Chapter one: Inelastic Moment Redistribution

1.1 Introduction

When designing a structure and its components, the designer must decide on the appropriate structural model. The choice of the model effects:

- the analysis of the structure, which is aimed at the determination of the stress (internal forces and moments), and
- the calculation of cross section resistance

Thus a model implies the use of a method of analysis combined with a method of cross section resistance calculation.

There are several possible combinations of methods of analysis and methods of cross-section calculation, for the ultimate limit state, involving either an elastic or plastic design approach; the possible combinations are:

a) Plastic-plastic model:

This is related to plastic design of structures. Full plasticity may be developed within cross-sections, so that plastic hinges can form. These have suitable moment rotation characteristics giving sufficient rotation capacity for the formation of a plastic mechanism, as the result of moment redistribution in the structure.

b) Elastic-plastic model:

For structures composed of sections which can achieve their plastic resistance, but have not sufficient rotation capacity to allow for a plastic mechanism in the structure. The stresses from the elastic analysis are compared with the plastic section capacity.

c) Elastic-elastic model:

When the cross section of a structure cannot achieve their plastic capacity both analysis and verification of cross section conducted elastically.

Elastic analysis of reinforced concrete beams gives reasonable results up to working loads. Beyond working loads the elastic analysis is not applicable because of the non linearity in the stress-strain curves for the materials and the cracks which develop in concrete. When beam is loaded beyond working loads, plastic hinges form at certain locations. On further loading of the beam, bending moments do not increase beyond the ultimate moment capacities of the these sections, however, rotations at the plastic hinges keep on increasing. A redistribution of moments takes place, the moment now being received by the less stressed sections. The rotation at a plastic hinges keeps on increasing with out any increase in the moment until the ultimate rotation capacity is reached beyond which the section collapses

1.2 Non-Linear Analysis of Indeterminate Structures

The linear elastic analysis of structures is based on the assumption that there is a linear relationship between the stress and strain in a member, i.e.

$$\sigma = E\varepsilon$$

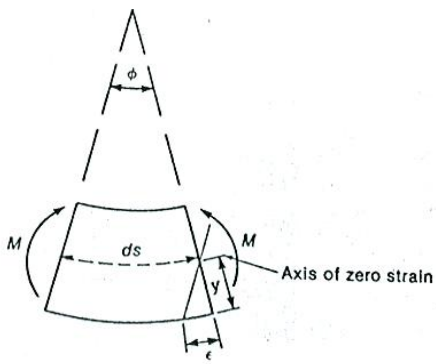
Where: E is the elastic modulus (young's modulus)
ε is strain

For a section in bending (Fig. 1.1a), the linear elastic moment curvature is given by:

$$\frac{M}{I} = \frac{E}{R} \Rightarrow M = EI * 1/R = EI\varphi \qquad (1^*)$$

Where: I is second moment of area of x-section
R is radius of curvature
φ = 1/R is curvature

Thus for a linear elastic section, the moment curvature relationship is linear (Fig1.1b).



(a)

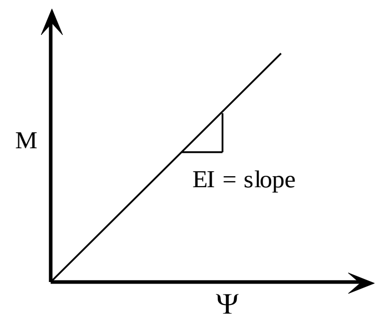


Fig. 1.1

The above equation and the second moment of area in this equation are applicable to beams of homogenous x-section. RC is not homogenous as it consists of two materials (concrete & steel) which have considerably different values of elastic modulus. However, it is possible to transform a RC section in to an equivalent homogenous concrete section and to calculate an equivalent second moment of area.

When the internal moment, M, is very small, the concrete is un cracked and the equivalent second moment of area is denoted by I_{ut}. Denoting the elastic modulus of concrete as E_c, equation (1*) becomes:

$$M = E_C I_{ut} \psi \qquad (2^*)$$

However, at quite low moment, the section cracks, the equivalent second moment of area drops to a much lower value, then equation (2*) becomes:

$$M = E_C I_{ct} \psi \qquad (3^*)$$

Where; I_{ct} is the equivalent second moment of area of the cracked section. This relationship is diagrammatically shown in Fig. 1.2.

Equation (3*) remains valid until the material behavior becomes non-linear. For a properly designed reinforced concrete section, the steel yields before the concrete crushes. This happens at an applied moment of M_y. As steel is ductile material, the section too is ductile, and beyond the yield point the curvature increases greatly for a relatively small increase in the moment. Complete failure of the section occurs when the concrete at the extreme fiber in compression crushes. The curvature at this stage is denoted by Ψ_{ult}.

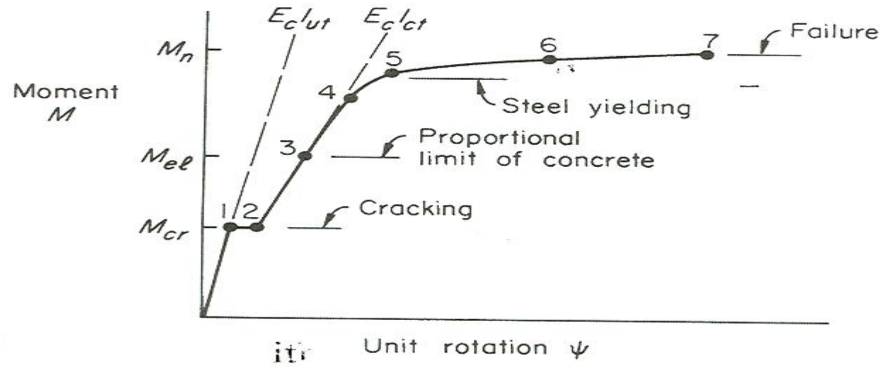


Fig. 1.2 Moment-curvature diagram for RC

Linear analysis is based on the assumption that $\Psi < \Psi_y$ at all sections of all members. Thus linear elastic analysis techniques, such as the stiffness methods and moment distribution, are based on this assumption. In this section non-linear methods of analysis are considered for which there is no such restriction on Ψ.

1.3 Plastic Hinges and Collapse Mechanisms

Indeterminate structures such as continuous beams or frames normally will not fail when the ultimate moment capacity of just one critical section is reached. A plastic hinge will form at that section permitting large rotation to occur at essentially constant resisting moment and thus transferring load to other locations along the span where the limiting resistance has not yet been reached. Normally in a continuous beam or frame, excess capacity will exist at those other locations because they would have been reinforce for moments resulting from different load distributions selected to produce maximum moments at those other locations.

As loading is further increased, additional plastic hinges may form at other locations along the span and eventually result in collapse of structure after a significant redistribution of moments.

Reconsider the following simplified moment curvature for an actual ductile RC beam.

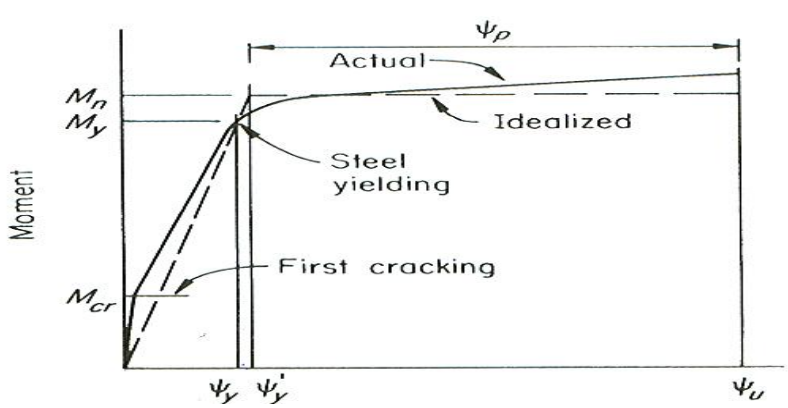


Fig. 1.3 Moment-curvature diagram for RC

The diagram is linear up to the cracking moment M_{cr}, after which nearly straight line of some what flatter slope is obtained. At the yielding moment M_y, the unit rotation starts to increase disproportionately. Further increased in applied moment causes extensive inelastic rotation until eventually the compressive strain limit of the concrete is reached at the ultimate rotation .After the ultimate moment M_u is reached, continued plastic rotation is assumed to occur with no change in applied moment. The beam behaves as if there were a hinge at that point.

If such a plastic hinge forms in a determinate structure, as shown in Fig.1.4, uncontrolled deflection takes place and the structure collapses

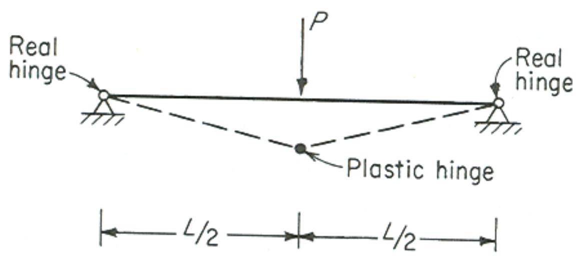


Fig. 1.4 Statically determinate member after the Formation of plastic hinge

As the formation of such hinges in indeterminate structures permits a redistribution of moments with in a beam or frame, stability may be maintained even after the formation of plastic hinges at several sections.

Consider a symmetrically reinforced propped cantilever shown in Fig. 1.5a. Let the load p be increased gradually until the elastic moment at the fixed support, 3/16 PL is equal to the plastic moment capacity of the section M_n. This load is

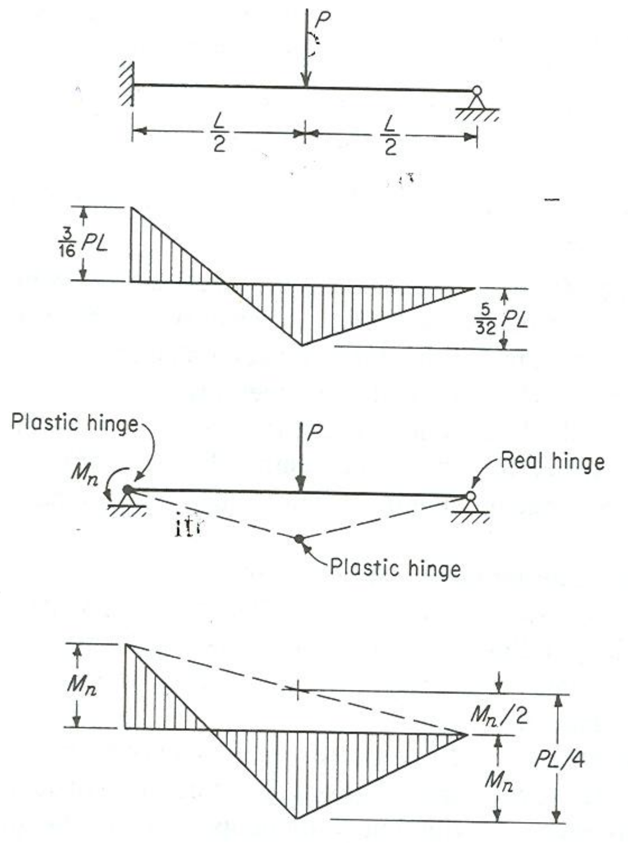
$$P = P_{el} = 16M_n/3L = 5.33M_n/L \qquad (a)$$

At this load, the positive moment under the load is 5/32PL, as shown in Fig.1.5b. The beam still responds elastically every where but at the left support. At that point the actual fixed support can be replaced for purpose of analysis with a plastic hinge offering a known resisting moment M_n. Because a redundant reaction has been replaced by a known moment, the beam is now determinate. The load can be increased further until the moment under the load also becomes equal to M_n, at which load the second hinge forms. The structure is converted in to a mechanism, as shown in Fig. 1.5c, and collapse occurs. The moment diagram at collapse load is shown in Fig.1.5d.

The magnitude of the load causing collapse is easily calculated from the geometry of Fig.1.5d.

$$M_n + M_n/2 = pL/4 \qquad (b)$$

By comparison of eqs (b) and (a), it is evident that an increase in P of 12.5 percent is possible beyond the load which caused the formation of the first plastic hinge, before the beam will actually collapse.



(a)

(b)

(c)

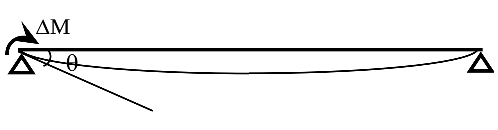
(d)

Fig. 1.5 Indeterminate beam with plastic hinges at support and mid span

Rotation Requirement

The amount of rotation required at plastic hinges for any assumed moment diagram can be found by considering the requirements of compatibility. The member must be bent, under the combined effects of elastic moment and plastic hinges, so that the correct boundary conditions are satisfied at the supports. Usually zero support deflection is to be maintained.

The angle of rotation caused by change in bending moment can be calculated using:



$$\theta = \frac{\Delta ML}{3EI}$$

Where: $EI \cong \frac{M_y}{\psi_y}$, and EI accounts for the cracked section at yielding of tension steel

Rotation Capacity

The capacity of concrete structures to absorb inelastic rotations at plastic hinge locations is not unlimited. The designer adopting full limit analysis in concrete must calculate not only the amount of rotation required at critical sections to achieve the assumed degree of moment redistribution but also the rotation capacity of the members at those sections to ensure that it is adequate.

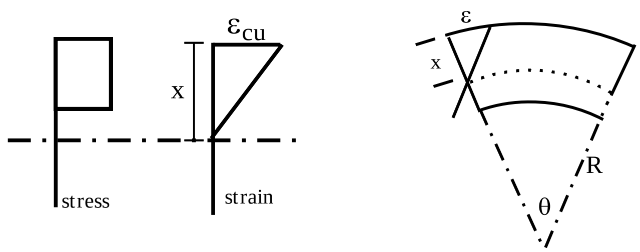
The rotation at the plastic hinge shall be checked and the calculated value shall not exceed certain ultimate rotation θ_p, given by:

$$\theta_p = (\psi_u - \psi_y)L_p$$

Where: ψ_u = curvature at the ultimate limit state

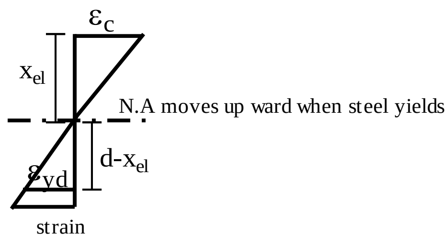
$$= \frac{\varepsilon_{cu}}{x}$$

$$\psi = \frac{1}{R}$$

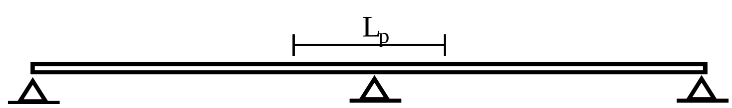


ψ_y = elastic curvature when yielding of steel just starts

$$\psi_y = \frac{\varepsilon_{yd}}{d - x_{el}}$$



L_p = length of plastic hinge (as the plastic rotation is not confined to one cross section but is distributed over a finite length)



$$X_{el} = \xi d$$

$$\xi = \alpha(\rho + \rho')\left[-1 + \sqrt{1 + \frac{2(\rho + \rho' \, d'/d)}{\alpha(\rho + \rho')^2}}\right]$$

$$\alpha = \frac{E_s}{E_c}$$

$$L_p = k_1 k_2 k_3 \left(\frac{z_o}{d}\right)^{1/4} d$$

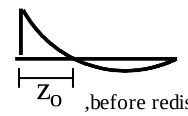
$$k_1 = \begin{cases} 0.7 & \text{for hot rolled steel} \\ 0.9 & \text{for cold worked steel} \end{cases}$$

$$k_2 = \begin{cases} 1.0 & \text{in case of bending} \\ 1+0.5\dfrac{N_d}{N_u} & \text{when normal force is present} \end{cases}$$

K_3 = 0.6, . . ., 0.9 depends on concrete grade

Concrete grade	C15	C25	C35	C45
K3	0.9	0.8	0.7	0.6

z_o is the distance between the plastic hinges and zero point of bending moment



,before redistribution

θ_p is the capacity to rotate after the formation of plastic hinge

1.4 Moment Redistribution as per EBCS-2

A limited amount of redistribution is permitted by EBCS depending up on the degree of ductility with out explicit calculation of rotation requirements and capacities.

The amount of redistribution allowed is dependant on the grade of the concrete and on the ductility characteristics of the reinforcement as well as the neutral axis.

Moments obtained from a linear analysis may be reduced by multiplying by the following reduction coefficient δ provided that the moments are increased in other sections in order to maintain equilibrium. Usually it is the maximum support moments which are reduced, so economizing in reinforcing steel and also reducing congestion at the columns.

a) For continuous beams and for beams in rigid jointed braced frames with span to effective depth ratio not greater than 20,

$$\delta = 0.44 + 1.25\, x/d$$

The neutral axis height, X, is calculated at the ultimate limit state and the term x/d refers to the section where the moment is reduced

b) For other continuous beams and rigid jointed braced frames

$$\delta \geq 0.75$$

c) For sway frames with slenderness ratio of column less than 25

$$\delta \geq 0.9$$