WOLLO UNIVERSITY
KIOT

SCHOOL OF MECHANICAL AND CHEMICAL ENGINERING

Mathematical Modelling of Physical Systems

Definition:

- A physical system is a collection of physical objects connected in some designed pattern to serve some prescribed objective.

- Based on types of motion mechanical systems are classified into two types:

  1. Translational mechanical system

  2. Rotational mechanical system

- The motion of the body/object during *translational motion is along a straight line or curved path*; whereas during rotational motion, *the motion of an object is about its own axis.*

1. Translational mechanical systems

- There are three basic elements involves in the analysis of translational motion. These are:

(i) Mass (ii) Spring (iii) Damper or dash pot

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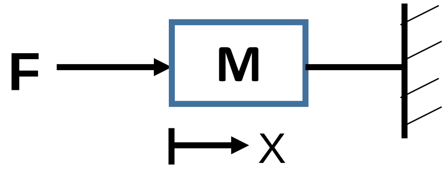
Mass:

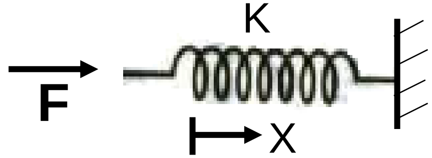


$f_m \propto a$

$$F = f_m = Ma = M\frac{d^2x}{dt^2}$$

Spring: let the linear spring constant for the spring be K. in this case the spring is subjected to force and it undergoes elastic deformation. The relation is $f_k \propto$ displacement.

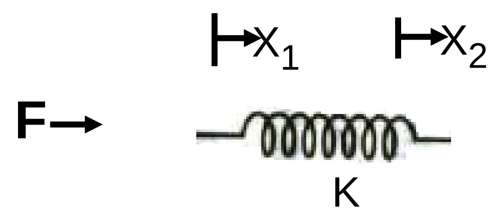
- If one end is fixed to the reference



$F_k \propto x$

$F = F_k = kx$

- If both ends are free



$F = F_k = k(x_1 - x_2)$

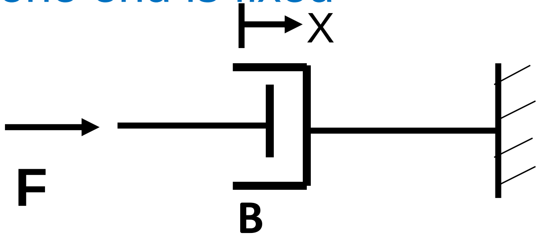
Cont'd…

Damper: Motion is opposed by friction. Types of frictional forces are:

(**i) Coulomb frictional force:** sliding friction b/n dry surface

(ii) Viscous friction force (F_b**) :** friction b/n moving surface by a viscous fluid or friction b/n solid body and a fluid medium.

- If one end is fixed



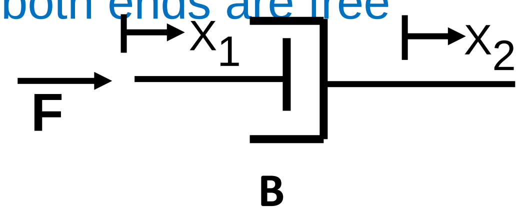
$F_b \propto V$ *(Velocity)*

$F_b = BV$

$F_b = F = B\dfrac{dx}{dt}$

Where: B is viscous friction constant

- If both ends are free



$F_b \propto B\dfrac{d(x_1 - x_2)}{dt}$

$F = F_b = B\dfrac{d}{dt}(x_1 - x_2)$

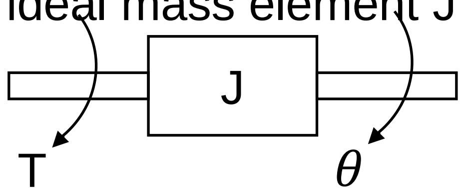
Cont'd…

2. Rotational Mechanical Systems

- Considering torque (T) and angular displacement (θ) here for object rotating about its own axis.

- The three elements of rotational motion are:
  - Moment Inertia of mass (J)
  - Damper (B)
  - Torsional Spring (K)

Mass: For any ideal mass element J

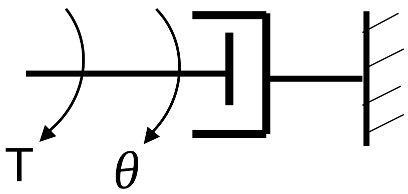


$$T_J \propto \frac{d^2\theta}{dt^2}$$

$$T = T_J = J\frac{d^2\theta}{dt^2}$$

Cont'd…

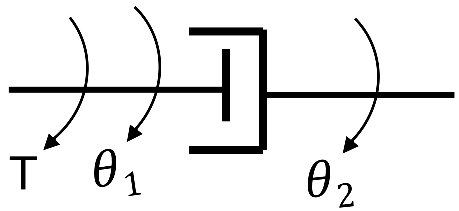
Damper: Opposing torque (T_b) due to friction of dash pot or damper is:



- If one end is fixed.

$$T_b \propto \frac{d\theta}{dt} \quad \Rightarrow \quad T = T_b = B\frac{d\theta}{dt}$$

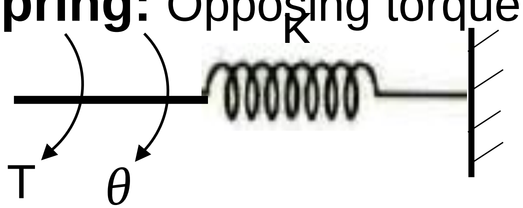
- If both ends are free.



$$T_b = B\frac{d\theta}{dt} \quad \Rightarrow \quad T = T_b = B\frac{d}{dt}(\theta_1 - \theta_2)$$

Since T is at θ_1, therefore, $\theta = \theta_1 - \theta_2$

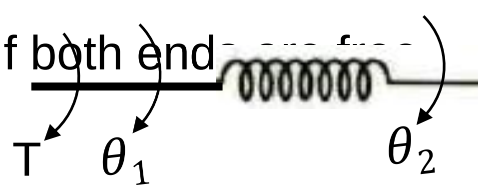
Spring: Opposing torque force due to elasticity of spring K is



$$T_K \propto \theta$$

Total torque ,$T = T_K = K\theta$

- If both ends are free



$$T = T_K = K(\theta_1 - \theta_2)$$

Modelling of Electrical Systems

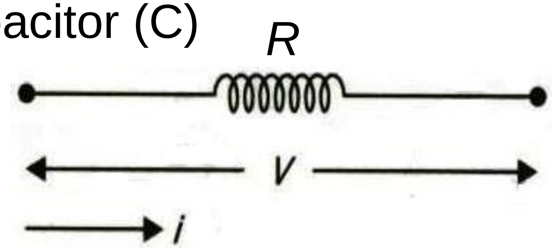
- Basic elements of electrical system are:

  (i) Resistor (R)

  (ii) Inductor (L)

  (iii) Capacitor (C)

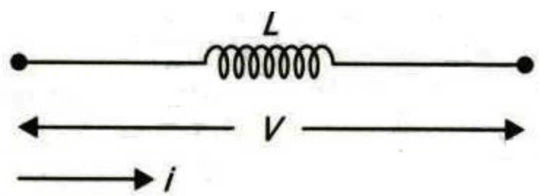
Resistor:



$$V = R\frac{dQ}{dt} = iR$$

Where *V* is voltage, Q is charge and ***i*** is current

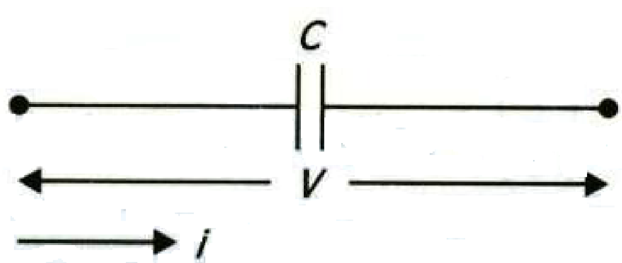
Inductor:



$$v = L\frac{di}{dt}$$

$$i = \frac{1}{L}\int_0^t V\,dt$$

Capacitor:



$$V = \frac{Q}{C} \Rightarrow v = \frac{1}{C}\int_0^t i\,dt$$

$$i = C\frac{dv}{dt}$$

Analogous Systems

- Two different physical systems having same mathematical model
- In analogous system a non – electrical systems is expressed in terms of its electrical counter part.
- There are two methods to get analogous systems:

(i) Force - voltage analogy (ii) Force - current analogy

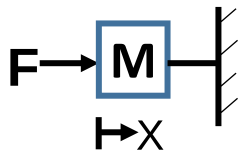
1) Translational mechanical system to Electrical System

(i) Force – Voltage analogy

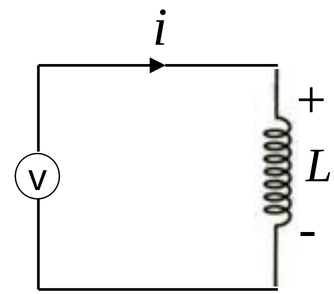
Mechanical system	Electrical system
Input: Force	Input: voltage source
Output: Velocity	Output: current through element

Mechanical system | **Electrical system**

1)

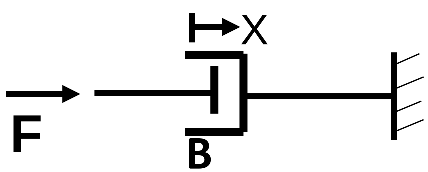


$$F = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$$

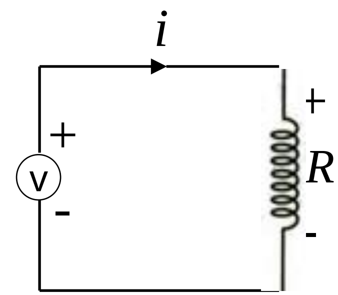


$$v = L\frac{di}{dt}$$

2)

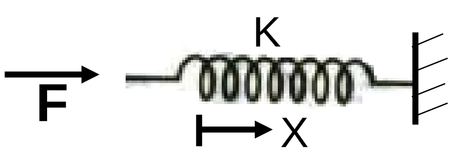


$$F = B\frac{dx}{dt} = BV$$

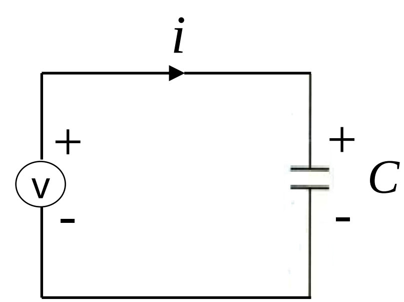


$$v = iR$$

3)



$$F = kx = k\int v\,dt$$



$$v = \frac{1}{C}\int v\,dt$$

Cont'd…

- **Note:**

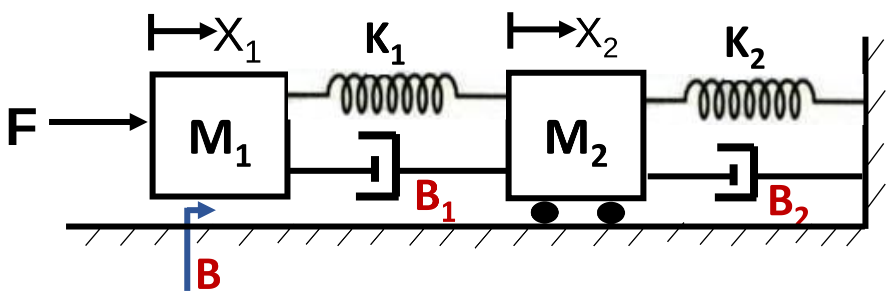
i. In mechanical system the elements having same velocity are said to be in series, similarly in electrical system the elements in series will have same current

ii. Each node (mass) in mechanical system corresponds to a closed loop in electrical system.

iii. Number of meshes in electrical system is equal to number masses in mechanical system

iv. The element connected between two masses in mechanical system is represented as a common element between two meshes in electrical analogous system.

Example: Element between two meshes in electrical analogous system



Mechanical system

Input: Force

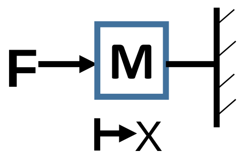
Output: Velocity

Electrical system

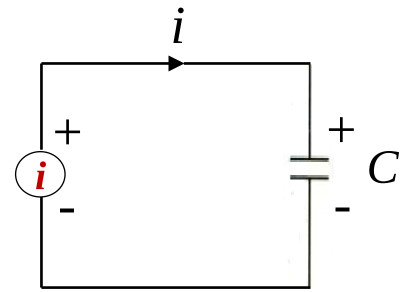
Input: Current source

Output: Voltage across element

1)

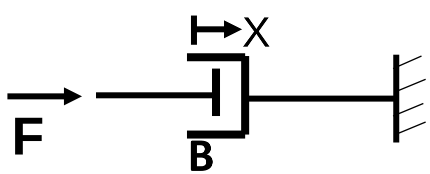


$$F = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$$

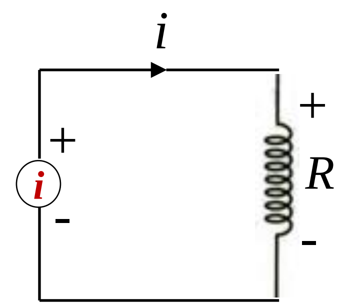


$$i = C\frac{dV}{dt}$$

2)

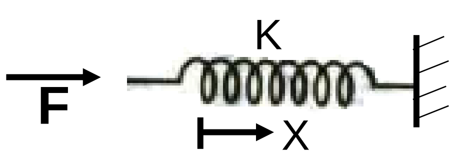


$$\mathrm{F} = \mathrm{B}\frac{dx}{dt} = BV$$

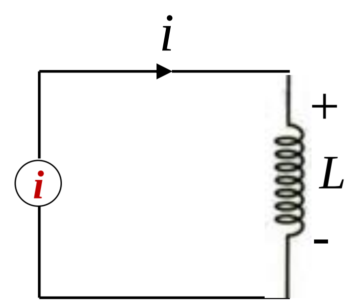


$$i = \frac{V}{R}$$

3)



$$\mathrm{F} = \mathrm{kx} = \mathrm{k}\int vdt$$



$$i = \frac{1}{L}\int Vdt$$

Note:

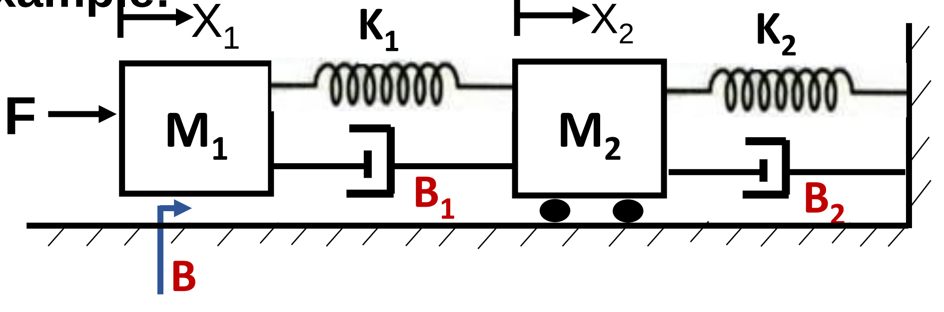
i) In mechanical system the elements in parallel will have same forces similarly, in electrical system parallel elements will have same voltage.

ii) Each mode (mass) in mechanical system corresponds to a node in electrical system.

iii) Number of nodes in electrical system is equal to number of **Mass** in mechanical system.

iv) The element connected b/n two mass in mechanical system is represented as common element b/n nodes in electrical system.

Example:



2) Rotational Mechanical system to Electrical System

(i)Torque – Voltage analogy

Mechanical rotational system

Input: Torque

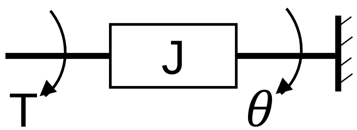
Output: angular velocity

Electrical system

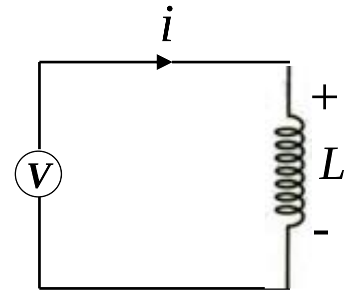
Input: voltage source

Output: current through element

1)

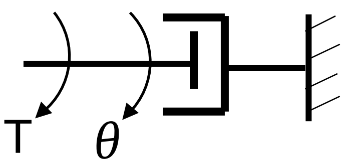


$$T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}$$

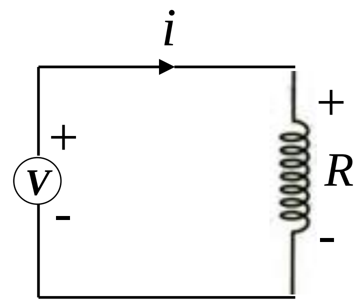


$$V = L\frac{di}{dt}$$

2)

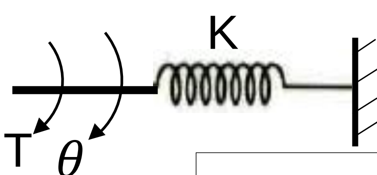


$$T = B\frac{d\theta}{dt} = B\omega$$

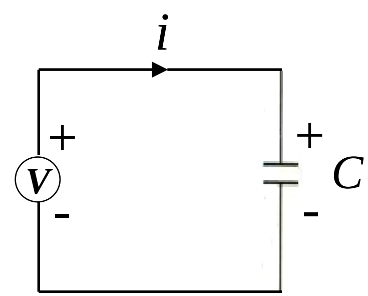


$$V = iR$$

3)



$$T = K\theta = K\int \omega dt$$



$$V = \frac{1}{C}\int i dt$$

2) Rotational Mechanical system to Electrical System

(ii)Torque – Current analogy

Mechanical rotational system

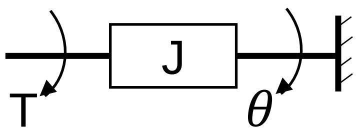
Input: Torque

Output: Angular velocity

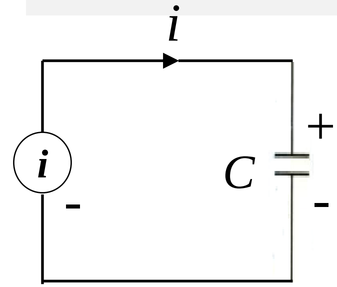
Electrical system

Input: Current source

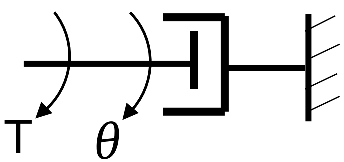
Output: Voltage across the element

1) 

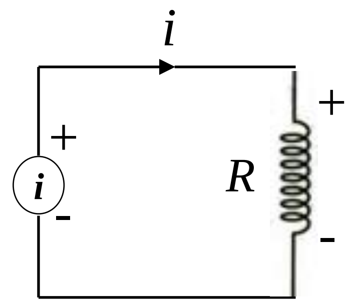
$$T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}$$



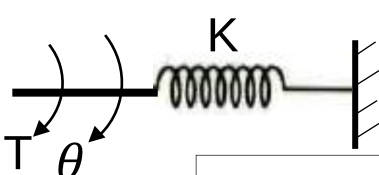
$$i = C\frac{dV}{dt}$$

2) 

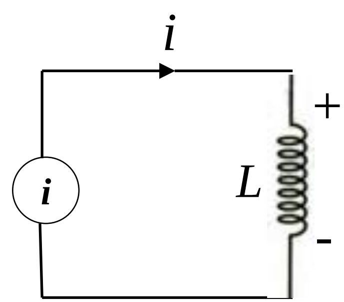
$$T = B\frac{d\theta}{dt} = B\omega$$



$$i = \frac{V}{R}$$

3) 

$$T = K\theta = K\int \omega dt$$



$$i = \frac{1}{L}\int V dt$$

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Summary: Counterparts of Translational and Rotational Motion

Force-voltage Analogy

Translational	Electrical	Rotational
Force (f)	Voltage (V)	Torque (T)
Mass (M)	Inductance (L)	Inertia (J)
Damper (D)	Resistance (R)	Damper (D)
Spring (K)	Elastance $\left(\frac{1}{C}\right)$	Spring (K)
Displacement (x)	Charge (q)	Displacement (θ)
Velocity (u)	Current (i)	Velocity (ω)

Force-current Analogy

Translational	Electrical	Rotational
Force (f)	Current (i)	Torque (T)
Mass (M)	Capacitance (C)	Inertia (J)
Spring (K)	Reciprocal of Inductance $\left(\frac{1}{L}\right)$	Spring (K)
Damper (D)	Conductance $\left(\frac{1}{R}\right)$	Damper (B)
Displacement (x)	Flux Linkage (ψ)	Displacement (θ)
Velocity $\left(u = \frac{dx}{dt}\right)$	Voltage (V) $= \frac{d\psi}{dt}$	Velocity $\left(\omega = \frac{d\theta}{dt}\right)$

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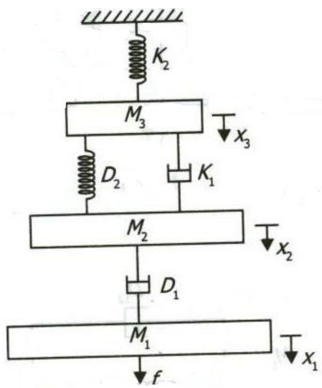
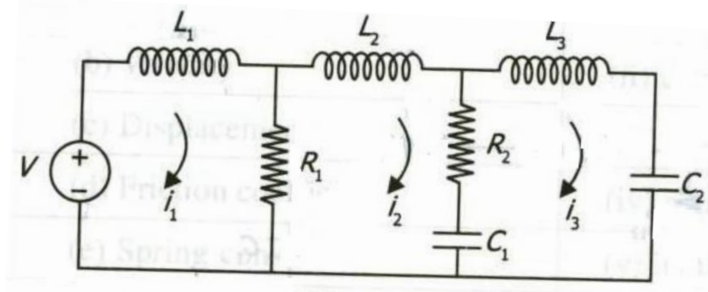
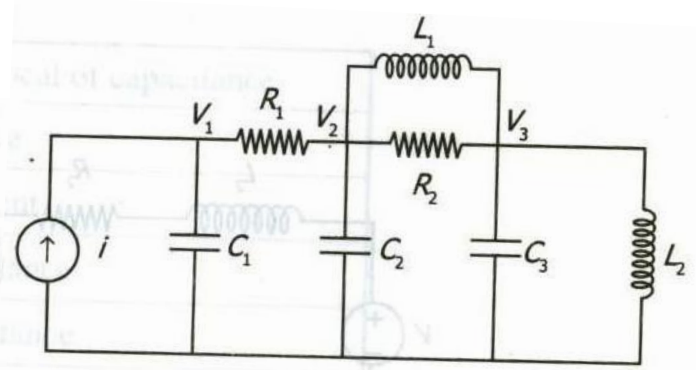
Examples: 1. Obtain the analogous electrical network of Fig below.

Using (a) *f-v* and (b) *f - I* analogy



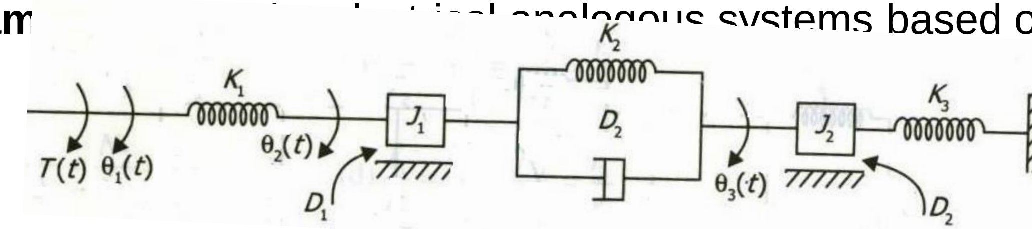


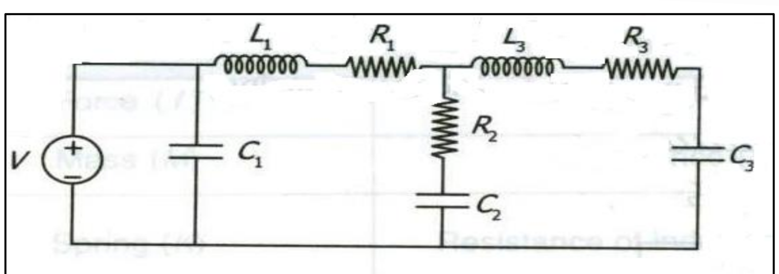
f-V analogy



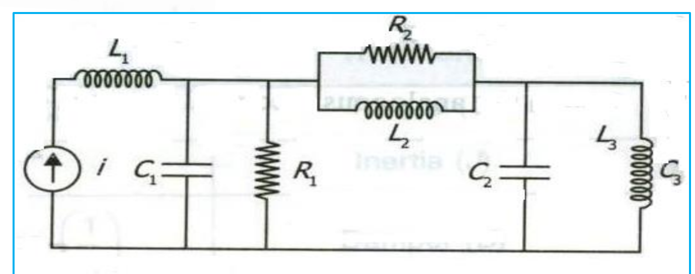
f-i analogy

Exam … ical analogous systems based on (a) *T-v* and (b) *T- i* analogy





T− V analogy



T− i analogy

Reading Assignment

1. Thermal system modeling
2. Hydraulics system modeling

End of Chapter

Thank you!