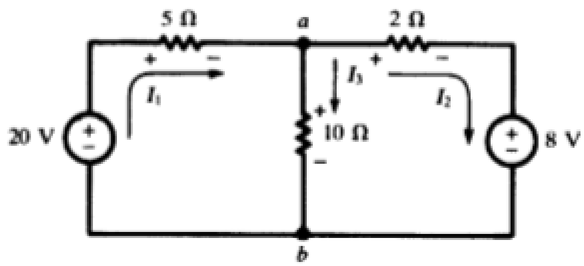
# 4. Analysis Methods

4.1 The Branch Current Method

In the branch current method a current is assigned to each branch in an active network. Then Kirchhoff's current law is applied at the principal nodes and the voltages between the nodes employed to relate the currents. This produces a set of simultaneous equations which can be solved to obtain the currents.

**EXAMPLE 1** Obtain the current in each branch of the network shown below using the branch current method.




Currents $I_1$, $I_2$, and $I_3$ are assigned to the branches as shown. Applying KCL at node $a$,

$$I_1 = I_2 + I_3 \qquad (1)$$

The voltage $V_{ab}$ can be written in terms of the elements in each of the branches; $V_{ab} = 20 - I_1(5)$, $V_{ab} = I_3(10)$ and $V_{ab} = I_2(2) + 8$. Then the following equations can be written

$$20 - I_1(5) = I_3(10) \qquad (2)$$

$$20 - I_1(5) = I_2(2) + 8 \qquad (3)$$

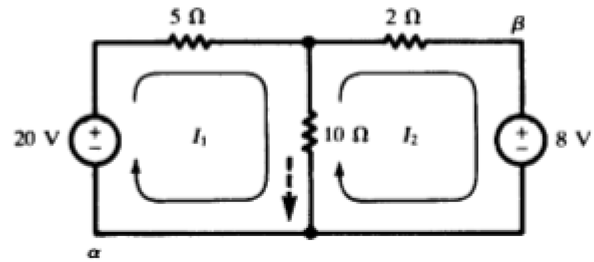
Solving the three equations (1), (2), and (3) simultaneously gives $I_1 = 2\,\text{A}$, $I_2 = 1\,\text{A}$, and $I_3 = 1\,\text{A}$.

Other directions may be chosen for the branch currents and the answers will simply include the appropriate sign. In a more complex network, the branch current method is difficult to apply because it does not suggest either a starting point or a logical progression through the network to produce the necessary equations. It also results in more independent equations than either the mesh current or node voltage method requires.

### 4.2 The Mesh Current Method

In the mesh current method a current is assigned to each window of the network such that the currents complete a closed loop. They are sometimes referred to as loop currents. Each element and branch therefore will have an independent current. When a branch has two of the mesh currents, the actual current is given by their algebraic sum. The assigned mesh currents may have either clockwise or counterclockwise directions, although at the outset it is wise to assign to all of the mesh currents a clockwise direction. Once the currents are assigned, Kirchhoff's voltage law is written for each loop to obtain the necessary simultaneous equations.

**EXAMPLE 2** Obtain the current in each branch of the network shown below using the mesh current method.




The currents I1 and I2 are chosen as shown on the circuit diagram. Applying KVL around the left loop, starting at point α,

$$-20 + 5I_1 + 10(I_1 - I_2) = 0$$

And around the right loop, starting at point β,

$$8 + 10(I_2 - I_1) + 2I_2 = 0$$

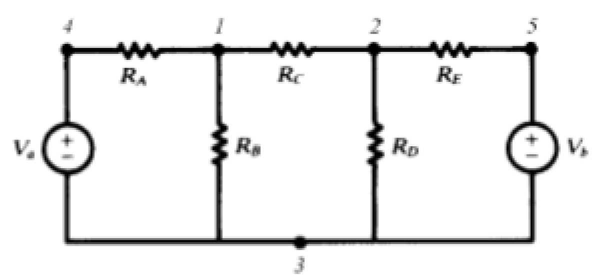
Rearranging terms,

$$15I_1 - 10I_2 = 20$$
$$-10I_1 + 12I_2 = -8$$

Solving these simultaneously results in I1 = 2A and I2 = 1A. The current in the center branch, shown dotted, is I1-I2=1A. In the above example this was branch current I3.

### 4.3 The Node Voltage Method

The network shown in Fig. below contains five nodes, where 4 and 5 are simple nodes and 1, 2, and 3 are principal nodes. In the node voltage method, one of the principal nodes is selected as the reference and equations based on KCL are written at the other principal nodes. At each of these other principal nodes, a voltage is assigned, where it is understood that this is a voltage with respect to the reference node. These voltages are the unknowns and, when determined by a suitable method, result in the network solution.




KCL requires that the total current out of node 1 be zero:

$$\frac{V_1 - V_a}{R_A} + \frac{V_1}{R_B} + \frac{V_1 - V_2}{R_C} = 0$$

Similarly, the total current out of node 2 must be zero:

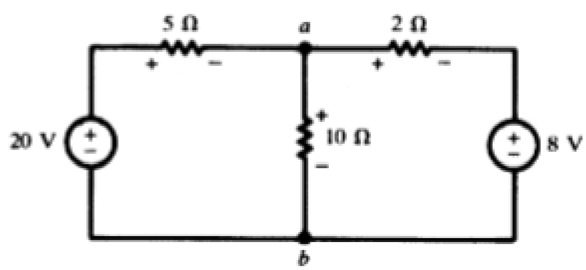
$$\frac{V_2 - V_1}{R_C} + \frac{V_2}{R_D} + \frac{V_2 - V_b}{R_E} = 0$$

$$\begin{bmatrix} \frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C} & -\frac{1}{R_C} \\ -\frac{1}{R_C} & \frac{1}{R_C} + \frac{1}{R_D} + \frac{1}{R_E} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_a/R_A \\ V_b/R_E \end{bmatrix}$$

Note the symmetry of the coefficient matrix. The 1,1-element contains the sum of the reciprocals of all resistances connected to note 1; the 2,2-element contains the sum of the reciprocals of all resistances connected to node 2. The 1,2- and 2,1-elements are each equal to the negative of the sum of the reciprocals of the resistances of all branches joining nodes 1 and 2.

On the right-hand side, the current matrix contains Va=RA and Vb=RE, the driving currents. Both these terms are taken positive because they both drive a current into a node.

**Example 3**: Solve the following circuit using the node voltage method.




$$\frac{V_1 - 20}{5} + \frac{V_1}{10} + \frac{V_1 - 8}{2} = 0$$

From which V1 = 10V. Then, I1 = (10-20)/5 = -2A (the negative sign indicates that current I1 flows into node 1); I2 = (10-8)/2 = 1A; I3 = 10/10 = 1A.

### 4.4 Source Transformation

Source transformation is another tool for simplifying circuits. Basic to these tools is the concept of equivalence. We recall that an equivalent circuit is one whose v-i characteristics are identical with the original circuit.

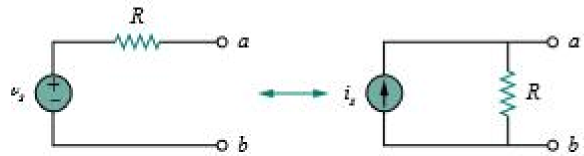



Figure 4.15 Transformation of independent sources.

A **source transformation** is the process of replacing a voltage source $v_s$ in series with a resistor $R$ by a current source $i_s$ in parallel with a resistor $R$, or vice versa.

The two circuits in Fig. 4.15 are equivalent—provided they have the same voltage-current relation at terminals $a$-$b$. It is easy to show that they are indeed equivalent. If the sources are turned off, the equivalent resistance at terminals $a$-$b$ in both circuits is $R$. Also, when terminals $a$-$b$ are short-circuited, the short-circuit current flowing from $a$ to $b$ is $i_{sc} = v_s/R$ in the circuit on the left-hand side and $i_{sc} = i_s$ for the circuit on the right-hand side. Thus, $v_s/R = i_s$ in order for the two circuits to be equivalent. Hence, source transformation requires that

$$v_s = i_s R \qquad \text{or} \qquad i_s = \frac{v_s}{R} \tag{4.5}$$

Source transformation also applies to dependent sources, provided we carefully handle the dependent variable. As shown in Fig. 4.16, a dependent voltage source in series with a resistor can be transformed to a dependent current source in parallel with the resistor or vice versa.

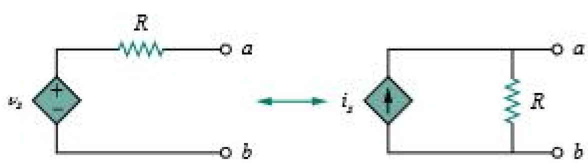



Figure 4.16 Transformation of dependent sources.

We should keep the following points in mind when dealing with source transformation.

1. Note from Fig. 4.15 (or Fig. 4.16) that the arrow of the current source is directed toward the positive terminal of the voltage source.

2. Note from Eq. (4.5) that source transformation is not possible when R = 0, which is the case with an ideal voltage source. However, for a practical, non-ideal voltage source, R differ from 0. Similarly, an ideal current source with R =∞ cannot be replaced by a finite voltage source.

***Example***:

Use source transformation to find $v_o$ in the circuit in Fig. 4.17.

**Solution:**

We first transform the current and voltage sources to obtain the circuit in Fig. 4.18(a). Combining the 4-Ω and 2-Ω resistors in series and transforming the 12-V voltage source gives us Fig. 4.18(b). We now combine the 3-Ω and 6-Ω resistors in parallel to get 2-Ω. We also combine the 2-A and 4-A current sources to get a 2-A source. Thus, by repeatedly applying source transformations, we obtain the circuit in Fig. 4.18(c).

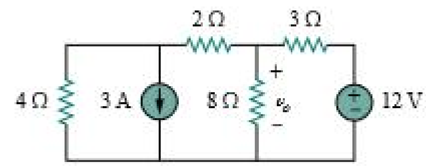



Figure 4.17 For Example 4.6.

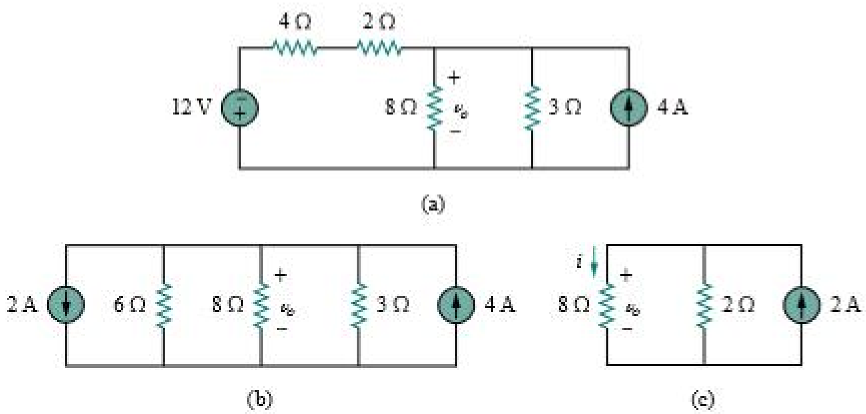



Figure 4.18 For Example 4.6.

We use current division in Fig. 4.18(c) to get

$$i = \frac{2}{2+8}(2) = 0.4$$

and

$$v_o = 8i = 8(0.4) = 3.2 \text{ V}$$

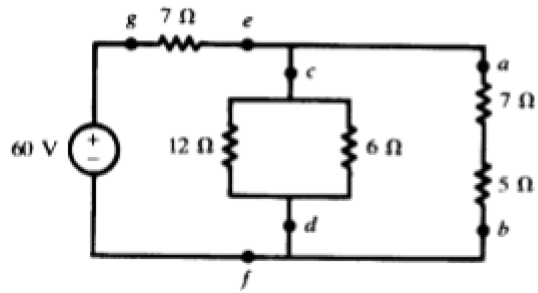
Alternatively, since the 8-Ω and 2-Ω resistors in Fig. 4.18(c) are in parallel, they have the same voltage $v_o$ across them. Hence,

$$v_o = (8 \parallel 2)(2 \text{ A}) = \frac{8 \times 2}{10}(2) = 3.2 \text{ V}$$

## 4.5 Network Reduction

The mesh current and node voltage methods are the principal techniques of circuit analysis. However, the equivalent resistance of series and parallel branches, combined with the voltage and current division rules, provide another method of analyzing a network. This method is tedious and usually requires the drawing of several additional circuits. Even so, the process of reducing the network provides a very clear picture of the overall functioning of the network in terms of voltages, currents, and power. The reduction begins with a scan of the network to pick out series and parallel combinations of resistors.

**EXAMPLE 4** Obtain the total power supplied by the 60-V source and the power absorbed in each resistor in the network shown below.




$$R_{ab} = 7 + 5 = 12\,\Omega$$

$$R_{cd} = \frac{(12)(6)}{12+6} = 4\,\Omega$$

These two equivalents are in parallel (fig a), giving

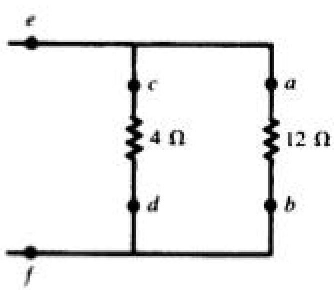



Fig. a

$$R_{ef} = \frac{(4)(12)}{4+12} = 3\,\Omega$$

Then this 3-Ω equivalent is in series with the 7-Ω resistor (Fig. b), so that for the entire circuit,

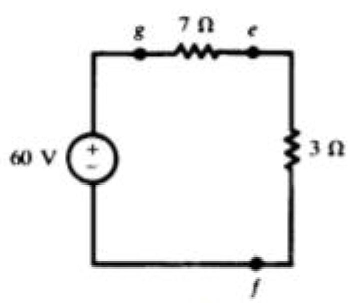



Fig. b

$$R_{eq} = 7 + 3 = 10\,\Omega$$

The total power absorbed, which equals the total power supplied by the source, can now be calculated as

$$P_T = \frac{V^2}{R_{eq}} = \frac{(60)^2}{10} = 360\text{ W}$$

This power is divided between $R_{ge}$ and $R_{ef}$ as follows:

$$P_{ge} = P_{7\Omega} = \frac{7}{7+3}(360) = 252\text{ W} \qquad P_{ef} = \frac{3}{7+3}(360) = 108\text{ W}$$

Power $P_{ef}$ is further divided between $R_{cd}$ and $R_{ab}$ as follows:

$$P_{cd} = \frac{12}{4+12}(108) = 81\text{ W} \qquad P_{ab} = \frac{4}{4+12}(108) = 27\text{ W}$$

Finally, these powers are divided between the individual resistances as follows:

$$P_{12\Omega} = \frac{6}{12+6}(81) = 27\text{ W} \qquad P_{7\Omega} = \frac{7}{7+5}(27) = 15.75\text{ W}$$

$$P_{6\Omega} = \frac{12}{12+6}(81) = 54\text{ W} \qquad P_{5\Omega} = \frac{5}{7+5}(27) = 11.25\text{ W}$$

**4.6 Superposition**

A linear network which contains two or more independent sources can be analyzed to obtain the various voltages and branch currents by allowing the sources to act one at a time, then superposing the results. This principle applies because of the linear relationship between current and voltage.

Superposition cannot be directly applied to the computation of power, because power in an element is proportional to the square of the current or the square of the voltage, which is nonlinear.

> The **superposition** principle states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltages across (or currents through) that element due to each independent source acting alone.

To apply the superposition principle, we must keep two things in mind:

1. We consider one independent source at a time while all other independent sources are turned off. This implies that we replace every voltage source by 0 V (or a short circuit), and every current source by 0 A (or an open circuit). This way we obtain a simpler and more manageable circuit.

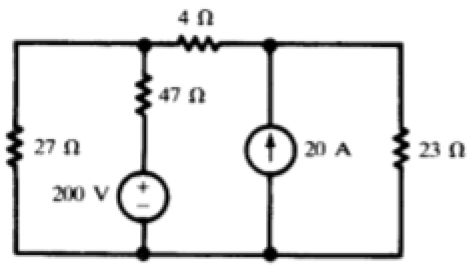
2. Dependent sources are left intact because they are controlled by circuit variables.

With these in mind, we apply the superposition principle in three steps:

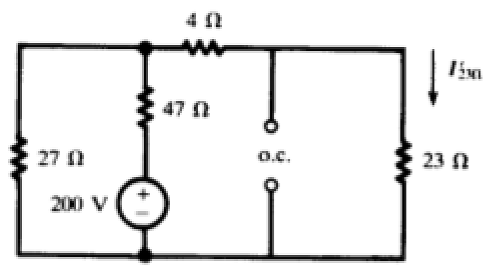
**Steps to Apply Superposition Principle:**

1. Turn off all independent sources except one source. Find the output (voltage or current) due to that active source using nodal or mesh analysis.
2. Repeat step 1 for each of the other independent sources.
3. Find the total contribution by adding algebraically all the contributions due to the independent sources.

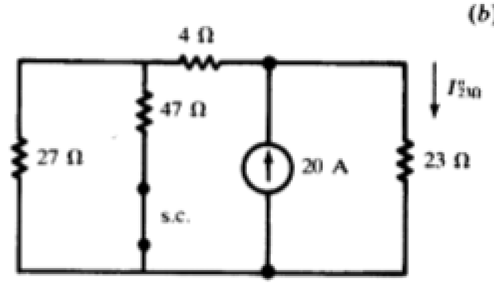
**Example 5:** Compute the current in the 23-Ω resistor of Fig. (a) by applying the superposition principle. With the 200-V source acting alone, the 20-A current source is replaced by an open circuit, Fig. (b).




(a)




(b)




(c)

$$R_{eq} = 47 + \frac{(27)(4+23)}{54} = 60.5\,\Omega$$

$$I_T = \frac{200}{60.5} = 3.31\text{ A}$$

$$I'_{23\Omega} = \left(\frac{27}{54}\right)(3.31) = 1.65\text{ A}$$

When the 20-A source acts alone, the 200-V source is replaced by a short circuit, as shown in (c)The equivalent resistance to the left of the source is

$$R_{eq} = 4 + \frac{(27)(47)}{74} = 21.15\,\Omega$$

Then
$$I''_{23\Omega} = \left(\frac{21.15}{21.15+23}\right)(20) = 9.58\text{ A}$$

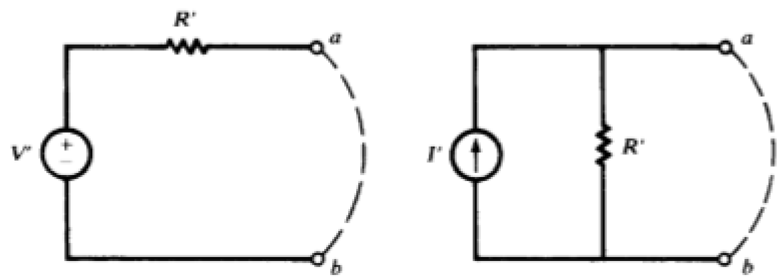
The total current in the 23-Ω resistor is

$$I_{23\Omega} = I'_{23\Omega} + I''_{23\Omega} = 11.23\text{ A}$$

**4.7 The´ venin’s and Norton’s Theorems**

Thevenin's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a voltage source $V_{Th}$ in series with a resistor $R_{Th}$, where $V_{Th}$ is the open-circuit voltage at the terminals and $R_{Th}$ is the input or equivalent resistance at the terminals when the independent sources are turned off.

A linear, active, resistive network which contains one or more voltage or current sources can be replaced by a single voltage source and a series resistance (The ´venin’s theorem), or by a single current source and a parallel resistance (Norton’s theorem). The voltage is called the The ´venin equivalent voltage, V0, and the current the Norton equivalent current, I’.




(b) Thévenin (c) Norton

**Example:**

Find the Thevenin equivalent circuit of the circuit shown in Fig. 4.27, to the left of the terminals $a$-$b$. Then find the current through $R_L = 6, 16,$ and $36\ \Omega$.

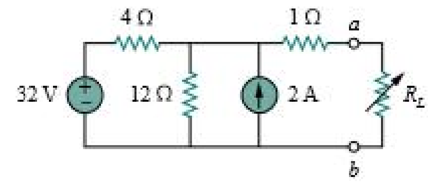



Figure 4.27 For Example 4.8.

**Solution:**

We find $R_{Th}$ by turning off the 32-V voltage source (replacing it with a short circuit) and the 2-A current source (replacing it with an open circuit). The circuit becomes what is shown in Fig. 4.28(a). Thus,

$$R_{Th} = 4 \parallel 12 + 1 = \frac{4 \times 12}{16} + 1 = 4\ \Omega$$

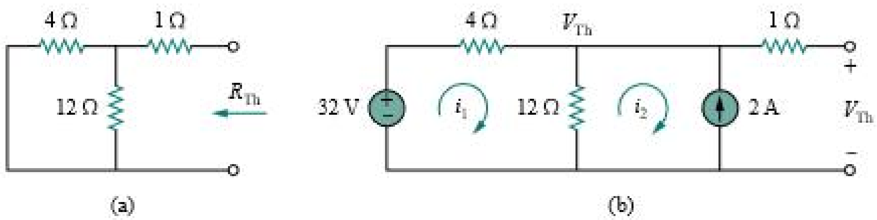



Figure 4.28 For Example 4.8: (a) finding $R_{Th}$, (b) finding $V_{Th}$.

To find $V_{Th}$, consider the circuit in Fig. 4.28(b). Applying mesh analysis to the two loops, we obtain

$$-32 + 4i_1 + 12(i_1 - i_2) = 0, \qquad i_2 = -2\ \text{A}$$

Solving for $i_1$, we get $i_1 = 0.5$ A. Thus,

$$V_{Th} = 12(i_1 - i_2) = 12(0.5 + 2.0) = 30\ \text{V}$$

Alternatively, it is even easier to use nodal analysis. We ignore the 1-Ω resistor since no current flows through it. At the top node, KCL gives

$$\frac{32 - V_{Th}}{4} + 2 = \frac{V_{Th}}{12}$$

or

$$96 - 3V_{Th} + 24 = V_{Th} \qquad \Longrightarrow \qquad V_{Th} = 30\ \text{V}$$

as obtained before. We could also use source transformation to find $V_{Th}$.

The Thevenin equivalent circuit is shown in Fig. 4.29. The current through $R_L$ is

$$I_L = \frac{V_{Th}}{R_{Th} + R_L} = \frac{30}{4 + R_L}$$

When $R_L = 6$,

$$I_L = \frac{30}{10} = 3 \text{ A}$$

When $R_L = 16$,

$$I_L = \frac{30}{20} = 1.5 \text{ A}$$

When $R_L = 36$,

$$I_L = \frac{30}{40} = 0.75 \text{ A}$$

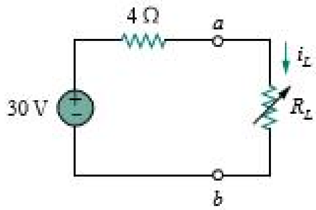



Figure 4.29 The Thevenin equivalent circuit for Example 4.8.

**Norton's theorem** states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a current source $I_N$ in parallel with a resistor $R_N$, where $I_N$ is the short-circuit current through the terminals and $R_N$ is the input or equivalent resistance at the terminals when the independent sources are turned off.

## Example:

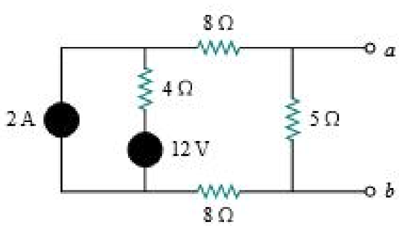



Figure 4.39 For Example 4.11.

Find the Norton equivalent circuit of the circuit in Fig. 4.39.

**Solution:**

We find $R_N$ in the same way we find $R_{Th}$ in the Thevenin equivalent circuit. Set the independent sources equal to zero. This leads to the circuit in Fig. 4.40(a), from which we find $R_N$. Thus,

$$R_N = 5 \parallel (8 + 4 + 8) = 5 \parallel 20 = \frac{20 \times 5}{25} = 4 \ \Omega$$

To find $I_N$, we short-circuit terminals $a$ and $b$, as shown in Fig. 4.40(b). We ignore the 5-Ω resistor because it has been short-circuited. Applying mesh analysis, we obtain

$$i_1 = 2 \text{ A}, \qquad 20i_2 - 4i_1 - 12 = 0$$

From these equations, we obtain

$$i_2 = 1 \text{ A} = i_{sc} = I_N$$

Alternatively, we may determine $I_N$ from $V_{Th}/R_{Th}$. We obtain $V_{Th}$ as the open-circuit voltage across terminals $a$ and $b$ in Fig. 4.40(c). Using mesh analysis, we obtain

$$i_3 = 2 \text{ A}$$

$$25i_4 - 4i_3 - 12 = 0 \quad \Longrightarrow \quad i_4 = 0.8 \text{ A}$$

and

$$v_{oc} = V_{\text{Th}} = 5i_4 = 4 \text{ V}$$

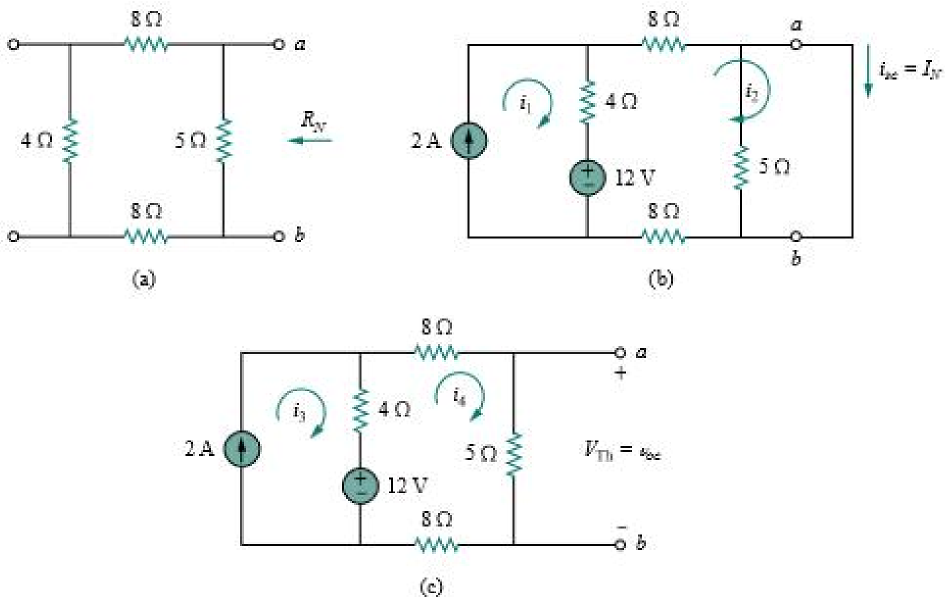



**Figure 4.40** For Example 4.11; finding: (a) $R_N$, (b) $I_N = i_{sc}$, (c) $V_{\text{Th}} = v_{oc}$.

Hence,

$$I_N = \frac{V_{\text{Th}}}{R_{\text{Th}}} = \frac{4}{4} = 1 \text{ A}$$

as obtained previously. This also serves to confirm Eq. (4.7) that $R_{\text{Th}} = v_{oc}/i_{sc} = 4/1 = 4\ \Omega$. Thus, the Norton equivalent circuit is as shown in Fig. 4.41.

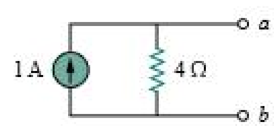



**Figure 4.41** Norton equivalent of the circuit in Fig. 4.39.

**EXAMPLE 4.8** Obtain the Thévenin and Norton equivalent circuits for the active network in Fig. 4-13(*a*).

With terminals *ab* open, the two sources drive a clockwise current through the 3-Ω and 6-Ω resistors [Fig. 4-13(*b*)].

$$I = \frac{20 + 10}{3 + 6} = \frac{30}{9} \text{ A}$$

Since no current passes through the upper right 3-Ω resistor, the Thévenin voltage can be taken from either active branch:

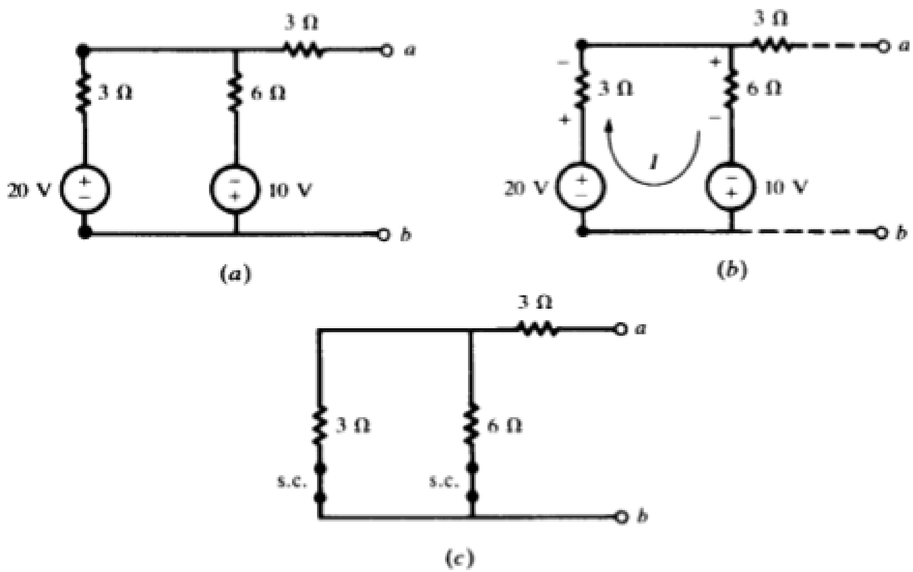



Fig. 4-13

$$V_{ab} = V' = 20 - \left(\frac{30}{9}\right)(3) = 10\,\text{V}$$

or

$$V_{ab} = V' = \left(\frac{30}{9}\right)6 - 10 = 10\,\text{V}$$

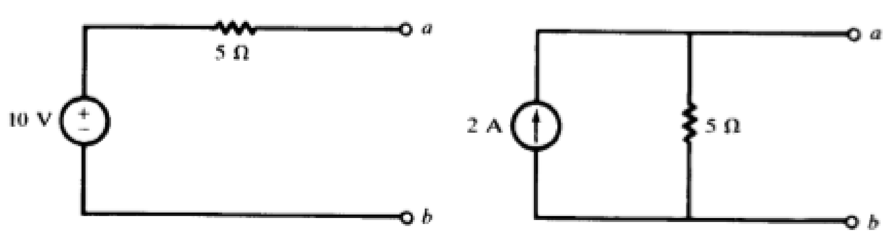
The resistance $R'$ can be obtained by shorting out the voltage sources [Fig. 4.13($c$)] and finding the equivalent resistance of this network at terminals $ab$:

$$R' = 3 + \frac{(3)(6)}{9} = 5\,\Omega$$

When a short circuit is applied to the terminals, current $I_{\text{s.c.}}$ results from the two sources. Assuming that it runs through the short from $a$ to $b$, we have, by superposition,

$$I_{\text{s.c.}} = I' = \left(\frac{6}{6+3}\right)\left[\frac{20}{3+\frac{(3)(6)}{9}}\right] - \left(\frac{3}{3+3}\right)\left[\frac{10}{6+\frac{(3)(3)}{6}}\right] = 2\,\text{A}$$

Figure 4-14 shows the two equivalent circuits. In the present case, $V'$, $R'$, and $I'$ were obtained independently. Since they are related by Ohm's law, any two may be used to obtain the third.




($a$) Thévenin Equivalent ($b$) Norton Equivalent

Fig. 4-14

The usefulness of Thévenin and Norton equivalent circuits is clear when an active network is to be examined under a number of load conditions, each represented by a resistor. This is suggested in

Fig. 4-15, where it is evident that the resistors $R_1, R_2, \ldots, R_n$ can be connected one at a time, and the resulting current and power readily obtained. If this were attempted in the original circuit using, for example, network reduction, the task would be very tedious and time-consuming.

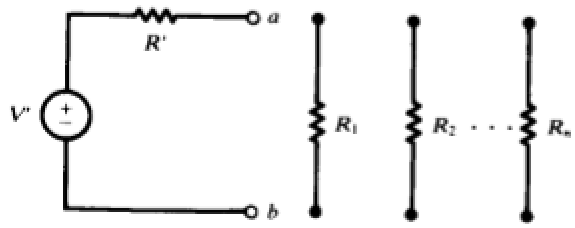



**Fig. 4-15**

## 4.8 Maximum Power Transfer Theorem

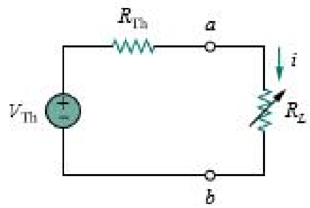



Figure 4.48 The circuit used for maximum power transfer.

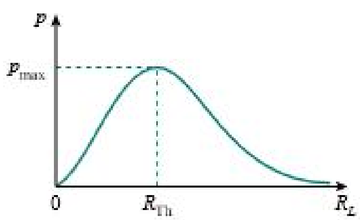



Figure 4.49 Power delivered to the load as a function of $R_L$.

The Thevenin equivalent is useful in finding the maximum power a linear circuit can deliver to a load. We assume that we can adjust the load resistance $R_L$. If the entire circuit is replaced by its Thevenin equivalent except for the load, as shown in Fig. 4.48, the power delivered to the load is

$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L}\right)^2 R_L \qquad (4.21)$$

For a given circuit, $V_{Th}$ and $R_{Th}$ are fixed. By varying the load resistance $R_L$, the power delivered to the load varies as sketched in Fig. 4.49. We notice from Fig. 4.49 that the power is small for small or large values of $R_L$ but maximum for some value of $R_L$ between 0 and $\infty$. We now want to show that this maximum power occurs when $R_L$ is equal to $R_{Th}$. This is known as the *maximum power theorem*.

> **Maximum power** is transferred to the load when the load resistance equals the Thevenin resistance as seen from the load ($R_L = R_{Th}$).

To prove the maximum power transfer theorem, we differentiate $p$ in Eq. (4.21) with respect to $R_L$ and set the result equal to zero. We obtain

$$\frac{dp}{dR_L} = V_{Th}^2 \left[\frac{(R_{Th} + R_L)^2 - 2R_L(R_{Th} + R_L)}{(R_{Th} + R_L)^4}\right]$$

$$= V_{Th}^2 \left[\frac{(R_{Th} + R_L - 2R_L)}{(R_{Th} + R_L)^3}\right] = 0$$

This implies that

$$0 = (R_{\text{Th}} + R_L - 2R_L) = (R_{\text{Th}} - R_L) \tag{4.22}$$

which yields

$$\boxed{R_L = R_{\text{Th}}} \tag{4.23}$$

showing that the maximum power transfer takes place when the load resistance $R_L$ equals the Thevenin resistance $R_{\text{Th}}$. We can readily confirm that Eq. (4.23) gives the maximum power by showing that $d^2p/dR_L^2 < 0$.

The maximum power transferred is obtained by substituting Eq. (4.23) into Eq. (4.21), for

$$\boxed{p_{\max} = \frac{V_{\text{Th}}^2}{4R_{\text{Th}}}} \tag{4.24}$$

Equation (4.24) applies only when $R_L = R_{\text{Th}}$. When $R_L \neq R_{\text{Th}}$, we compute the power delivered to the load using Eq. (4.21).

**Example**:

Find the value of $R_L$ for maximum power transfer in the circuit of Fig. 4.50. Find the maximum power.

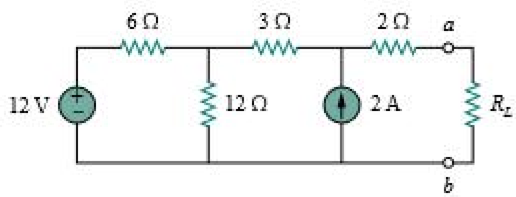



Figure 4.50 For Example 4.13.

**Solution:**

We need to find the Thevenin resistance $R_{\text{Th}}$ and the Thevenin voltage $V_{\text{Th}}$ across the terminals $a$-$b$. To get $R_{\text{Th}}$, we use the circuit in Fig. 4.51(a) and obtain

$$R_{\text{Th}} = 2 + 3 + 6 \parallel 12 = 5 + \frac{6 \times 12}{18} = 9\ \Omega$$

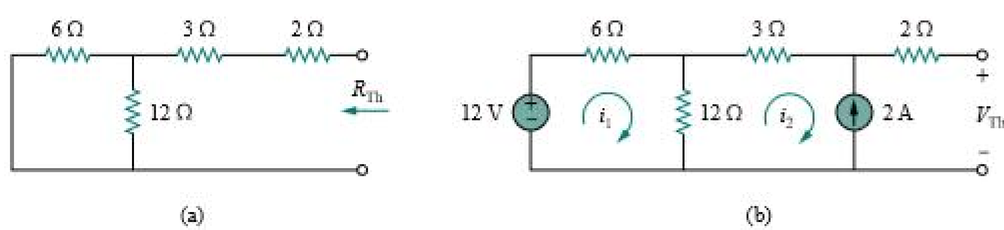



Figure 4.51 For Example 4.13: (a) finding $R_{\text{Th}}$, (b) finding $V_{\text{Th}}$.

To get $V_{\text{Th}}$, we consider the circuit in Fig. 4.51(b). Applying mesh analysis,

$$-12 + 18i_1 - 12i_2 = 0, \qquad i_2 = -2 \text{ A}$$

Solving for $i_1$, we get $i_1 = -2/3$. Applying KVL around the outer loop to get $V_{\text{Th}}$ across terminals $a$-$b$, we obtain

$$-12 + 6i_1 + 3i_2 + 2(0) + V_{\text{Th}} = 0 \qquad \Longrightarrow \qquad V_{\text{Th}} = 22 \text{ V}$$

For maximum power transfer,

$$R_L = R_{\text{Th}} = 9\ \Omega$$

and the maximum power is

$$p_{\max} = \frac{V_{\text{Th}}^2}{4R_L} = \frac{22^2}{4 \times 9} = 13.44 \text{ W}$$