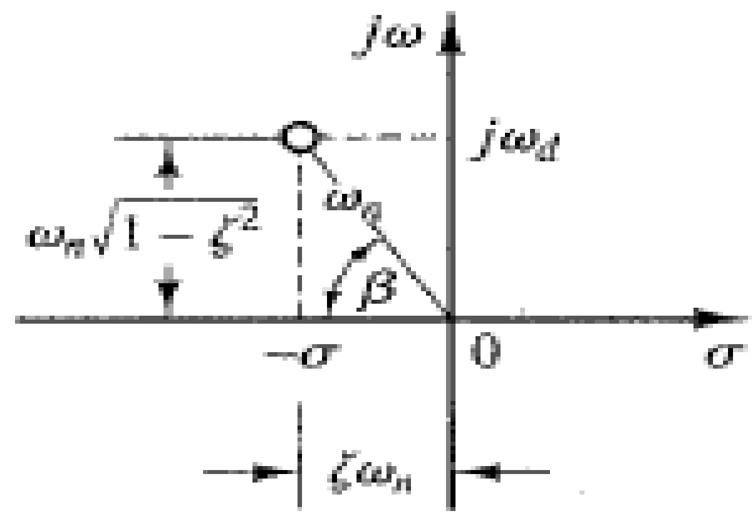
$$\rightarrow \tan\omega_d t_r = -\frac{\sqrt{(1-\zeta^2)}}{\zeta}$$ but $\omega_d = \omega_n\sqrt{(1-\zeta^2)}$ and $\sigma = \zeta\omega_n$

$$\rightarrow \tan\omega_d t_r = -\frac{\omega_n\sqrt{(1-\zeta^2)}}{\zeta\omega_n} = -\frac{\omega_d}{\sigma}$$

$$\rightarrow \omega_d t_r = \tan^{-1}(-\frac{\omega_d}{\sigma})$$

$$\rightarrow t_r = \frac{1}{\omega_d}\tan^{-1}(-\frac{\omega_d}{\sigma})$$

$$\rightarrow t_r = \frac{\pi-\beta}{\omega_d}$$



For the smallest value of t_r, ω_d must be large.

Cont. …

ii. **Peak time (t_p):** it can be obtained by differentiating c(t) with respect to time and letting this derivate equal to zero.

Exercise 4.2: *Proof that the peak time corresponds to the first peak overshoot for second order underdamped system is given by*

$$t_p = \frac{\pi}{\omega_d}$$

Cont. …

iii. **Maximum overshoot (M_p):** the maximum overshoot occurs at the peak time. Assuming that the final value of the response is unity, M_p can be obtained from

$$M_p = \frac{c(t_p) - c(\infty)}{C(\infty)} \text{ but } c(\infty) = 1$$

$$\rightarrow M_p = \frac{c(t_p) - 1}{1} = c(t_p) - 1 \quad \text{where } t_p = \frac{\pi}{\omega_d}$$

$$\rightarrow M_p = 1 - e^{-\zeta\omega_n * \pi/\omega_d}\left\{\cos\omega_d * \pi/\omega_d + \frac{\zeta}{\sqrt{(1-\zeta^2)}}\sin\omega_d * \pi/\omega_d\right\} - 1$$

$$\rightarrow M_p = -e^{-\zeta\omega_n * \pi/\omega_d}\left\{\cos\pi + \frac{\zeta}{\sqrt{(1-\zeta^2)}}\sin\pi\right\} = -e^{-\zeta\omega_n * \pi/\omega_d}(-1 + 0)$$

$$\rightarrow M_p = e^{-\zeta\omega_n * \pi/\omega_d} = e^{-(\frac{\sigma}{\omega_d})\pi}$$

Cont. …

- The maximum percent overshoot is

$$M_p = e^{-(\frac{\sigma}{\omega_d})\pi} * 100\%$$

iv. Settling time (t_s): for an underdamped second order system, the transient responses are obtained from.

$$c(t) = 1 - e^{-\zeta\omega_n t}\left\{\cos\omega_d t + \frac{\zeta}{\sqrt{(1-\zeta^2)}}\sin\omega_d t\right\}$$

- The settling time corresponding to a 2% or 5% tolerance may be measured from the determinant factors that can lead the response to the steady state value. The determinant factors of c(t) for settling are

$$1 - e^{-\zeta\omega_n t}$$

Cont. …

For 2% tolerance

$$1 - e^{-\zeta\omega_n t_s} = 0.98$$

$$e^{-\zeta\omega_n t_s} = 0.02$$

$$t_s = \frac{4}{\zeta\omega_n}$$

For 5% tolerance

$$1 - e^{-\zeta\omega_n t_s} = 0.95$$

$$e^{-\zeta\omega_n t_s} = 0.05$$

$$t_s = \frac{3}{\zeta\omega_n}$$

Cont. …

Exercise 4.3

1) Find the time response of the following system for unit step input

   a) $\frac{C(s)}{R(s)} = \frac{25}{s^2+7s+25}$ b) $\frac{C(s)}{R(s)} = \frac{20}{s^2+7s+25}$

2) For a system having $\frac{C(s)}{R(s)} = \frac{50}{s^2+7s+25}$, find its time response specifications and the expression for the output for unit step input.

3) For a system given by $\frac{d^2y(t)}{dt^2} + \frac{5dy(t)}{dt} + 16y(t) = 9r(t)$, where y(t) is the output and r(t) is the input. Determine,

a) Time response specifications.

b) Time response of the system for unit step input.

4) A system has $G(s) = \frac{k}{s(Ts+1)}$ with unity feedback when k and T are constant. Determine the factor by which k should be multiplied to reduce the overshoot from 85% to 35%.

4.3. Stability of control systems

4.3.1. Definition of stability

- Stability implies that small change in system input or small change in system initial condition or small change in system parameter do not produce large change in system output.

4.3.2. Analysis of stability

4.3.2.1. By using the location of the pole in the complex plane

- The stability of a linear closed-loop system can be determined from the location of the closed-loop poles in the **s** plane. If any of these poles lie in the right-half **s** plane, the system is unstable.

Cont. …

- If any poles lie in the right-half **s** plane, then with increasing time they give rise to the dominant mode, and the transient response increases monotonically or oscillates with increasing amplitude. For such a system, as soon as the power is turned on, the output may increase with time.
- If no saturation takes place in the system and no mechanical stop is provided, then the system may eventually be subjected to damage and fail since the response of a real physical system cannot increase indefinitely.
- Therefore, closed-loop poles in the right-half *s* plane are not permissible in the usual linear control system.

Cont. …

- If all closed-loop poles lie to the left-half s plane ($j\omega$ axis), the system is stable. For such a system the transient response eventually reaches equilibrium.
- Whether a linear system is stable or unstable is a property of the system itself and does not depend on the input or driving function of the system. The poles of the input, or driving function, do not affect the property of stability of the system, but they contribute only to steady-state response terms in the solution.
- Thus, the problem of absolute stability can be solved readily by choosing no closed-loop poles in the right-half **s** plane, including the $j\omega$ axis (closed-loop poles on the $j\omega$ axis will yield oscillations, the amplitude of which is neither decaying nor growing with time).

Cont. …

- **Note:** All closed-loop poles lie in the left-half s plane does not guarantee satisfactory transient-response characteristics. If dominant complex-conjugate closed-loop poles lie close to the $j\omega$ axis, the transient response may exhibit excessive oscillations or may be very slow.

- Therefore, to guarantee fast, yet well-damped, transient response characteristics, it is necessary that the closed-loop poles of the system lie in a particular region in the complex plane, such as the region bounded by the shaded area shown in fig.4.9.

Cont. …

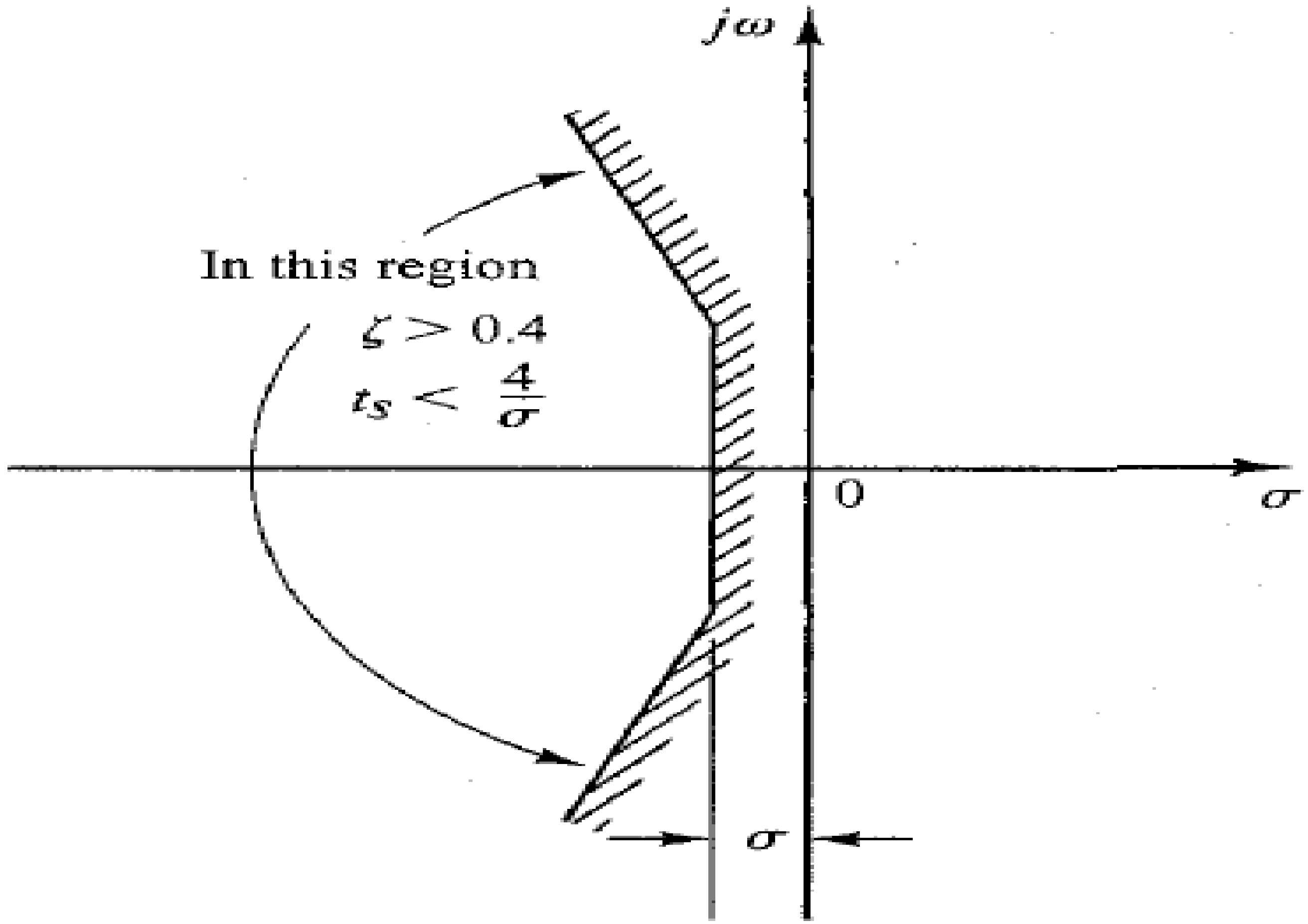


Fig.4.9: Region in the complex plane satisfying the conditions $\zeta > 0.4$ and $t_s < {}^4/_{\sigma}$.

4.3.2.2. By using Routh's stability criterion

- The most important problem in linear control systems concerns stability. Most linear closed-loop systems have closed-loop transfer functions of the form

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

where the a's and b's are constants and $m \leq n$.

- Routh's stability criterion, enables us to determine the number of closed-loop poles that lie in the right-half s plane without having to factor the denominator polynomial. This stability criterion applies to polynomials with only a finite number of terms.

Cont. …

❖ The procedure in Routh's stability criterion is as follows:

1) Write the polynomial in s in the following form:

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

where the coefficients are real quantities. We assume that $+a_n \neq 0$; that is, any zero root has been removed.

2) The necessary but not sufficient condition for stability is that all the coefficients of the denominator of transfer function should present and all have a positive sign. (If all a's are negative, they can be made positive by multiplying both sides of the equation by -1.)

➢ If any of the coefficients are zero or negative in the presence of at least one positive coefficient, there is a root or roots that are imaginary or that have positive real parts. In this case, the system is not stable.

Cont. …

3) If all coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the following pattern:

$$
\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
s^0 & g_1 & & & &
\end{array}
$$

- The process of forming rows continues until we run out of elements. (The total number of rows is $n + 1$.) The coefficients b_1, b_2, b_3, and so on, are evaluated as follows until the remaining ones are all zero.

Cont. …

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$\vdots$$

- The same pattern of cross-multiplying the coefficients of the two previous rows is followed in evaluating the c's, d's, e's, and so on. That is,

Cont. …

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

$$\vdots$$

And

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

$$d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}$$

$$\vdots$$

- This process is continued until the nth row has been completed. The complete array of coefficients is triangular. Routh's stability criterion states

- The number of roots of the denominator of transfer function with positive real parts is equal to the number of changes in sign of the coefficients of the first column of the array.

Cont. …

- The necessary and sufficient condition that all roots of the denominator of transfer function lie in the left-half **s** plane is that all the coefficients of the denominator should be positive and all terms in the first column of the array have positive signs.

Example 4.2: Investigate the stability of a system with characteristics polynomials

a) $s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$

b) $P(s) = s^4 + s^3 + s^2 + 2s + 4$

c) $P(s) = s^3 + s^2 + 2s + 1$

Cont. …

Solution: a) All the coefficients are positive numbers.

Forming the array

s^4	1	3	5
s^3	2	4	0
s^2	b_1	b_2	
s^1	c_1		
s^0	d_1		

$$b_1 = \frac{2*3-1*4}{2} = 1,\ b_2 = \frac{2*5-1*0}{2} = 5$$

$$c_1 = \frac{b_1*4-2*b_2}{b_1} = \frac{1*4-2*5}{1} = -6$$

$$d_1 = \frac{c_1*b_2-b_1*0}{c_1} = \frac{-6*5-1*0}{-6} = 5$$

Therefore

s^4	1	3	5
s^3	2	4	0
s^2	1	5	
s^1	-6		
s^0	5		

The number of changes in sign of the coefficients in the first column is 2. This means that there are two roots on right half of s plane. So the system is unstable.

Cont. …

Special Cases

- There are two cases in which the Routh criterion, as it has been presented above can not be applied. For these two cases certain modifications are necessary so that the above procedure is applicable.

Case1: A zero element in the first column of the Routh array

- In this case the Routh array cannot be completed because the element below the zero elements in the first column will become infinite. To overcome this difficult,

- **Method1**: If a first-column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then the zero term is replaced by a very small positive number ϵ and the rest of the array is evaluated.

Cont. …

- **Method2:** we multiply the characteristics polynomial p(s) with a factor $(s+a)$, where $a > 0$ and $-a$ is not the root of p(s). The conclusions regarding stability of the new polynomial $\hat{p}(s) = (s + a)p(s)$ are obviously the same as those of the original polynomial p(s).

Example 4.3: consider the following equation

$$p(s) = s^4 + 2s^3 + s^2 + 2s + 4$$

Construct the Routh array as,

s^4	1	1	4
s^3	2	2	0
s^2	0	4	
s^1	∞		
s^0			

Since the third element in the first column of the Routh array is zero, it is clear that the Routh array can not completed

Cont. …

Method1: Replace the zero term by a very small positive number ϵ and then evaluate the rest of the array.

s^4	1	1	4
s^3	2	2	0
s^2	$\in$	4	
s^1	$\in - \frac{8}{\epsilon}$		
s^0	4		

There are two sign changes of the coefficients in the first column. It show that p(s) has two roots in the right half complex plane and therefore the system is unstable.

Method2: If we multiply the polynomial p(s) by the factor $(s + 1)$,

we have,

$$\hat{p}(s) = (s + 1)p(s) = s^5 + 3s^4 + 3s^3 + 3s^2 + 6s + 4$$

Next construct the Routh array

Cont. …

s^5	1	3	6
s^4	3	3	4
s^3	2	$\frac{14}{3}$	0
s^2	-4	4	
s^1	20	0	
s^0	4		

According to the above Routh array, one observes that the polynomial $\hat{p}(s)$ and p(s) have two roots in the right half complex plane and therefore the system with characteristics polynomial P(s) is unstable

Case2: **A zero row in the Routh array**

- In this case the Routh array cannot be completed, because in computing the rest of the elements that follow zero rows, indeterminate form $^0/_0$ will appear. To overcome this difficulty:

Cont. …

1) Form the auxiliary polynomial q(s) of the row which precedes the zero row.

2) Take the derivative of q(s) with respect to s and replace the zero row with the coefficients of $\dot{q}(s)$.

3) Complete the construction of the Routh array in the usual manner.

Example 4.4: investigate the stability of a system with characteristic polynomial

$$p(s) = s^4 + 3s^3 + 4s^2 + 3s + 3$$

Cont. …

Solution: construct the Routh array as follow,

s^4	1	4	3
s^3	3	3	0
s^2	3	3	
s^1	0		
s^0	$0/0$		

Since the row s^1 of the Routh array involves only zeros, it is clear that the Routh array cannot be completed. At this point form auxiliary polynomial $q(s) = 3s^2 + 3$ of the row s^2. Taking the derivative of q(s), $\dot{q}(s) = 6s$. The replace the zero row (i.e. s^1) with the coefficient of $\dot{q}(s)$ and complete the Routh array in the usual manner to yield

s^4	1	4	3
s^3	3	3	0
s^2	3	3	
s^1	6		
s^0	3		

Since the first column of the new Routh array doesn't have sign change the system is stable.

Cont. …

- **Relative Stability Analysis:** Routh's stability criterion provides the answer to the question of absolute stability. This, in many practical cases is not sufficient. We usually require information about the relative stability of the system.

- **Absolute stability:** indicates stability or unstability of a system under consideration.

- **Relative stability:** indicates how close a system is to instability.

- Routh's stability criterion is of limited usefulness in linear control system analysis mainly because it does not suggest how to improve relative stability or how to stabilize an unstable system.

Cont. …

- It is possible, however, to determine the effects of changing one or two parameters of a system by examining the values that cause instability.

Example4.5: Let us determine the range of k such that the system with characteristic polynomial $p(s) = s^4 + s^3 + 2s^2 + s + 2k$ is stable.

Solution:

s^4	1	2	2k
s^3	1	1	0
s^2	1	2k	
s^1	1 – 2k	0	
s^0	2k		

For the system to be stable all elements of the first column of Routh array must have the same sign. Hence there must be

$2k > 0$ and $1 - 2k > 0$

$k > 0$ and $k < 0.5$

Hence the system is stable for $0 < k < 0.5$

Exercise 4.4:

1) Determine the stability of a system with closed loop transfer function:

a) $G(s) = \frac{1}{2s^5+3s^4+2s^3+3s^2+2s+1}$

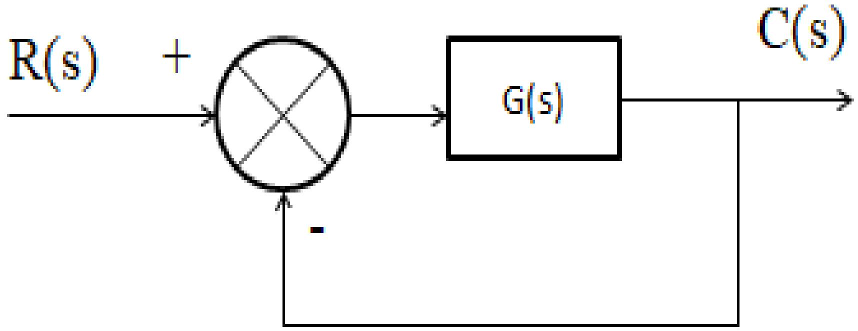
b) $G(s) = \frac{10}{s^5+2s^4+3s^3+6s^2+5s+3}$

c) $G(s) = \frac{1}{2s^5+7s^4+6s^3+42s^2+8s+56}$

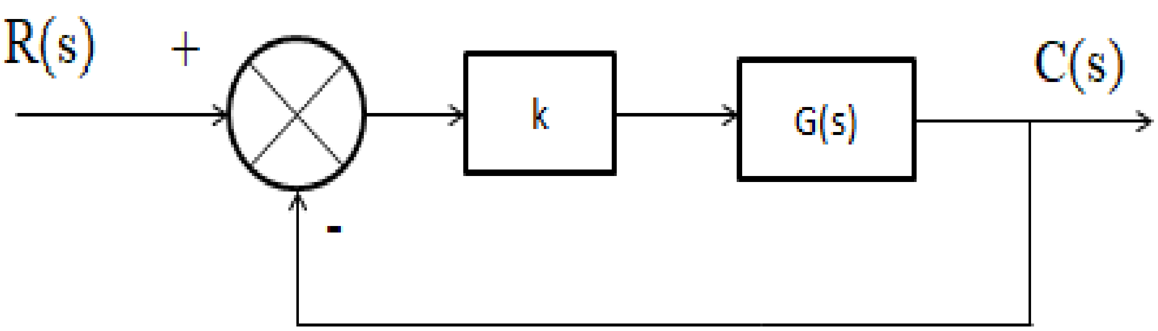
d) $G(s) = \frac{1}{s^3+5s^2+6s+30}$

Cont. …

2) For a unity feedback system with $G(s) = \frac{10}{s(s+1)(s+2)}$ determine the stability using Routh criterion.

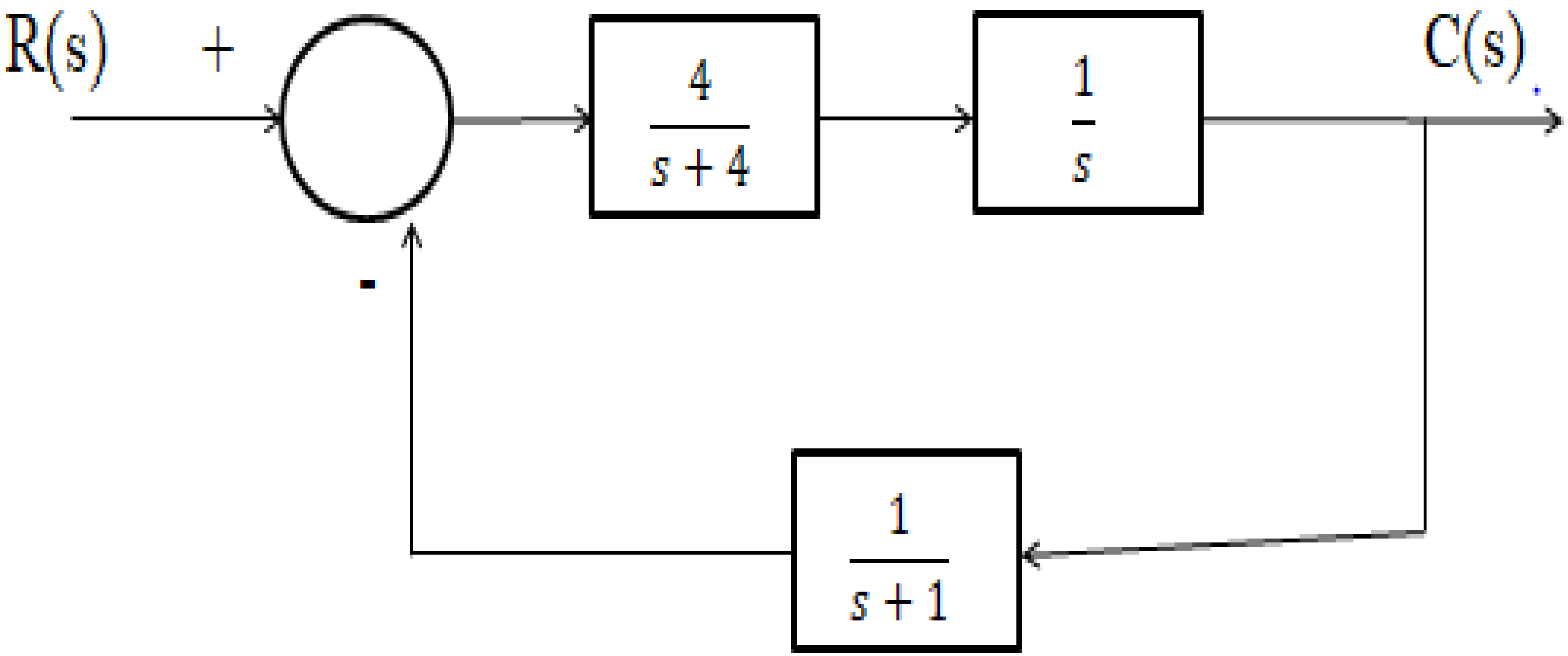


3) For a unity feedback system given below with $G(s) = \frac{s+6}{s(s+1)(s+3)}$ determine the range of k that makes the system stable.



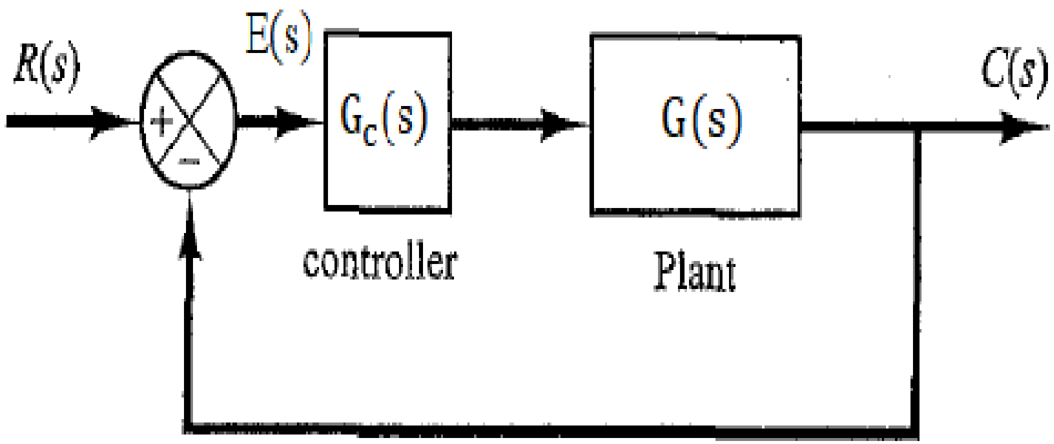
Cont. …

5) Investigate the stability of the given closed loop control system by using Routh criterion.



4.4. Effect of derivative and integral control actions on system performance

❖ Consider the system shown in the figure below:



where

$G_c(s)$ –Tf of the controller

$G_c(s)$ –Tf of the plant

➢ The transfer function from E(s) into R(s) can be obtained from

$$E(s) = R(s) - C(s) \quad \text{but } C(s) = G_c(S)G(s)E(s)$$

$$\rightarrow E(s) = R(s) - G_c(S)G(s)E(s)$$

$$\rightarrow E(s) = \frac{1}{1+G_c(S)G(s)} R(s)$$

Cont. …

➢ For a unit step reference input, the error of the system will becomes

$$E(s) = \frac{1}{1 + G_c(S)G(s)} * \frac{1}{s}$$

❖ Let us consider the transfer function of the plant be $G(s) = \frac{1}{Ts+1}$, then

i. The steady state error of the system, when the proportional control action is added into the system is:

$$e_{ss}(t) = \lim_{s\to 0} sE(s) \text{ where } G_c(s) = k, \qquad G_c(s)G(s) = \frac{k}{Ts+1}$$

$$e_{ss}(t) = \lim_{s\to 0} s * \frac{1}{\left(1+\frac{k}{Ts+1}\right)} * \frac{1}{s} = \lim_{s\to 0} \frac{Ts+1}{Ts+k+1} = \frac{1}{k+1}$$

Cont. …

- Such a system with a proportional controller in the feed forward path always has a steady state error in the step response. This steady state error is called an offset. Such an error can be eliminated by including an integral control action on the system.

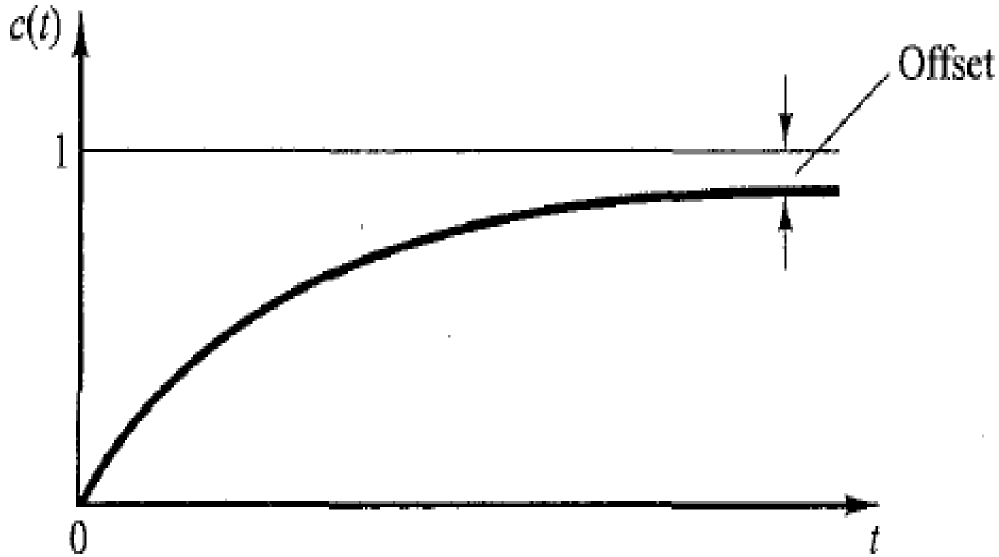


Fig4.10: unit step response and offset.

Cont. …

ii. The steady state error of the system when an integral control action is added into the system is:

$$e_{ss}(t) = \lim_{s\to 0} sE(s) \text{ where } G_c(s) = \frac{k}{s}, \qquad G_c(s)G(s) = \frac{k}{s(Ts+1)}$$

$$e_{ss}(t) = \lim_{s\to 0} s * \frac{1}{(1+\frac{k}{s(Ts+1)})} * \frac{1}{s} = 0$$

- Note that integral control action, while removing offset or steady-state error, may lead to oscillatory response of slowly decreasing amplitude or even increasing amplitude, both of which are usually undesirable.

4.5. Steady state errors in unity feedback control systems

❖ Consider the unity-feedback control system with the following open-loop transfer function G(s):

$$G(s) = \frac{k(T_a s + 1)(T_b s + 1) \ldots (T_m s + 1)}{s^N (T_1 s + 1)(T_2 s + 1) \ldots (T_p s + 1)}$$

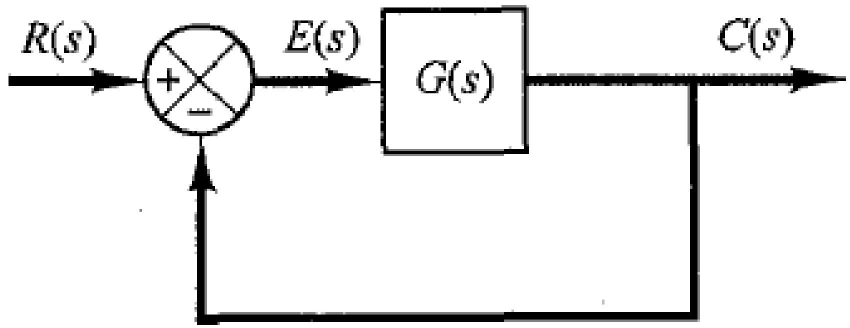
➢ It involves the term s^N in the denominator, representing a pole of multiplicity N at the origin. The present classification scheme is based on the number of integrations indicated by the open-loop transfer function. A system is called type 0, type 1, type 2, . . . ,if $N = 0$, $N = 1$, $N = 2$, . . . ,respectively.

➢ Note: As the type number is increased, accuracy is improved; however, increasing the type number aggravates the stability problem

Cont. …

❖ Steady-State Errors:

➢ consider the system shown in figure below



➢ The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

➢ The transfer function between the error signal e(t) and the input signal r(t) is

Cont. …

$$E(s) = R(s) - C(s)$$

$$\rightarrow \frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = 1 - \frac{G(s)}{1+G(s)} = \frac{1}{1+G(s)} \quad \therefore\ E(s) = \frac{1}{1+G(s)} * R(s)$$

- By using final value theorem, the steady state error is

$$e_{ss} = \lim_{t\rightarrow\infty} e(t) = \lim_{s\rightarrow 0} SE(s) = \lim_{s\rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

i. The steady-state error of the system for a unit-step input is

$$e_{ss} = \lim_{s\rightarrow 0} SE(s) = \lim_{s\rightarrow 0} \frac{s}{1+G(s)} * \frac{1}{s} = \frac{1}{1+G(0)}$$

- The static position error constant k_p, is defined by

$$k_p = \lim_{s\rightarrow 0} G(s) = G(0)$$

Cont. …

- Thus, the steady-state error in terms of the static position error constant k_p, is given by

$$e_{ss} = \frac{1}{1 + k_p}$$

- For a type 0 system,

$$k_p = \lim_{s \to 0} \frac{k(T_a s + 1)(T_b s + 1)}{(T_1 s + 1)(T_2 s + 1) \ldots} = k$$

- For a type 1 or higher system,

$$k_p = \lim_{s \to 0} \frac{k(T_a s+1)(T_b s+1)}{s^N (T_1 s+1)(T_2 s+1) \ldots} = \infty \quad \text{for } N \geq 1$$

Cont. …

- For a unit-step input, the steady-state error e_{ss}, may be summarized as follows:

$$e_{ss} = \frac{1}{1+k}, \qquad \text{for type 0 systems}$$

$$e_{ss} = 0, \qquad \text{for type 1 or higher systems}$$

ii. The steady-state error of the system with a unit-ramp input is given by

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} * \frac{1}{s^2} = \lim_{s\to 0} \frac{1}{s+sG(s)} = \lim_{s\to 0} \frac{1}{sG(s)}$$

- The static velocity error constant k_v is defined by

$$k_v = \lim_{s\to 0} sG(s)$$

Cont. …

- Thus, the steady-state error in terms of the static velocity error constant k_v, is given by

$$e_{ss} = \frac{1}{k_v}$$

- For a type 0 system

$$k_v = \lim_{s\to 0} \frac{sk(T_a s+1)(T_b s+1)}{(T_1 s+1)(T_2 s+1)\ldots} = 0$$

- For a type 1 system

$$k_v = \lim_{s\to 0} \frac{sk(T_a s + 1)(T_b s + 1)}{s(T_1 s + 1)(T_2 s + 1)\ldots} = k$$

- For type 2 or higher system

$$k_v = \lim_{s\to 0} \frac{sk(T_a s+1)(T_b s+1)}{s^N (T_1 s+1)(T_2 s+1)\ldots} = \infty, \qquad \text{for } N \geq 2$$

Cont. …

- The steady-state error e_{ss}, for the unit-ramp input can be summarized as follows:

$$e_{ss} = \infty, \quad \text{for type 0 system}$$

$$e_{ss} = \frac{1}{k}, \quad \text{for type 1 system}$$

$$e_{ss} = 0, \quad \text{for type 2 and higher system}$$

iii. The steady-state error of the system with a unit-parabolic input (acceleration input), is defined by

$$e_{ss} = \lim_{s\to 0}\frac{s}{1+G(s)} * \frac{1}{s^3} = \lim_{s\to 0}\frac{1}{s^2+s^2G(s)} = \lim_{s\to 0}\frac{1}{s^2G(s)} \quad \text{where} \quad r(t) = \begin{cases} \frac{t^2}{2}, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases}$$

Cont. …

- The static acceleration error constant k_a, is defined by

$$k_a = \lim_{s\to 0} s^2 G(s)$$

- The steady-state error interms of k_a is then

$$e_{ss} = \frac{1}{k_a}$$

- For a type 0 system,

$$k_a = \lim_{s\to 0} \frac{s^2 k(T_a s + 1)(T_b s + 1)}{(T_1 s + 1)(T_2 s + 1)\ \ldots} = 0$$

- For a type 1 system,

$$k_a = \lim_{s\to 0} \frac{s^2 k(T_a s + 1)(T_b s + 1)}{s(T_1 s + 1)(T_2 s + 1)\ \ldots} = 0$$

Cont. …

❖ For a type 2 system,

$$k_a = \lim_{s\to 0} \frac{s^2 k(T_a s + 1)(T_b s + 1)}{s^2 (T_1 s + 1)(T_2 s + 1) \ldots} = k$$

❖ For a type 3 or higher system,

$$k_a = \lim_{s\to 0} \frac{s^2 k(T_a s+1)(T_b s+1)}{s^N (T_1 s+1)(T_2 s+1)\ldots} = \infty, \qquad \text{for } N \geq 3$$

➢ Thus, the steady-state error for the unit parabolic input is

$e_{ss} = \infty$, for type 0 and 1 systems

$e_{ss} = \frac{1}{k}$, for type 2 systems

$e_{ss} = 0$, for type 3 and higher systems

Cont. …

Table 4.1: Summary (Steady-State Error in Terms of Gain k)

	Step Input $r(t) = 1$	Ramp Input $r(t) = t$	Acceleration Input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\frac{1}{1+K}$	∞	∞
Type 1 system	0	$\frac{1}{K}$	∞
Type 2 system	0	0	$\frac{1}{K}$

Exercise 4.5

1) Identify the type of system given below

a) $G(s) = \frac{k(1+3s)}{s^2}$ and $H(s) = \frac{5s+1}{s^2+6s+7}$

b) $G(s) = \frac{k(1+3s)}{s}$ and $H(s) = 1$

c) $G(s) = \frac{4}{s^2+6s+7}$ and $H(s) = s + 3$

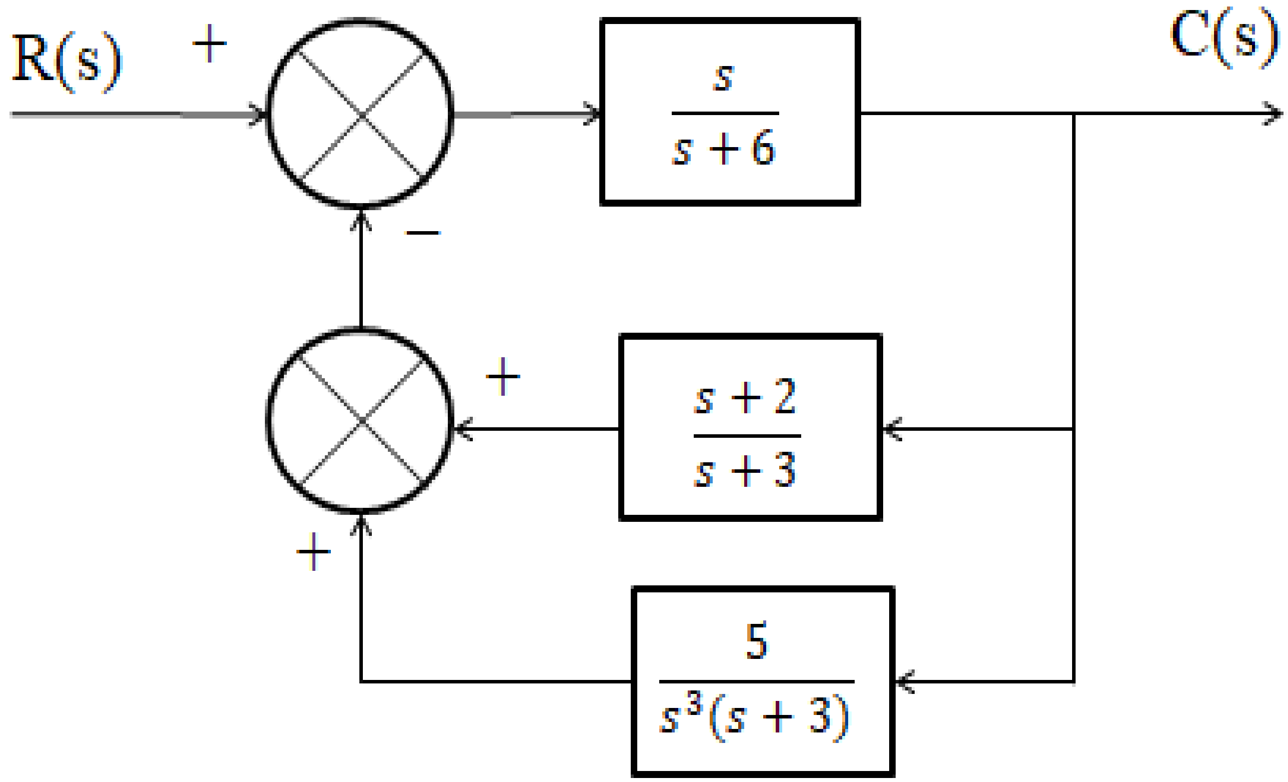
2) For a system having $G(s)H(s) = \frac{k(s+4)}{s(s^3+5s^2+6s)}$, find

i. Type of the system

ii. Error coefficient

iii. Error due to parabolic input (s^2)

3) Find the error coefficient for the system given by block diagram.



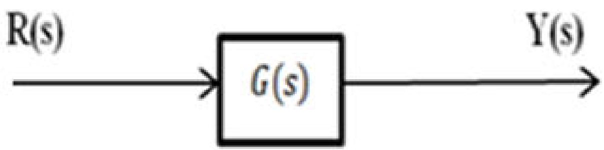
4.6. Feedback Characteristics of Control Systems

- Feedback is used in control systems to reduce error, the sensitivity of the system due to parameter variations and unwanted internal and external disturbances.

4.6.1. Effect of feedback on overall gain

- Consider an open-loop and closed loop control system shown below.

open-loop

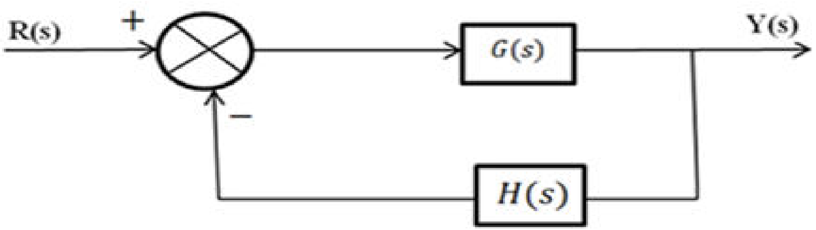


- For open-loop the overall transfer function is,

$$\frac{Y(s)}{R(S)} = G(s)$$

- The gain of such system is $G(s)$.

closed-loop



- For closed-loop the overall transfer function is,

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

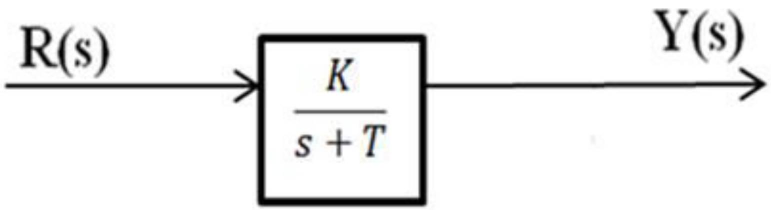
- The overall gain of the system is $\frac{G(s)}{1+G(s)H(s)}$.

Cont. …

- Due to introduction of negative feedback, the gain $G(s)$ is reduced by the factor of $\frac{1}{1+G(s)H(s)}$. Therefore, due to negative feedback the overall gain reduces.

4.6.2. Effect of feedback on stability

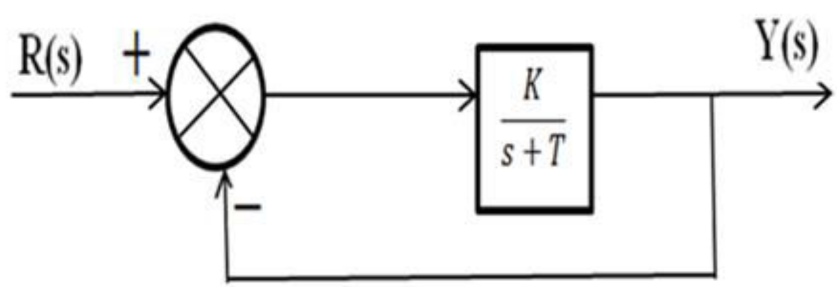
- Consider open-loop and a unity negative feedback control system shown below.



- The overall transfer function is,

$$G(s) = \frac{K}{s + T}$$

- The open-loop pole is located at $s = -T$



- The overall transfer function of a unity negative feedback is,

$$G(s) = \frac{k}{s+(k+T)}$$

- The closed-loop pole is located at $s = -(k + T)$

Cont. …

- Due to introduction of feedback, the time constant decrease and the transient response decays more rapidly.
- Since the stability of a system depends upon the location of poles in the s-plane, feedback may improve the stability of the system.
- Open-loop pole is stable; the closed-loop pole may be unstable.
- To control the stability of the system, a proper design and application of the feedback is required.

4.6.3. Effect of feedback on system sensitivity

- The parameters of a control system change due to environmental changes and other disturbances. This change has adverse affect on the system performance.

Cont. …

- If the variable in a system be T due to the variation in parameter K of the system, the sensitivity of the system parameter T to the parameter K is given by

$$S = \frac{\%\ change\ in\ T}{\%\ change\ in\ K}$$

Mathematically, it can be written as

$$\int_K^T = \frac{\partial T/_T}{\partial K/_K} = \frac{K}{T}\frac{\partial T}{\partial K}$$

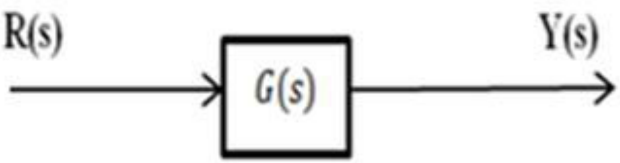
- Let T(s) be the overall transfer function of a control system and G(s) is its forward path gain. The sensitivity of overall transfer function T(s) with respect to the variations in G(s) is given by

Cont. …

$$\int_G^T = \frac{\partial T(s)/T(s)}{\partial G(s)/G(s)} = \frac{G(s)}{T(s)}\frac{\partial T(s)}{\partial G(s)}$$

- It describes relative variation of overall transfer function due to variation of G(s).

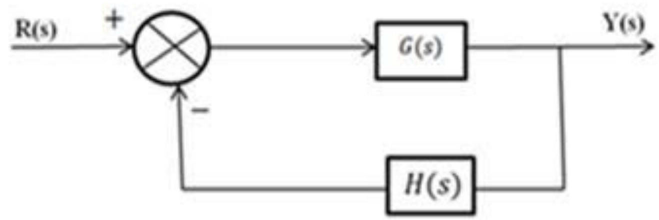
Open-loop system



$$T(s) = \frac{Y(s)}{R(s)} = G(s)$$

$$\int_G^T = \frac{G(s)}{T(s)}\frac{\partial T(s)}{\partial G(s)} = \frac{G(s)}{G(s)}\frac{\partial G(s)}{\partial G(s)} = 1x1 = 1$$

closed-loop system



$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + H(s)G(s)}$$

$$\frac{\partial T(s)}{\partial G(s)} = \frac{[1 + H(s)G(s)] - H(s)G(s)}{[1 + H(s)G(s)]^2} = \frac{1}{[1 + H(s)G(s)]^2}$$

$$\int_G^T = \frac{G(s)}{T(s)}\frac{\partial T(s)}{\partial G(s)} = \frac{G(s)}{\frac{G(s)}{1 + H(s)G(s)}} x \frac{1}{[1 + H(s)G(s)]^2} = \frac{1}{1 + H(s)G(s)}$$

Cont. …

- The sensitivity function gets reduced by a factor $\frac{1}{1+H(s)G(s)}$ compared to open-loop sensitivity function due to the presence of negative feedback. In closed loop systems, $\int_G^T$ is less sensitive to the variation of G(s).

4.6.4. Effect of feedback on steady state error

- The error signal is given by

$$\mathrm{e(t) = r(t) - y(t) \ or \ E(s) = R(s) - Y(s)}$$

Where, r(t) is the reference input and y(t) is the actual output.

- The steady state error is given by

$$\mathrm{e}_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \text{ or } e_{ss} = r(t) - y_{ss}(t)$$

where $y_{ss}(t)$- steady state output.

Cont. …

Open-loop system

$$Y(s) = G(s)R(s)$$

$$E(s) = R(s) - Y(s) = R(s) - G(s)R(s)$$

$$E(s) = (1 - G(s))R(s)$$

The steady state error

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$e_{ss} = \lim_{s\to 0} s(1 - G(s))R(s)$$

If r(t) unit step input ($R(s) = \frac{1}{s}$)

$$e_{ss} = \lim_{s\to 0}(1 - G(s)) = 1 - G(0)$$

Closed-loop system

- Consider a unity feedback system

$$Y(s) = \frac{G(s)}{1 + G(s)} R(s)$$

$$E(s) = R(s) - Y(s) = R(s) - \frac{G(s)}{(1 + G(s)} R(s)$$

$$E(s) = \frac{R(s)}{1+G(s)}$$

The steady state error

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{R(s)}{1 + G(s)} = \frac{1}{1 + G(0)}$$

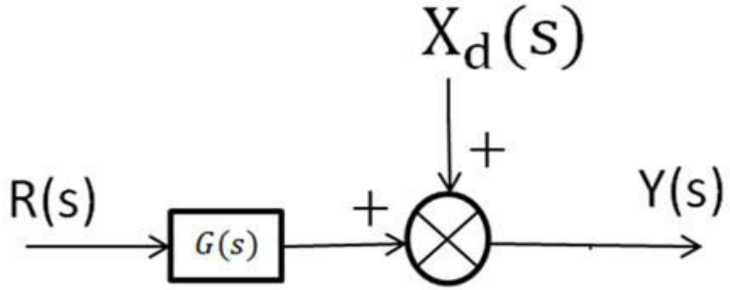
Since $R(s) = \frac{1}{s}$

- Since $G(0)$ may be large value, the steady state error of open-loop system will be large.
- In the case of negative unity feedback the steady state error is reduced to $\frac{1}{1+G(0)}$.

4.6.5. Effect of feedback on disturbance

- Consider an open-loop and closed loop control system shown below.

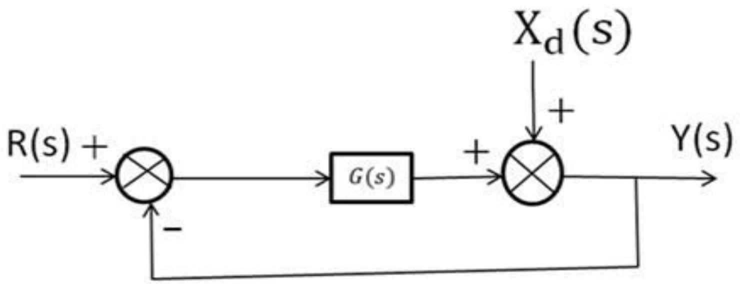
Open-loop system



$$Y(s) = R(s)G(s) + X_d(s)$$

Closed-loop system

- Consider a unity feedback system with the disturbance applied at the output.



$$Y_s = Y_r(s) + Y_d(s)$$

$$Y(s) = \frac{G(s)R(s)}{1 + G(s)} + \frac{X_d(s)}{1 + G(s)}$$

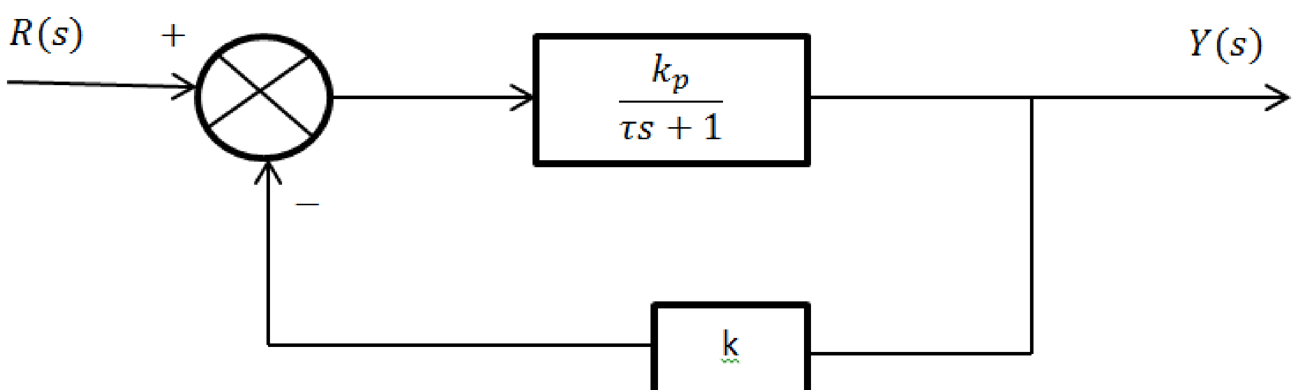
- Therefore, in the case of open loop the effect of feedback is large but in the case of closed loop by changing the value of G(s), the effect of disturbance on the output can be reduced.

Cont. …

❖ **Note:** Here in a negative unity feedback system the disturbance is reduced by the factor of $\frac{1}{1+G(s)}$ when compared to open-loop system. But based on the position of disturbance applied to a system and type of feedback the reduction factor is not the same.

Exercise 4.2

1) Consider a feedback system given below.



Determine $\int_{k_p}^{T}$, $\int_{\tau}^{T}$ and $\int_{K}^{T}$ where, $T = \frac{Y(s)}{R(s)}$

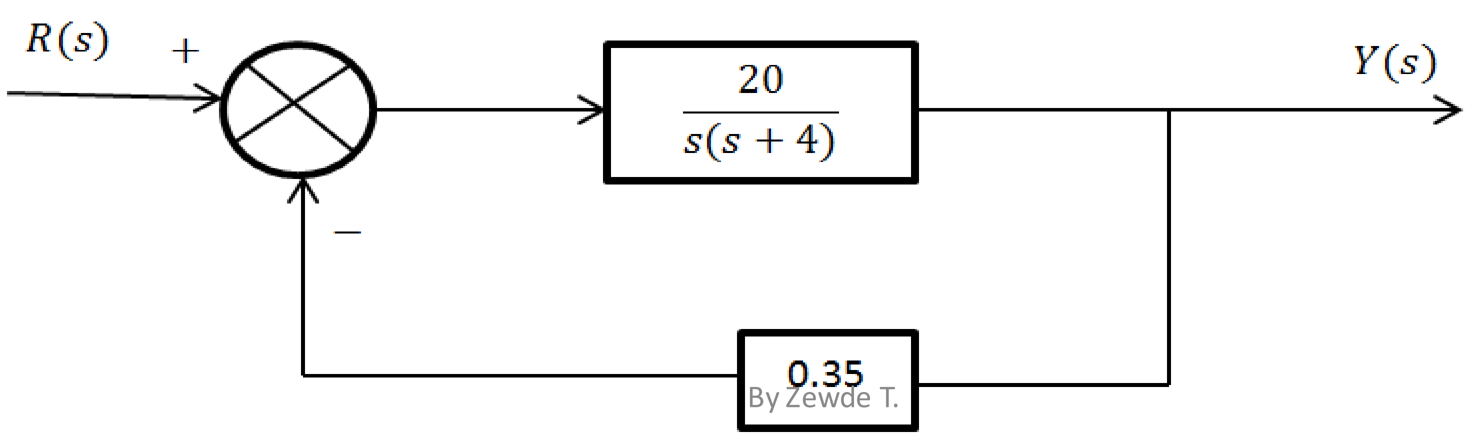
Cont. …

2) Find the sensitivity of over all transfer function of a system given below with respect to ($s = j\omega, \omega = 1.2\,{}^{rad}/_{sec}$).

a) Forward path transfer function. (**Ans.** $\int_G^T = 0.682$)

b) Feedback path transfer function. (**Ans.** $\int_H^T = 0.953$)

c) Determine the steady state system output and steady state error for unit step input.

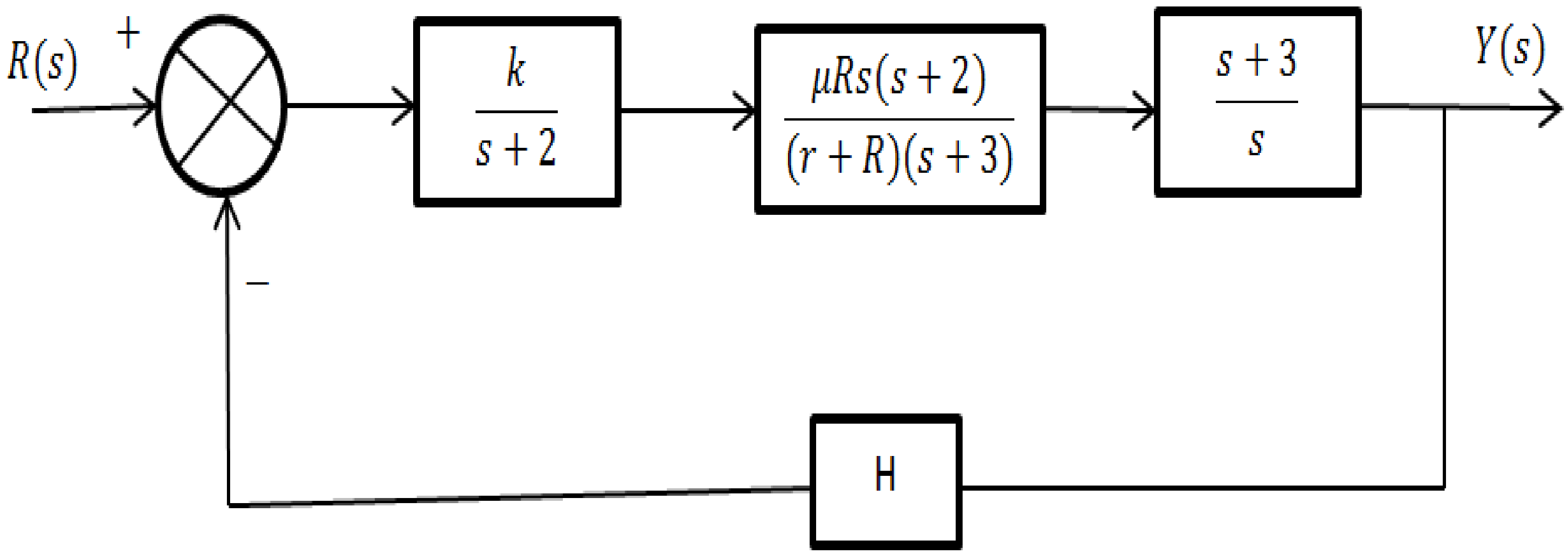


Cont. …

3) Find the following if $R = 10k\Omega$ and $r = 8k\Omega$.

   a) The value of k for 4% system sensitivity due to variation of μ, $H = 0.3$ and $\mu = 12$.

   b) The value of k for 3% system sensitivity due to variation of H, $\mu = 18$ and $H = 0.25$.



From your Text book refer the following:

- **Root locus analysis**
  - Introduction.
  - Root locus plots for negative feedback systems.
  - Root locus plot of positive feedback systems.
  - Conditionally stable systems.
- **Frequency response Analysis**
  - Introduction.
  - Presenting frequency-response characteristics in graphical forms.
  - Polar plot.
  - Nyquist plot.
  - Bode plot.
  - Nichols plot (Log-magnitude Versus phase plot).
  - Nyquist stability criterion.

Cont. …

- **Control Systems Design by Root locus and frequency response method**

- Design considerations, Lead compensation, Lag compensation, Lead-lag compensation, parallel Compensation.
- Introduction, Lead compensation, Lag compensation, Lead-lag compensation.