Chapter 3
Stability Analysis

Topics:

Polar Plot

Nyquist Plot

Bode Plot

Frequency Response:

- The magnitude and phase of $G(j\omega)$ can be found from the pole-zero plot for a system

$$G(s)=\frac{k\left(s+z_1\right)\left(s+z_2\right).......\left(s+z_m\right)}{\left(s+p_1\right)\left(s+p_2\right)............\left(s+p_n\right)}$$

Step

1. Plot the position of each pole and zero
2. Mark the position $s=j\omega$.
3. Draw lines from each pole and each zero to the point $s=j\omega$.
4. Measure the length and angles of each of the lines
5. The frequency response function is then:

$$\left|G(j\omega)\right|=\frac{k\times\text{product of the lengths of the lines from zeros}}{\text{product of the lengths of the lines from poles}}$$

$$\angle G(j\omega)=\text{sum of angles of lines from zeros - sum of angles of lines from poles}$$

Bode Plot

- A plot of logarithm of $|G(j\omega)|$ and phase angle of $G(j\omega)$ both plotted against frequency in logarithmic scale.
- The standard procedure is to plot $20log|G(j\omega)|$ and $\phi(\omega)$ versus ω.
- The unit of $20log|G(j\omega)|$ is decibel, dB.
- Allows a rapid and reasonably accurate plot to be made.
- Composite systems will be handled with relative ease.
- Possible to make the plots as accurate as required in a given frequency range.
- Plotted for a unit change in frequency $log(\omega_2/\omega_1)=1$ *or* $\omega_2=10\omega_1$.
- This range of frequency is called a *decade*.

- Consider the transfer function

$$G(s)=k\frac{(s+\omega_1)}{(s+\omega_2)}\Rightarrow G(j\omega)=k\frac{(j\omega+\omega_1)}{(j\omega+\omega_2)}=\frac{k\left(1+j\frac{\omega}{\omega_1}\right)}{(1+j\omega/\omega_2)}.\left(\frac{\omega_1}{\omega_2}\right)$$

$$|G(j\omega)|=k\frac{\omega_1}{\omega_2}\frac{|1+j\omega/\omega_1|}{|1+j\omega/\omega_2|}$$

$$|G(j\omega)|_{db}=20\log\left(k\frac{\omega_1}{\omega_2}\right)+20\log\left[1+\left(\frac{\omega}{\omega_1}\right)^2\right]^{1/2}-20\log\left[1+\left(\frac{\omega}{\omega_2}\right)^2\right]^{1/2}$$

- So, we can add contribution due to the individual magnitude terms.
- The phase graph when there are a number of elements is just the sum of the separate elements.
- The graph covers a greater range of frequencies and it is approximated as a straight line.
- Since Bode plots for a system can be built up from the plots for the individual elements within the transfer function for that system it is useful to consider the plots for element commonly found in the transfer function.

A multiple pole at the origin:

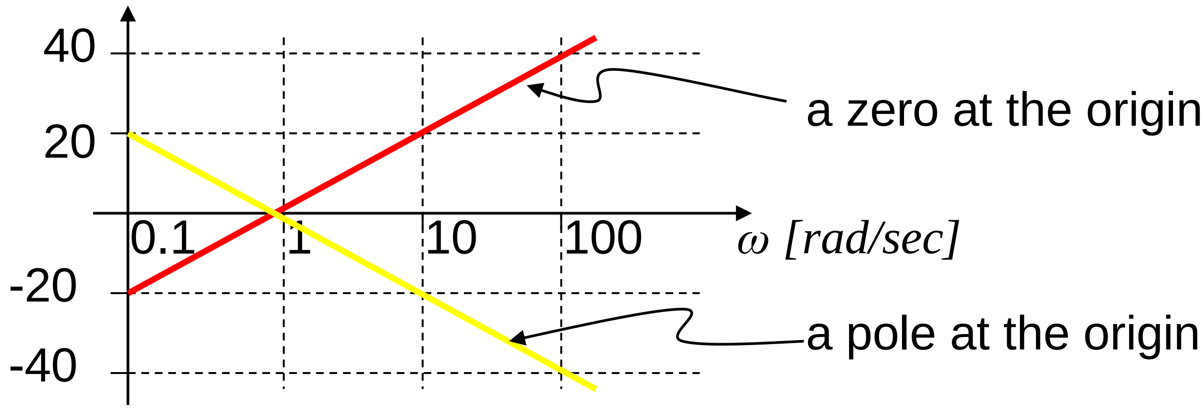
- $G(s)=1/s^m \rightarrow G(j\omega)= 1/(j\omega)^m$
- $\left|G(j\omega)\right|_{dB} = -20 \log \omega^m = -20m \log \omega;\ \phi = -\pi/2$

- That is a straight line of slope -*20m dB/decade*.

A zero at the origin:

- $G(s)=s \rightarrow G(j\omega)= j\omega$
- $\left|G(j\omega)\right|_{db} = 20 \log \omega;\ \phi=\pi/2$

- That is a straight line of slope *20 dB/decade*.



A Real Pole:

$$G(s)=\frac{1}{1+\tau s}\Rightarrow G(j\omega)=\frac{1}{1+\tau(j\omega)}$$

$$\left|G(j\omega)\right|_{dB}=20\log\left[\frac{1}{\sqrt{1+(\omega\tau)^2}}\right]=-10\log\left[1+(\omega\tau)^2\right]$$

$$\tan\phi = -\omega\tau \Rightarrow \phi = -\tan^{-1}(\omega\tau)$$

$$\text{for } \omega << 1/\tau ; (\omega\tau)^2 \cong 0$$

$$\therefore \left|G(j\omega)\right|_{db} = -10\log 1 = 0db$$

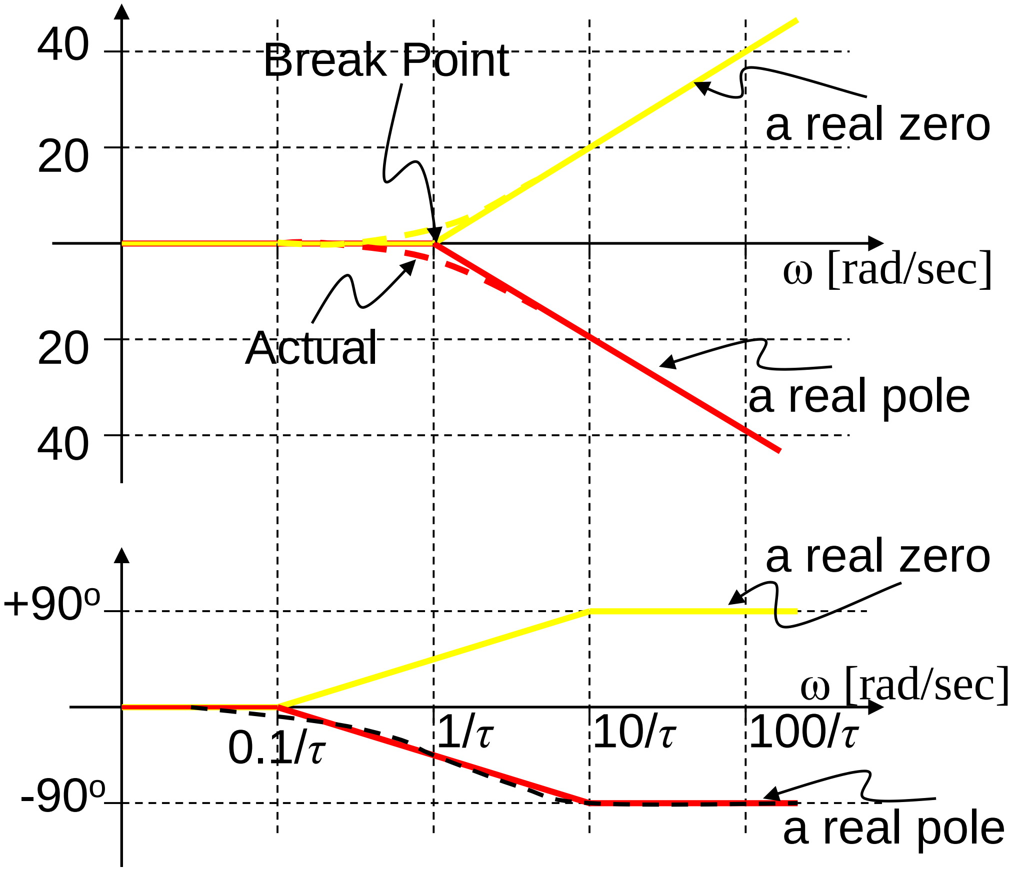
- This tells us that at low frequency there is a straight line.
- For $\omega >> 1/\tau$:

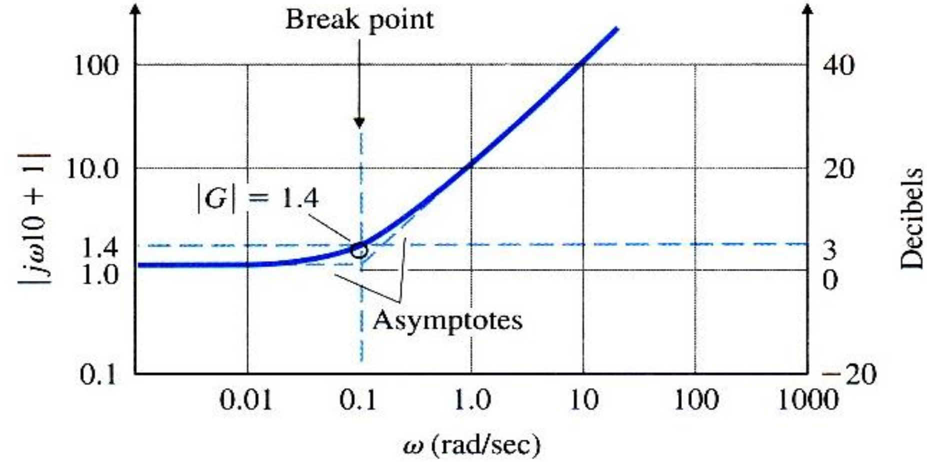
$$\left|G(j\omega)\right|_{dB} = -20 \log \omega\tau$$

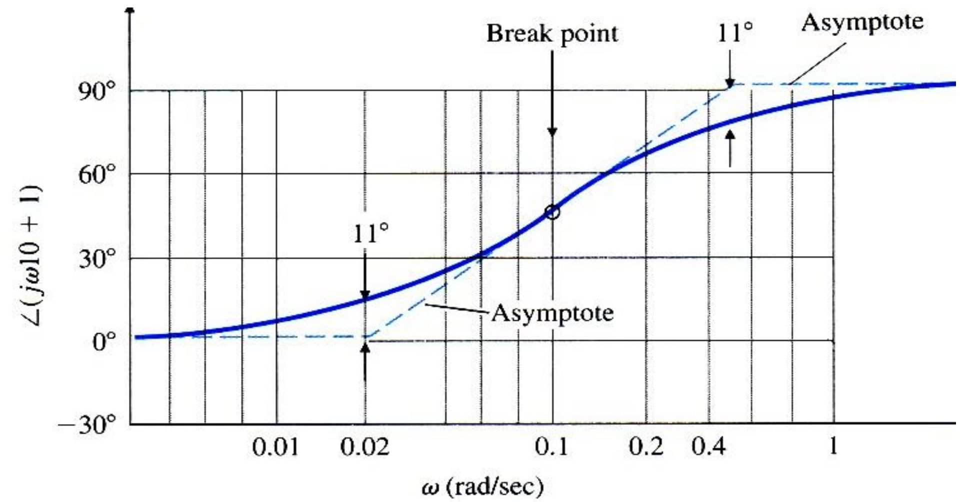
- We get a straight line of slope -*20db/decade*
- when $\omega\tau = 1$.

$$\left|G(j\omega)\right|_{dB} \approx 0$$

- $\omega = 1/\tau$ is called ***break point*** or ***corner frequency***





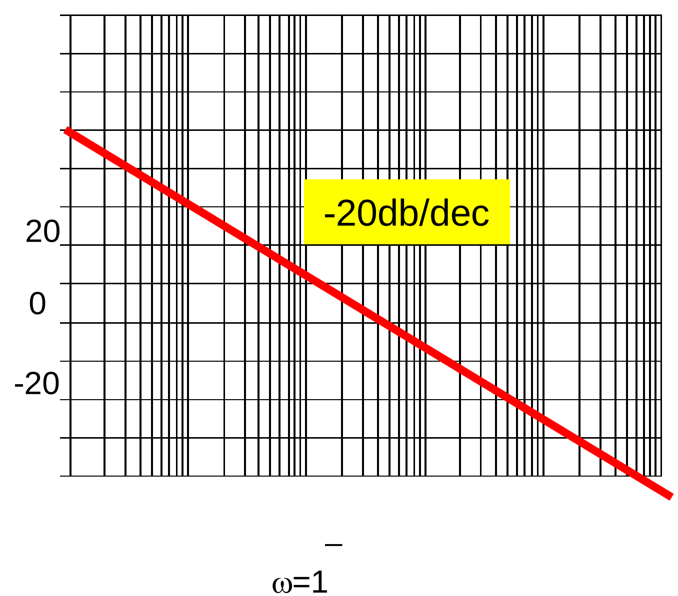


A real zero:

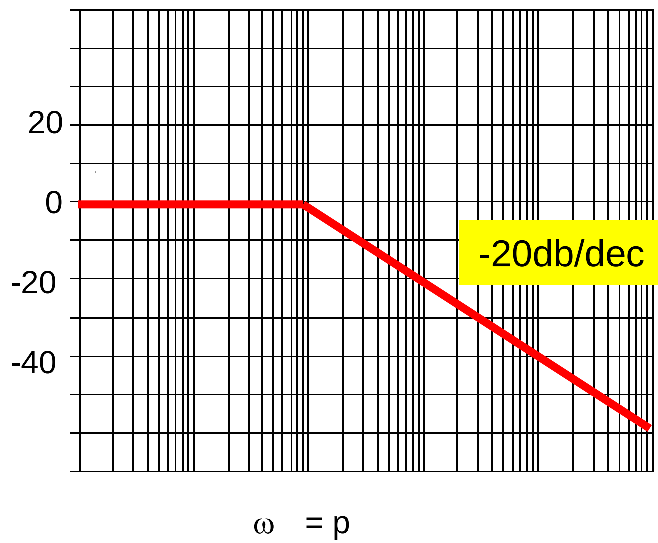
- $G(s)=(1+\tau s) \rightarrow G(j\omega)=(1+j\omega\tau)$
- $\left|G(j\omega)\right|_{db}=20\log\sqrt{1+(\omega\tau)^2}=10\log(1+(\omega\tau)^2)$
- $\tan\phi=\omega\tau$

The gain term, $20\log K_{B,}$ is just so many dB and this is a straight line on Bode paper, independent of omega (radian frequency).

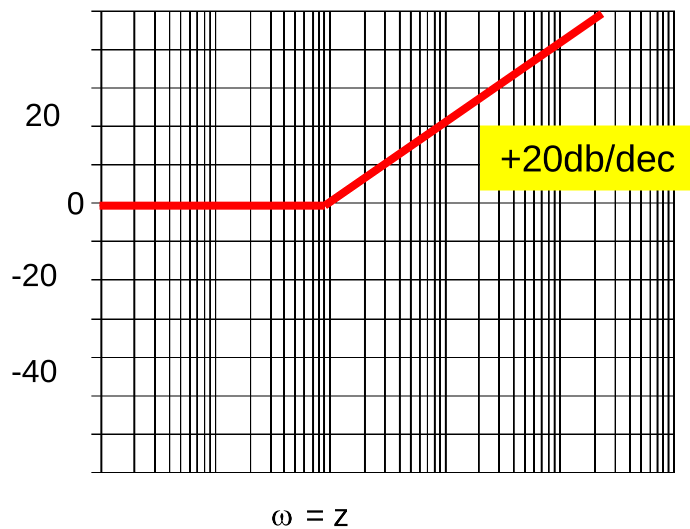
The term, $-20\log|jw| = -20\log w$, when plotted on semi-log paper is a straight line sloping at -20dB/decade. It has a magnitude of 0 at w = 1.



The term, - 20log|(jw/p + 1), is drawn with the following approximation: If w < p we use the approximation that –20log|(jw/p + 1)| = 0 dB, a flat line on the Bode. If w > p we use the approximation of –20log(w/p), which slopes at -20dB/dec starting at w = p. Illustrated below. It is easy to show that the plot has an error of -3dB at w = p and – 1 dB at w = p/2 and w = 2p. One can easily make these corrections if it is appropriate.



When we have a term of $20\log|(jw/z + 1)|$ we approximate it be a straight line of slop 0 dB/dec when $w < z$. We approximate it as $20\log(w/z)$ when $w > z$, which is a straight line on Bode paper with a slope of + 20dB/dec. Illustrated below.



Relative stability

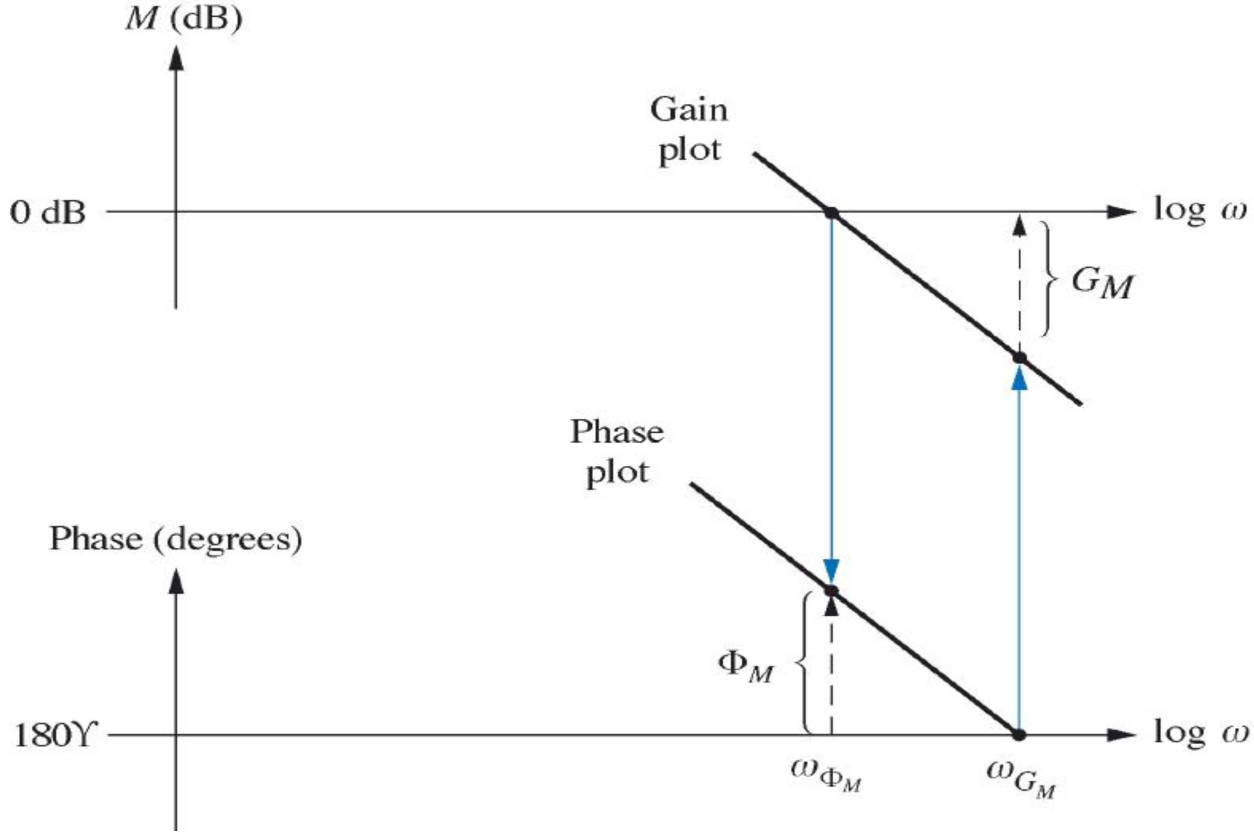
*A transfer function is called **minimum phase** when all the poles and zeros are LHP and **non-minimum-phase** when there are RHP poles or zeros.*

Minimum phase system → Stable

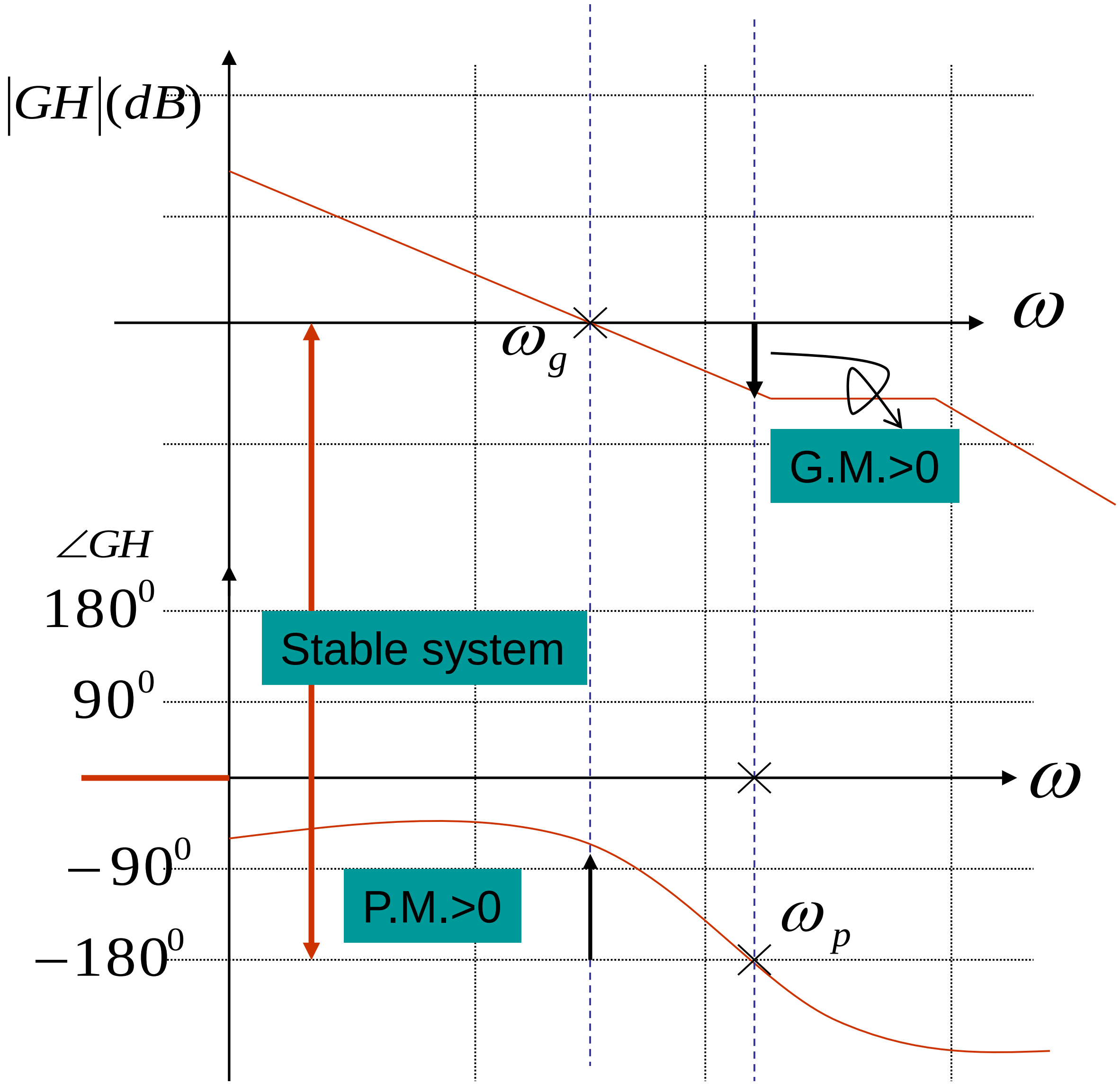
*The **gain margin** (GM) is the distance on the bode magnitude plot from the amplitude at the phase crossover frequency up to the 0 dB point.*
Gain Margin, (GM) = - (dB of GH measured at the phase crossover frequency)

*The **phase margin** (PM) is the distance from -180 up to the phase at the gain crossover frequency.*
Phase Margin (PM) =180+phase of GH measured at the gain crossover frequency

GAIN & PHASE MARGINS IN BODE PLOT



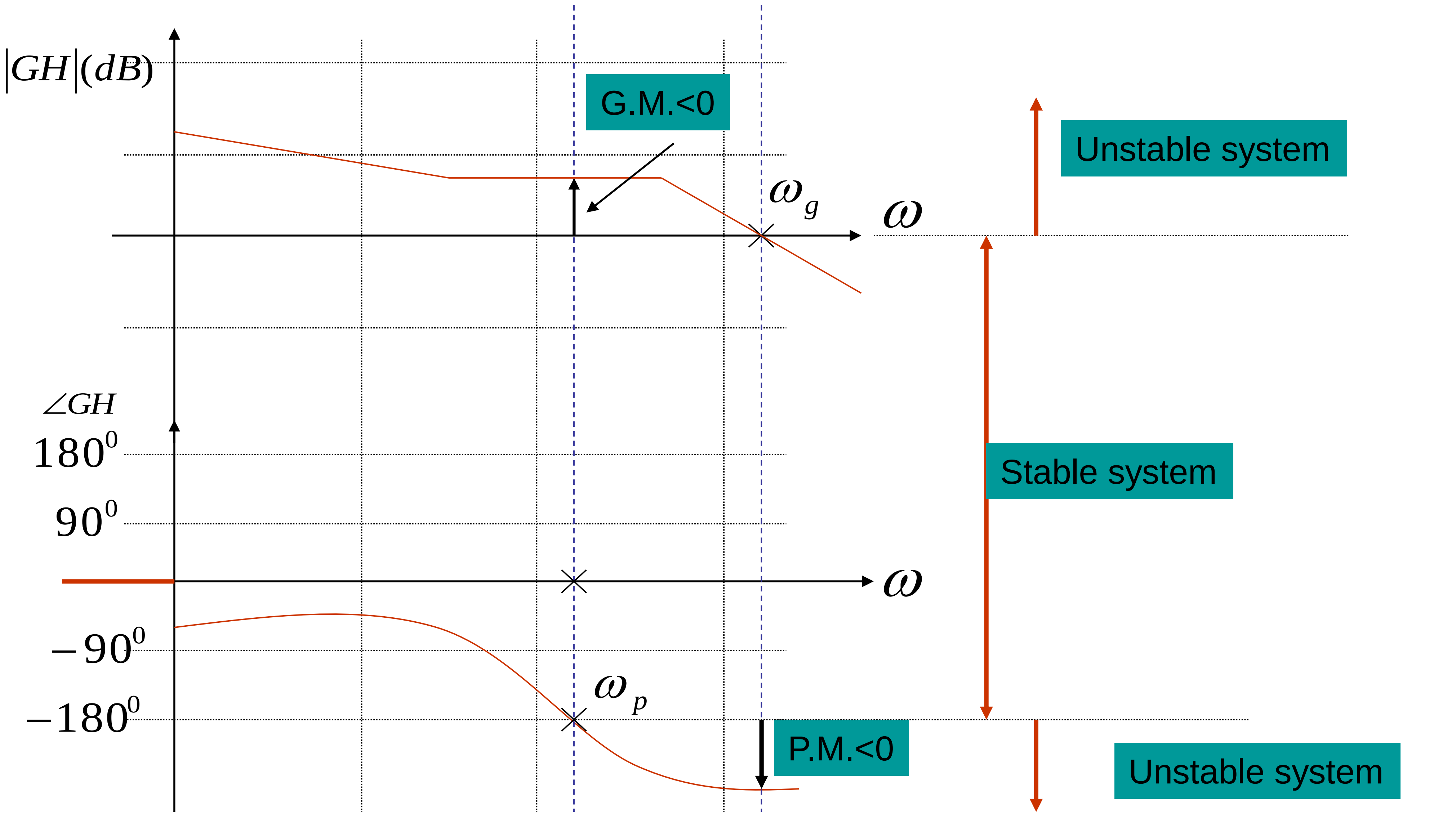
- GM – gain margin
- ΦM – phase margin
- ω_{GM}- Gain crossover frequency
- $\omega_{\Phi M}$- Phase crossover frequency
- Note that, negative gain or phase margin means that the system is not stable



$$(-1,0) \Rightarrow \begin{cases} 0dB \\ -180^0 \end{cases}$$

Gain crossover frequency: ω_g

phase crossover frequency: ω_p



|GH|(dB)
G.M.<0
Unstable system
ω_g
ω
∠GH
180°
90°
Stable system
ω
−90°
ω_p
−180°
P.M.<0
Unstable system

1Q.) Draw Bode plot for the function shown.

$$G(s) = \frac{1}{s+2} \qquad G(j\omega) = \frac{1}{j\omega+2} = \frac{2-j\omega}{\omega^2+4}$$

The magnitude frequency response is

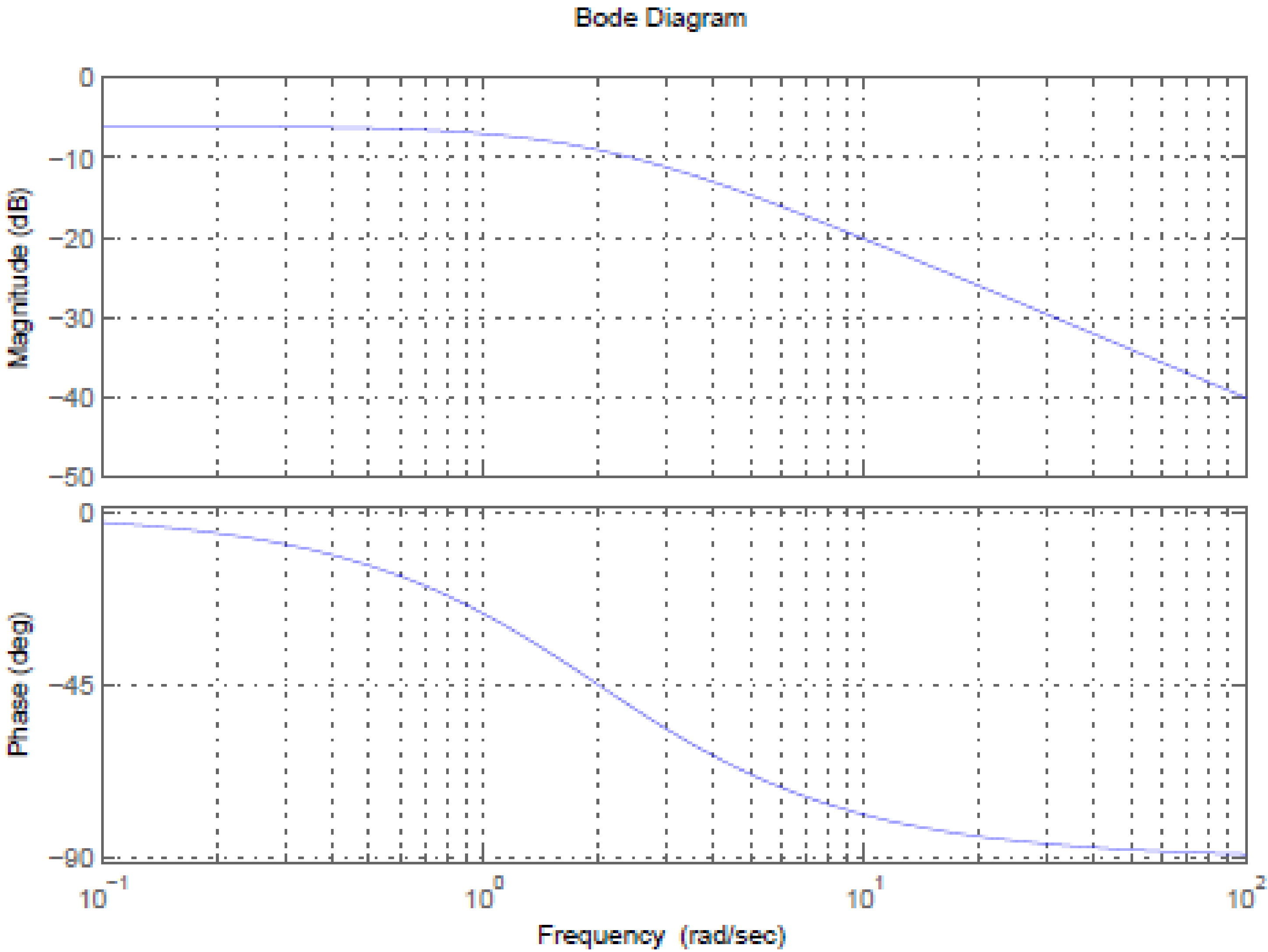
$$M(\omega) \;=\; |G(j\omega)| = \frac{1}{\sqrt{\omega^2+4}}$$

The phase frequency response is

$$\phi(\omega) \;=\; \angle G(j\omega) = -\tan^{-1}(\omega/2)$$

Frequency response for system $G(s) = \frac{1}{s+2}$

ω	$20\log\frac{1}{\sqrt{\omega^2+4}}$	$-\tan^{-1}(\omega/2)$
0	-6.0206	0
0.1	-6.0314	-2.8624
0.2	-6.0638	-5.7106
0.4	-6.1909	-11.3099
0.6	-6.3949	-16.6992
0.8	-6.6652	-21.8014
1	-6.9897	-26.5651
2	-9.0309	-45.0000
4	-13.0103	-63.4349
6	-16.0206	-71.5651
8	-18.3251	-75.9638
10	-20.1703	-78.6901
20	-26.0638	-84.2894
40	-32.0520	-87.1376
60	-35.5678	-88.0908
80	-38.0645	-88.5679
100	-40.0017	-88.8542



The figure above shows the separate plots for magnitude and phase response diagrams, where the magnitude diagram is $20\log\frac{1}{\sqrt{\omega^2+4}}$ vs. $\log\omega$, and the phase diagram is $-\tan^{-1}(\omega/2)$ vs. $\log\omega$.

2Q.) Given the transfer function make plot of Bode magnitude for function given below

G(s) = 50000*(s+10)/(s+1)(s+500)

$$G(jw) = \frac{50,000(jw+10)}{(jw+1)(jw+500)}$$

First: Always, always, always get the poles and zeros in a form such that the constants are associated with the jw terms. In the above example we do this by factoring out the 10 in the numerator and the 500 in the denominator.

$$G(jw) = \frac{50,000x10(jw/10+1)}{500(jw+1)(jw/500+1)} = \frac{100(jw/10+1)}{(jw+1)(jw/500+1)}$$

Second: When you have neither poles nor zeros at 0, start the Bode at $20\log_{10}K = 20\log_{10}100 = 40$ dB in this case.

continued…

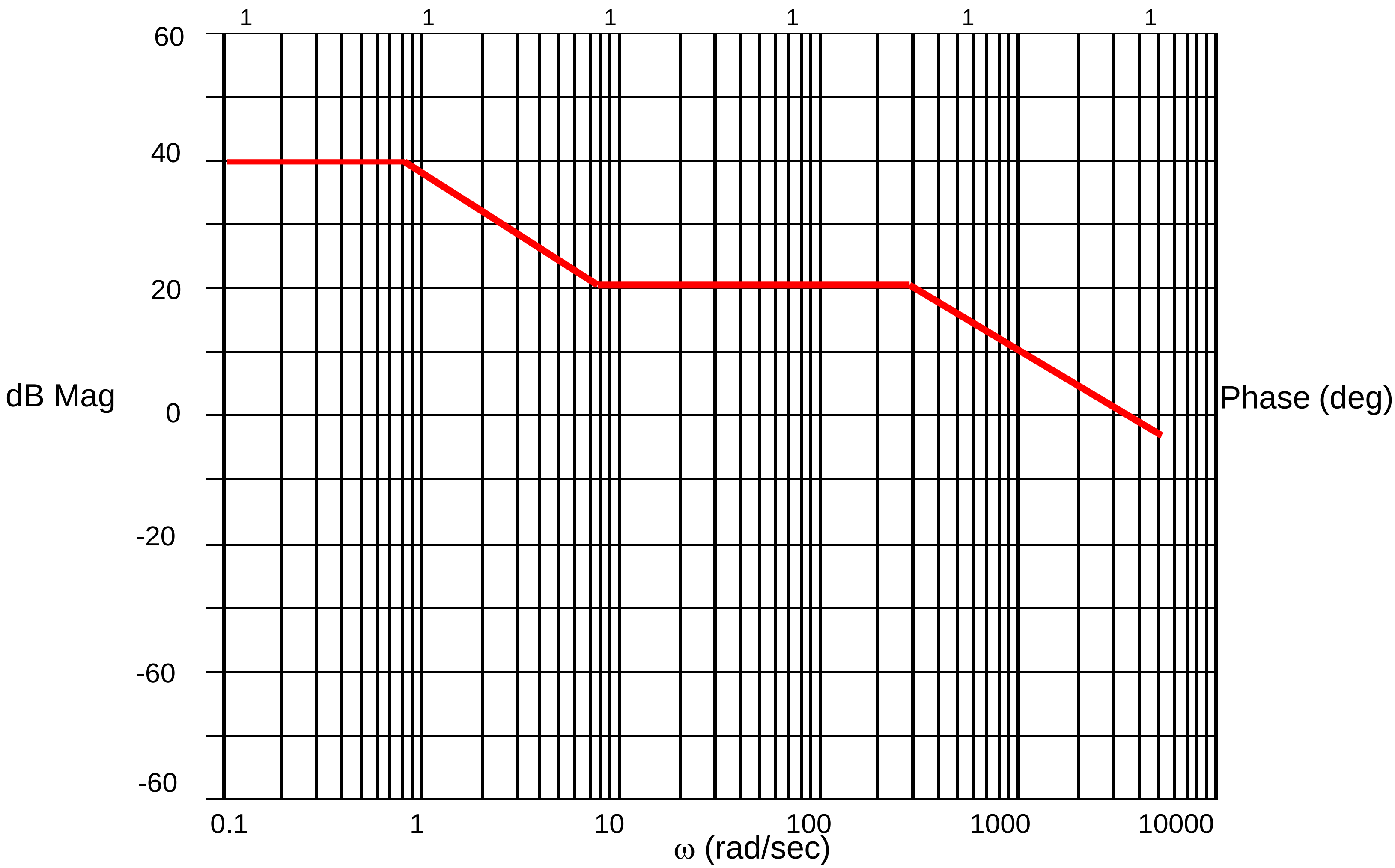
Third: Observe the order in which the poles and zeros occur.
This is the secret of being able to quickly sketch the Bode.
In this example we first have a pole occurring at 1 which causes the Bode to break at 1 and slope – 20 dB/dec. Next, we see a zero occurs at 10 and this causes a slope of +20 dB/dec which cancels out the – 20 dB/dec, resulting in a flat line (0 db/dec). Finally, we have a pole that occurs at w = 500 which causes the Bode to slope down at – 20 dB/dec.

We are now ready to draw the Bode.

Before we draw the Bode we should observe the range over which the transfer function has active poles and zeros. This determines the scale we pick for the w (rad/sec) at the bottom of the Bode.

The dB scale depends on the magnitude of the plot and experience is the best teacher here.

Bode Plot Magnitude for 100(1 + jw/10)/(1 + jw/1)(1 + jw/500)



Phase for Bode Plots

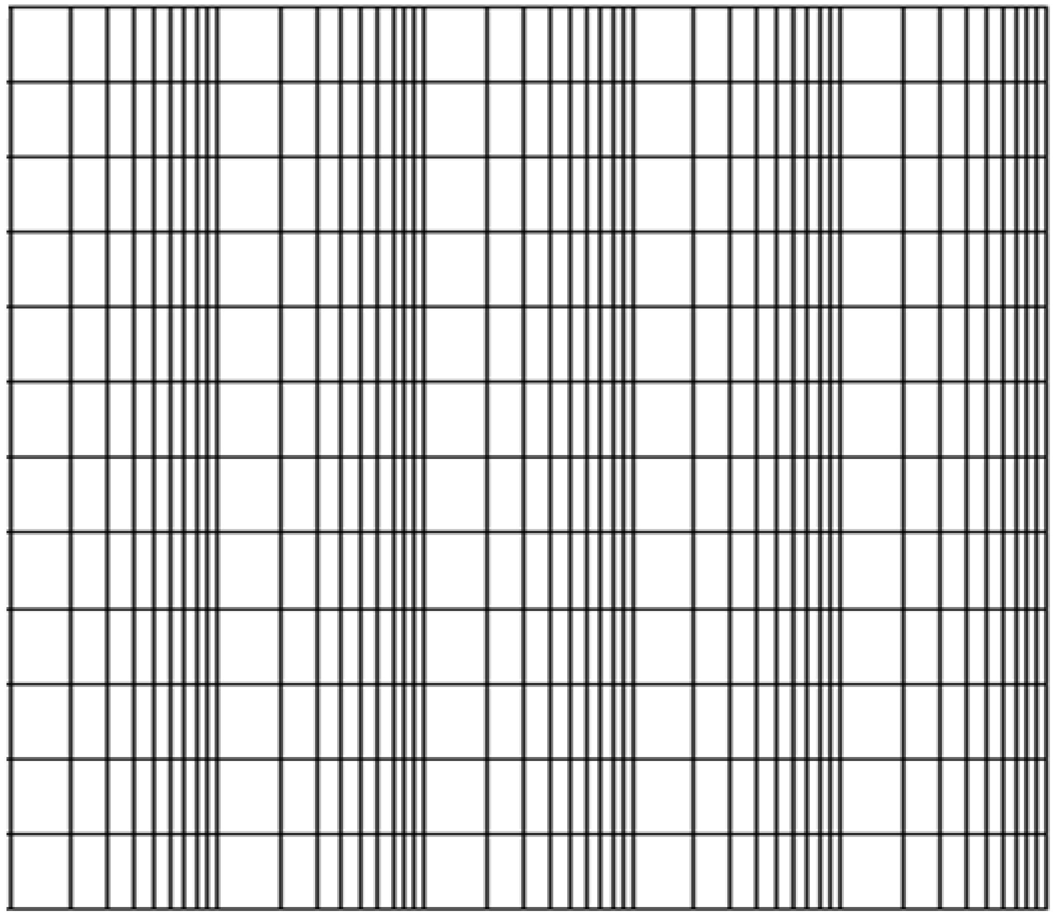
We express the angle as follows:

$$\angle G(jw) = \tan^{-1}(w/10) - \tan^{-1}(w/1) - \tan^{-1}(w/500)$$

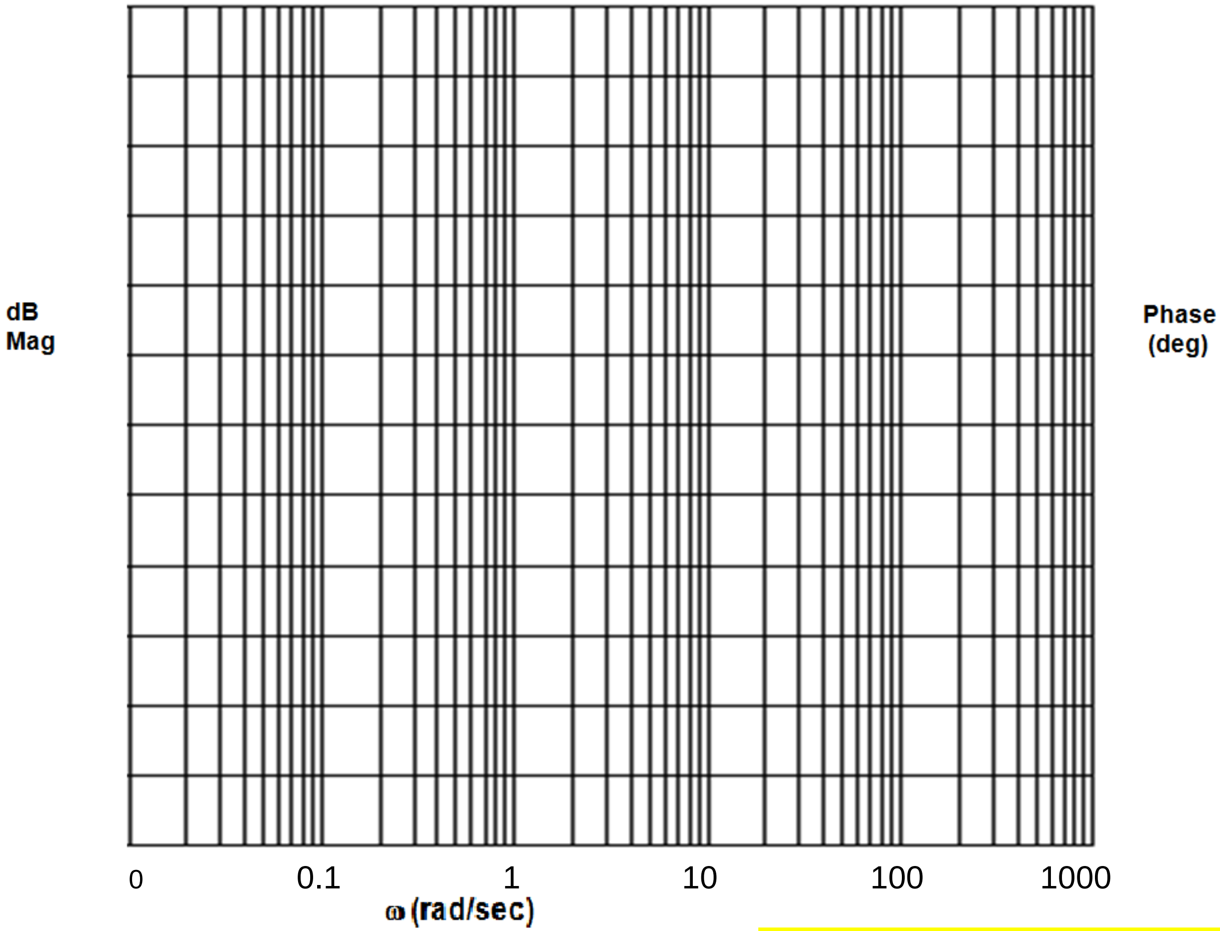
We are essentially taking the angle of each pole and zero.
Each of these are expressed as the $\tan^{-1}$(j part/real part)

Usually, about 10 to 15 calculations are sufficient to determine a good idea of what is happening to the phase.

Appendix
(Logarithmic graph sheet or semilog graph sheet)



Appendix (For Bode Plot)



This is a sheet of 5 cycle, semi-log paper.
This is the type of paper usually used for preparing Bode plots.

Polar plot

The polar plot is easily useful for investigating frequency response.

Example

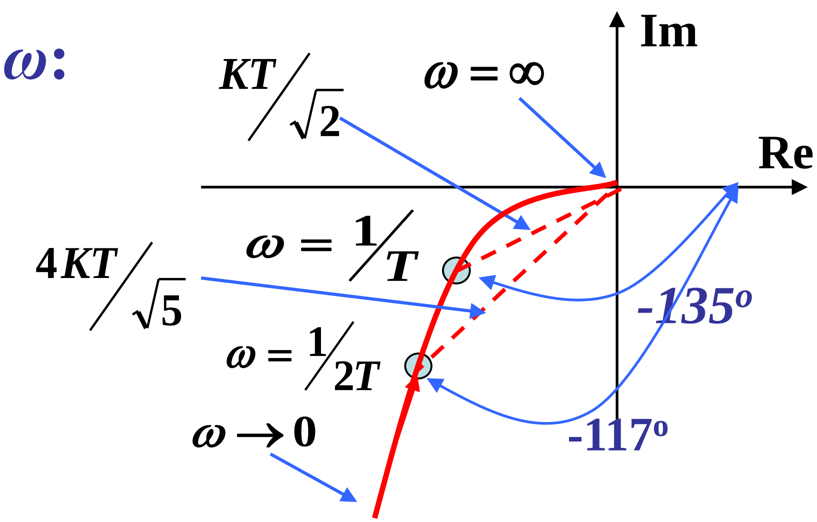
$$G(s)=\frac{K}{s(Ts+1)} \Rightarrow G(j\omega)=G(s)\big|_{s=j\omega}=\frac{K}{j\omega(j\omega T+1)}$$

The magnitude and phase response:

$$A(\omega)=|G(j\omega)|=\frac{K}{\omega\sqrt{1+(\omega T)^2}}\ ;\ \varphi(\omega)=\angle G(j\omega)=-[90^o+tg^{-1}(\omega T)]$$

Calculate $A(\omega)$ and $\varphi(\omega)$ for different ω:

$\omega=$	0	$1/2T$	$1/T$	∞
$A(\omega)=$	∞	$4KT/\sqrt{5}$	$KT/\sqrt{2}$	0
$\varphi(\omega)=$	-90^o	-117^o	-135^o	-180^o



3Q.) Draw Polar plot for the function shown.

$$G(s) = \frac{1}{s+2}$$

Frequency response for system $G(s) = \frac{1}{s+2}$

ω	$20 \log \frac{1}{\sqrt{\omega^2+4}}$	$-\tan^{-1}(\omega/2)$
0	-6.0206	0
0.1	-6.0314	-2.8624
0.2	-6.0638	-5.7106
0.4	-6.1909	-11.3099
0.6	-6.3949	-16.6992
0.8	-6.6652	-21.8014
1	-6.9897	-26.5651
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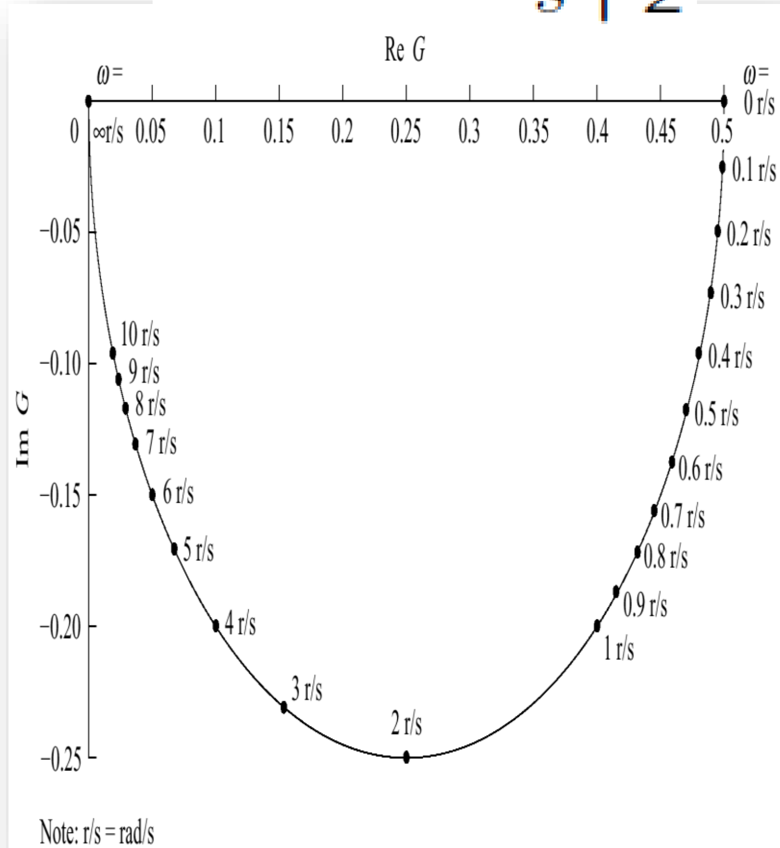


The figure above shows the polar plot for the frequency response. It is a plot of

$$\frac{1}{\sqrt{\omega^2 + 4}} \angle -\tan^{-1}(\omega/2)$$

for different ω.

4Q.) Make a Polar plot of the function shown below and find the gain margin:

$$G(s) = \frac{1}{s(s+2)^2}$$

$$G(j\omega) = \frac{1}{j\omega(j\omega+2)^2}$$

$$G(j\omega) = \frac{1}{j\omega(j\omega+2)^2}\frac{(2-j\omega)^2}{(2-j\omega)^2} = -\frac{j(2-j\omega)^2}{\omega(\omega^2+4)^2}$$

$$= -\frac{j(4-\omega^2-j4\omega)}{\omega(\omega^2+4)^2}$$

$$= \frac{-4\omega + j(\omega^2-4)}{\omega(\omega^2+4)^2}$$

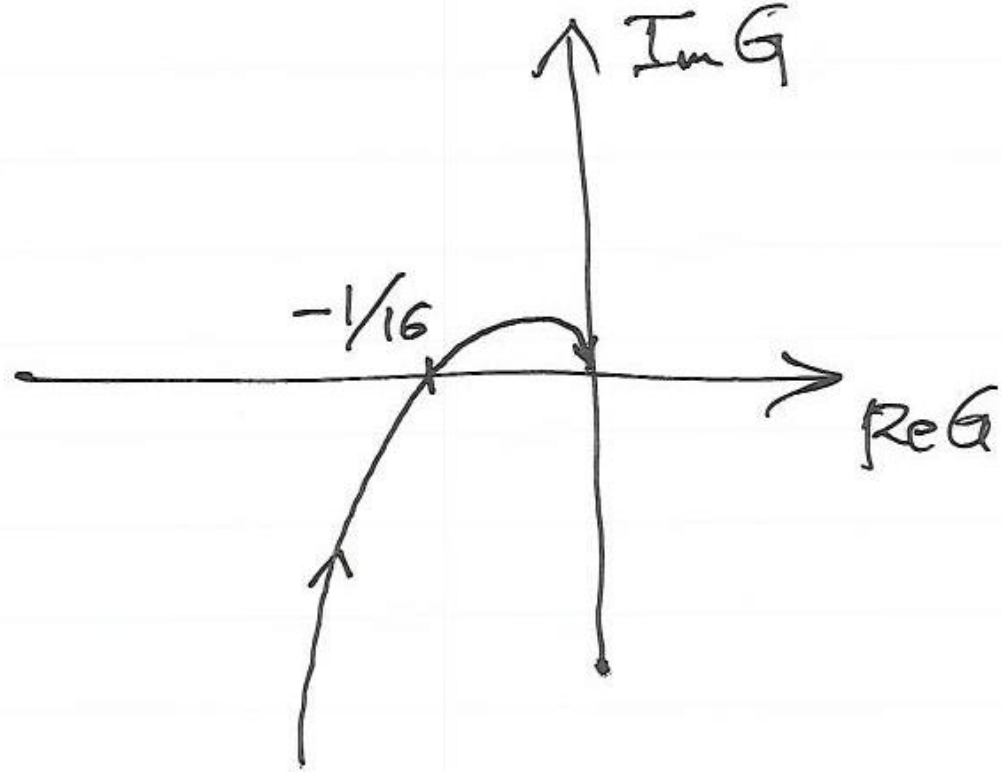
$$\mathrm{Re}\,G(j\omega) = -\frac{4}{(\omega^2+4)^2}, \quad \mathrm{Im}\,G(j\omega) = \frac{\omega^2-4}{\omega(\omega^2+4)^2}$$

Continued...

$\mathrm{Im}\, G(j\omega^*) = 0 \quad \text{for } \omega^* = 2$

$\mathrm{Re}\, G(j\omega^*) = -\frac{1}{16}$

$GM = 16$



The Nyquist-criterion

A method to investigate the stability of a system in terms of the open-loop frequency response.

The argument principle(Cauchy's theorem)

Assume:

$$G(s)H(s)=\frac{K_1(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)} \qquad n>m$$

$$here: \quad z_i \rightarrow open\text{-}loop\ zeros; \quad p_j \rightarrow open\text{-}loop\ poles.$$

Make :

$$F(s)=1+G(s)H(s)=1+\frac{K_1(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}$$

$$=\frac{(s-p_1)(s-p_2)...(s-p_n)+K_1(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}$$

$$=\frac{K_F(s-s_1)(s-s_2)...(s-s_n)}{(s-p_1)(s-p_2)...(s-p_n)} \qquad s_i \rightarrow zeros\ of\ the\ F(s)$$

Note: $s_i \rightarrow$ **the zeros of the** $F(s)$**, also the roots of the** $1+G(s)H(s)=0$

The argument principle

here: ***N*** —— ***number of the F(s) locus encircling the origin of the F(s)-plane in the counterclockwise direction.***

P —— ***number of the zeros of the F(s) encircled by the path Γ in the s-plane.***

Z —— ***number of the poles of the F(s) encircled by the path Γ in the s-plane.***

Nyquist criterion

If we choose the closed path *Γ* so that the *Γ* encircles the entire right hand of the s-plane but not pass through any zeros or poles of F(s) shown in Fig.

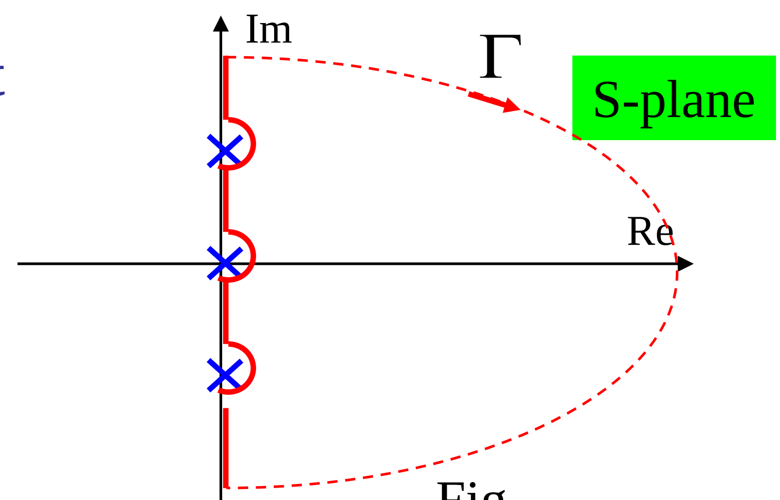


Fig.

The path *Γ* is called the Nyquist-path.

Nyquist criterion

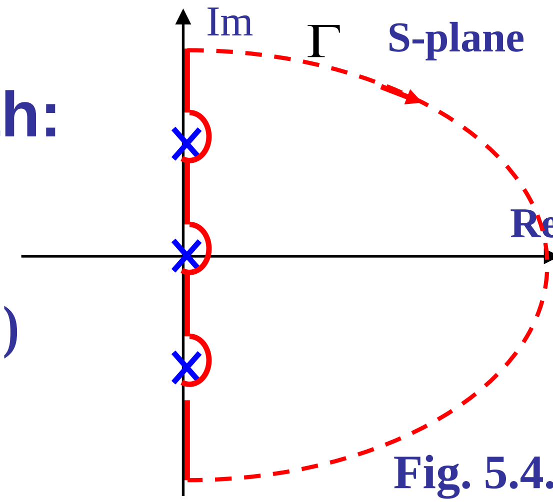


Fig. 5.4.2

When s travels along the the Nyquist-path:

$$F(s)=1+G(s)H(s)\big|_{s=j\infty}=1\leftarrow G(s)H(s)=0$$

$$F(s)\big|_{s=\pm j\omega}=1+G(j\omega)H(j\omega)\rightarrow G(j\omega)H(j\omega)=-1+F(s)$$

Because the origin of the *F(s)*-plane is equivalent to the point (*-1, j0*) of the *G(jω)H(jω)*-plane, we have another statement of the argument principle:

When *ω* vary from -∞ (*or 0*) →+ ∞, *G(jω)H(jω)* Locus mapped in the *G(jω)H(jω)*-plane will encircle the point (*-1, j0*) in the counterclockwise direction:

$$N=P-Z\ \ [or\ N=(P-Z)/2\ \ \text{for}\ \omega\ \text{from}\ 0\rightarrow+\infty\ ]$$

here: *P* — the number of the poles of G(s)H(s) in the right hand of the s-plane.

Z — the number of the zeros of F(s) in the right hand of the s-plane.

Nyquist-criterion

If the systems are stable, should be *Z = 0*, then we have:

The sufficient and necessary condition of the stability of the linear systems is : When *ω* vary from -∞ (*or 0*) →+ ∞, the G(j*ω*)H(j*ω*) Locus mapped in the G(j*ω*)H(j*ω*)-*plane* will encircle the point (-1, j0) as *P* (or *P/2*) times in the counterclockwise direction.

——Nyquist criterion

Here: P — the number of the poles of G(s)H(s) in the right hand of the s-plane.

Discussion :

i) If the open loop systems are stable, that is *P = 0*, then:

for the stable open-loop systems, The sufficient and necessary condition of the stability of the closed-loop systems is :

When *ω* vary from -∞ (*or 0*) →+ ∞, the *G(jω)H(jω)* locus mapped in the *G(jω)H(jω)-plane* will not encircle the *point* (-1, *j0*).

Nyquist-criterion

ii) Because that the *G(jω)H(jω)* locus encircles the point (*-1, j0*) means that the *G(jω)H(jω)* locus traverse the left real axis of the point (*-1, j0*) , we make:

G(jω)H(jω) **Locus traverses the left real axis of the point (*-1, j0*) in the counterclockwise direction —"*positive traversing*".**

G(jω)H(jω) **Locus traverses the left real axis of the point (*-1, j0*) in the clockwise direction —"*negative traversing*".**

Then we have another statement of the Nyquist criterion:

The sufficient and necessary condition of the stability of the linear systems is : When *ω* vary from -∞ (*or 0*) →+ ∞ , the number of the net "positive traversing" is *P* (*or P/2*).

***Here*: *the net "positive traversing" —— the difference between the number of the "positive traversing" and the number of the "negative traversing"*.**

Nyquist-criterion

Example 1

The polar plots of the open loop systems are shown in Fig. below determine whether the systems are stable.

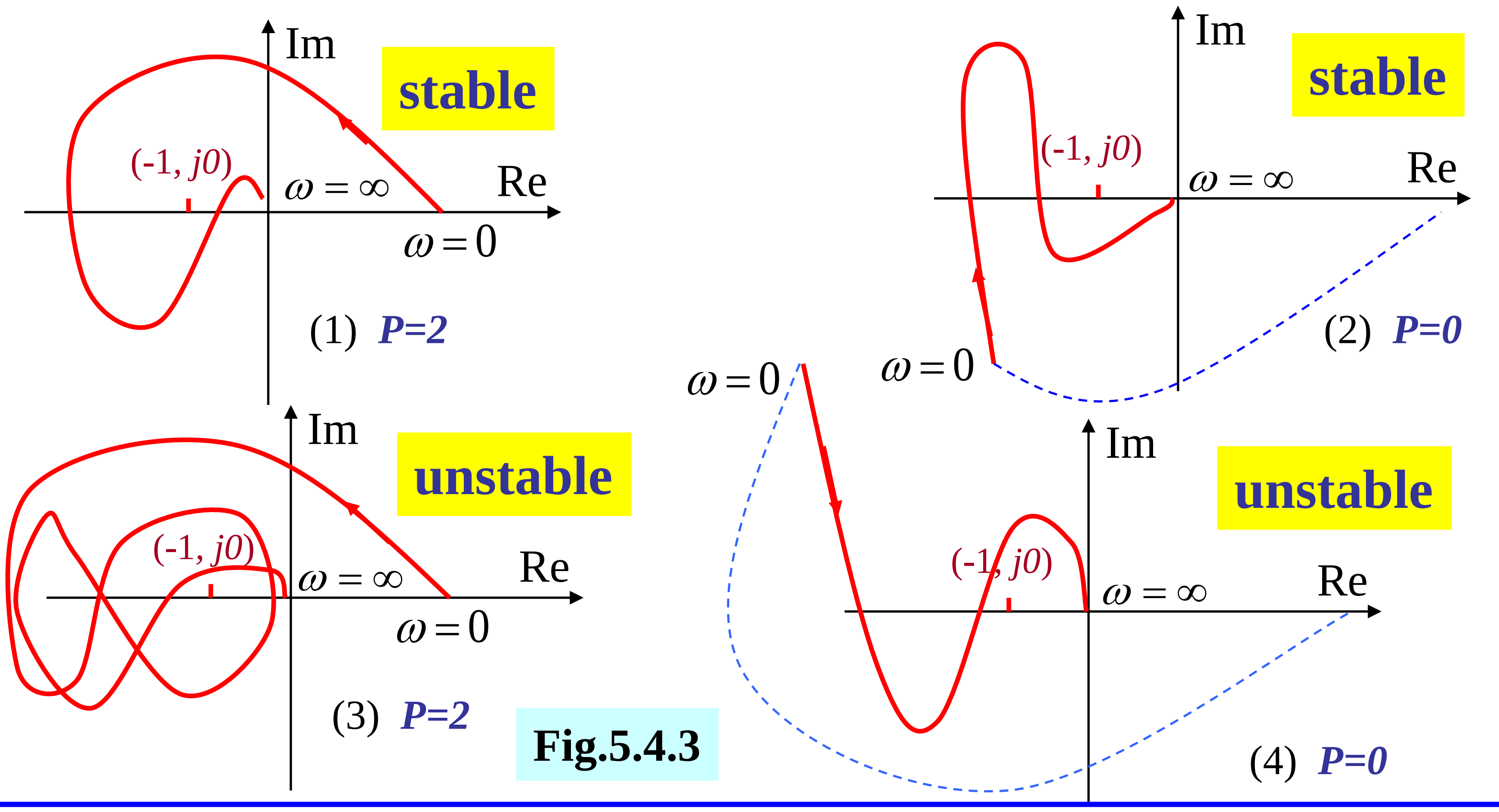


Fig.5.4.3

Nyquist-criterion

Note: **the system with the poles (or zeros) at the imaginary axis**

Example 2 $$G(s)H(s) = \frac{10}{s(s+1)(0.5s+1)}$$

There is a pole $s = 0$ at the origin in this system, but the Nyquist path can not pass through any poles of $G(s)H(s)$.

Idea: **We consider a semicircular detour around the pole ($s = 0$) represented by setting** $s = \varepsilon e^{j\phi}$ $(\varepsilon \to 0)$

at the $s = 0$ point we have:

$$\omega = 0^- \Rightarrow s = \varepsilon e^{-j90^o} \Rightarrow G(j0^-)H(j0^-) = \frac{1}{\varepsilon e^{-j90^o}} = \frac{1}{\varepsilon} e^{j90^o}$$

$$\omega = 0 \Rightarrow s = \varepsilon e^{j0^o} \Rightarrow G(j0)H(j0) = \frac{1}{\varepsilon e^{j0^o}} = \frac{1}{\varepsilon} e^{j0^o}$$

$$\omega = 0^+ \Rightarrow s = \varepsilon e^{j90^o} \Rightarrow G(j0^+)H(j0^+) = \frac{1}{\varepsilon e^{j90^o}} = \frac{1}{\varepsilon} e^{-j90^o}$$

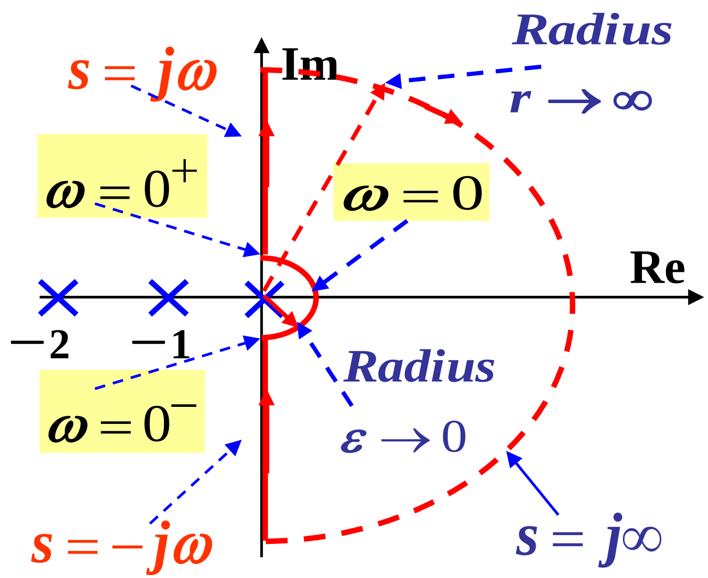
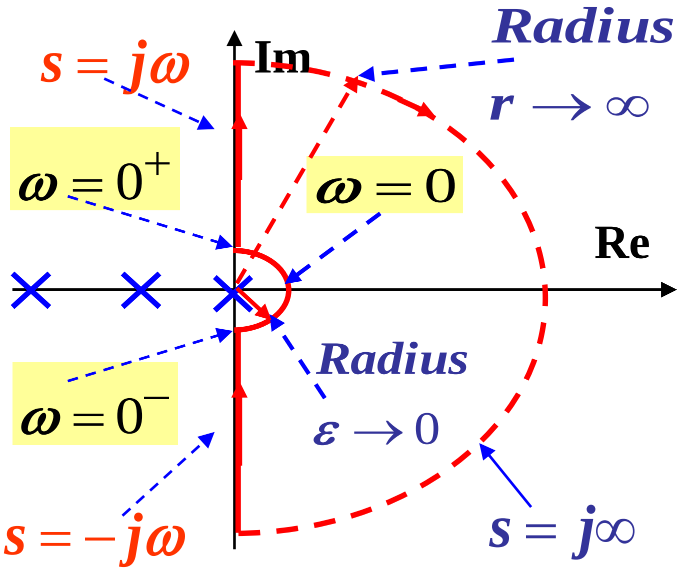


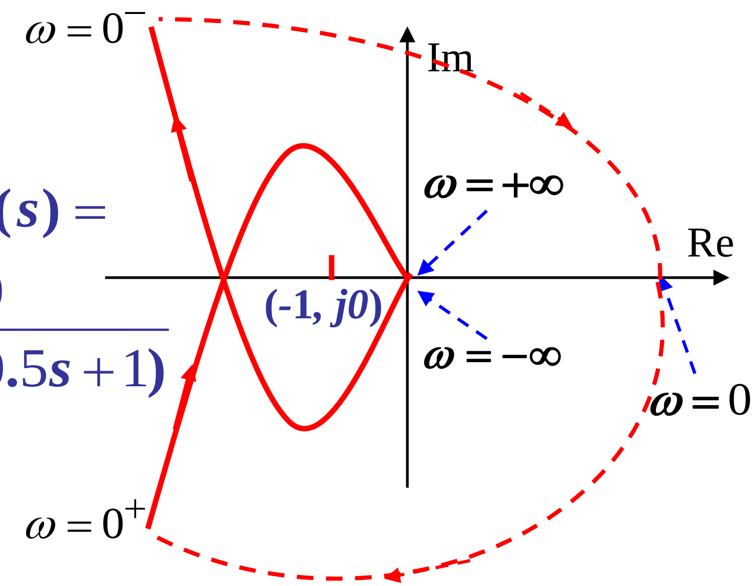
Fig. 5.4.4

Nyquist-criterion

It is obvious that there is a phase saturation of the *G*(*jω*)*H*(*jω*) at *ω*=0, and the magnitude of the *G*(*jω*)*H*(*jω*) is infinite at *ω*=0.



$$G(s)H(s) = \frac{10}{s(s+1)(0.5s+1)}$$



In terms of above discussion , we can plot the system's polar plot shown as Fig.

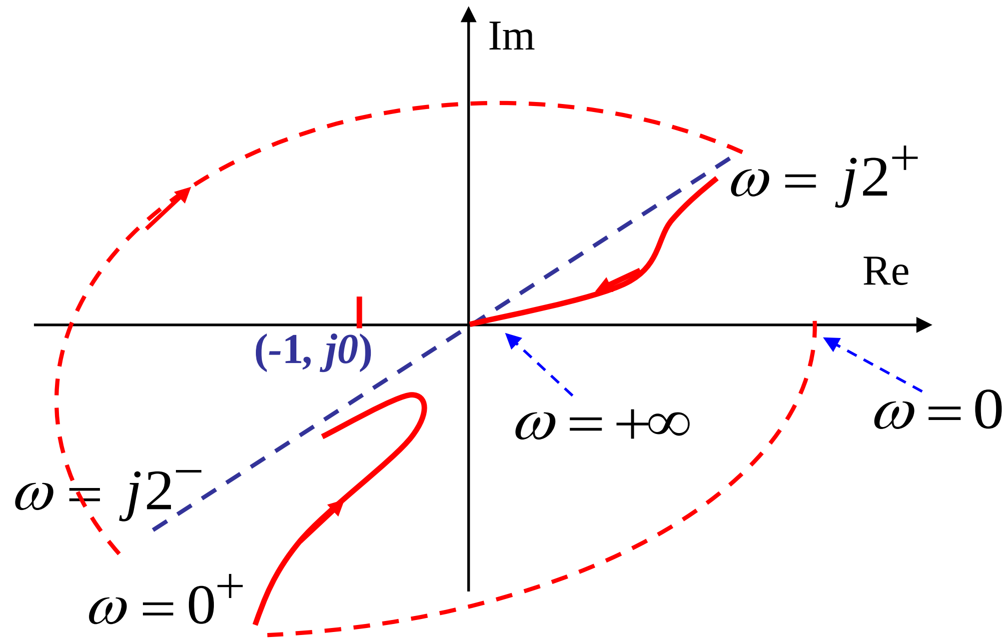
The closed loop system is unstable.

Example.3

$$G(s)H(s)=\frac{10}{s(s+1)(s^2+4)}=\frac{10}{s(s+1)(s-j2)(s+j2)}$$

Determine the stability of the system applying Nyquist criterion.

Solution



The closed loop system is unstable.

Application of the Nyquist criterion in the Bode diagram

Application of the Nyquist criterion in the Bode diagram

$G(j\omega)H(j\omega)$ locus traverses the left real axis of the point (-1, *j0*) in $G(j\omega)H(j\omega)$*-plane* → $L(\omega)\geq 0dB$ and $\varphi(\omega)=-180^o$ in Bode diagram

We have the Nyquist criterion in the Bode diagram :

The sufficient and necessary condition of the stability of the linear closed loop systems is : When ω vary from $0\rightarrow+\infty$, the number of the net "positive traversing" is $P/2$.

***Here*: the net "positive traversing" —the difference between the number of the "positive traversing" and the number of the "negative traversing" in all $L(\omega)\geq 0dB$ ranges of the open-loop system's Bode diagram.**

"positive traversing" — $\varphi(\omega)$ traverses the "-180^o line" from below to above in the open-loop system's Bode diagram; "negative traversing" — $\varphi(\omega)$ traverses the "-180^o line" from above to below.

Nyquist criterion and the relative stability

(Relative stability of the control systems)

In frequency domain, the relative stability could be described by the “gain margin” and the “phase margin”.

1. Gain margin K_g

$$K_g = \frac{1}{\left|G(j\omega)H(j\omega)\right|}\Bigg|_{\omega=\omega_g} \qquad K_g(dB) = -20\log\left|G(j\omega)H(j\omega)\right|\Big|_{\omega=\omega_g}$$

$$\omega_g:\ \angle G(j\omega)H(j\omega)\Big|_{\omega=\omega_g} = -180^0 \rightarrow \textit{Phase-crossover frequency}$$

2. Phase margin γ_c

$$\gamma_c = \angle G(j\omega)H(j\omega)\Big|_{\omega=\omega_c} - (-180^0) = \angle G(j\omega_c)H(j\omega_c) + 180^0$$

$$\omega_c:\ \left|G(j\omega)H(j\omega)\right|\Big|_{\omega=\omega_c} = 1 \rightarrow \textit{Gain-crossover frequency}$$

3. Geometrical and physical meanings of the K_g and γ_c

Nyquist criterion and the relative stability

The geometrical meanings is shown in Fig.

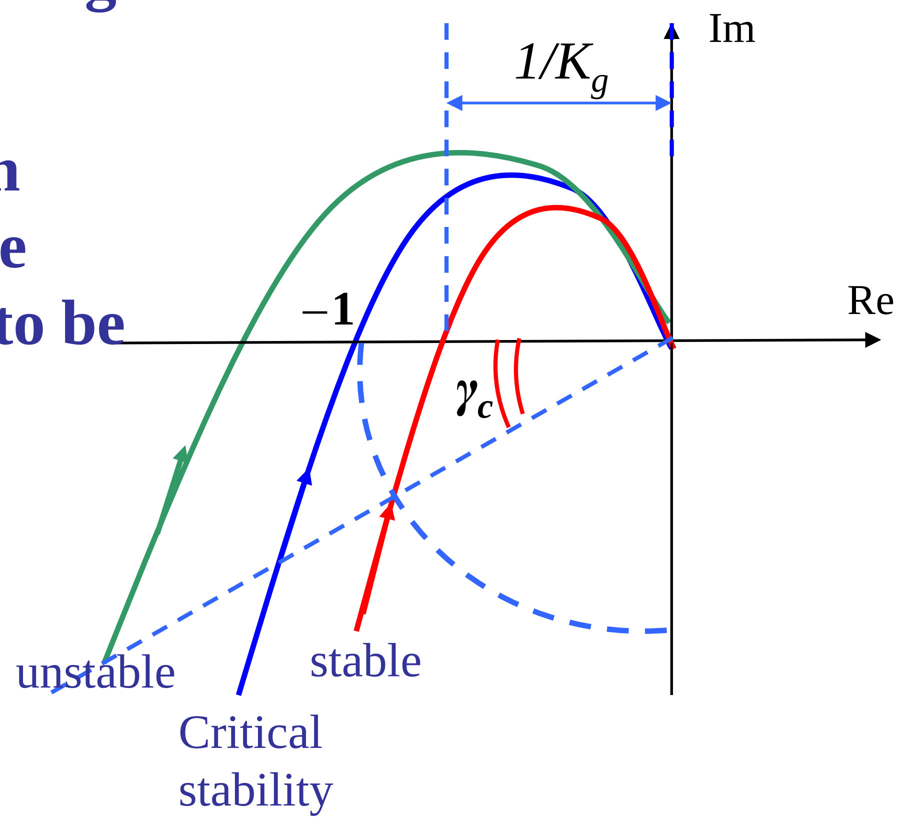
The physical signification :

***K_g*— amount of the open-loop gain in decibels that can be allowed to increase before the closed-loop system reaches to be unstable.**

For the minimum phase system:

$K_g>1$

the closed loop system is stable .



***γ_c*—amount of the phase shift of $G(j\omega)H(j\omega)$ to be allowed before the closed-loop system reaches to be unstable.**

For the minimum phase system: ***$\gamma_c>0$ the closed loop system is stable .***