Wollo University
Collage of Business and Economics
Department of Accounting and Finance

Distance Module for Degree Program

Mathematics for Finance (ACFN1041)

Prepared By: Seid Mohammed (MSc.)

Editor: Naod Mekonnen (MSc.)

Distance Education Program

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Preface

Dear Students! Mathematics for Finance is mathematics used by commercial enterprises to record and manage business operations. Commercial organizations use mathematics in accounting, inventory management, marketing, sales forecasting and financial analysis. Mathematics typically used in commerce includes elementary arithmetic, elementary algebra, statistics and probability. Business operations can be made more effective by the use of more advanced mathematics such as calculus, matrix algebra and linear programming.

Why This Course? Dear student, welcome to the learning task Mathematics for Finance and its application. This learning task is designed to expose you to the basic concepts and area of the application of mathematics in businesses. It is divided into five chapter: The first section deals with the linear equations and its applications; the second section is about the matrix algebra and its applications; the third section deals with linear programming, the fourth section is dedicated to mathematics of finance and the fifth section is about elements and application of calculus. You will find learning activities in each sections. To successfully accomplish this learning task you are expected to study and practice the provided examples and exercises.

Thus this course is designed for accounting and finance students. The course will assist students reaching a level of increased competence in mathematics and expanded understanding of the application of mathematical concept in business activities. Emphasis is placed upon learning mathematical concepts through practical applications to common business problem.

About the Course

Course Code	AcFn1041
Course Title	Mathematics for Finance
Degree Program	BA Degree in Accounting and Finance
Module	Computational and Quantitative Methods for Finance
ETCTS Credits	6
Credit Hour	4
Course Objectives & Competences to be Acquired	At the end of this course, students should be able to • Differentiate the various techniques of mathematics that can be employed in solving business problems • Identify the way mathematical techniques are utilized • Appreciate the importance of mathematics in solving real world business problems • Use different mathematical techniques for supporting managerial Decisions • Analyze real managerial problems using mathematical tools
Course Description	Mathematics for Finance is one of the preliminary quantitative aids to decision making that offers the decision-maker a method of evaluating every possible alternative (act or course of action) by using various techniques to know the potential outcomes. This course is designed to expose finance students to the basic concepts and area of managerial application of mathematics for decision making. Topics include: linear equations and their applications, matrix algebra and its applications, Markov chain analysis, linear programming, mathematics of finance, elements and application of calculus.
Evaluation Type	Weight
Assignment	35%
Tutorial Attendance	5%
Final exam	60%
Total	100%

Contents

Chapter One: Linear Equations and their Interpretive Application

Chapter objectives

Dear students! after successful completion of this chapter, you are expected to:

- Understand linear equation and its components
- Develop linear equations using different method
- Demonstrate how linear equations can be used in cost -volume - profit (CVP) analysis for manufacturing business
- Determine the Quantity and sales volume required to earn a target profit
- Understand how to apply linear equations in CVP analysis for merchandising business
- Compute the break-even point in terms of Quantity and sales revenue for manufacturing and merchandising business

1.1. Meaning of linear equation

Before directly proceeding to the definition of linear equations, let's start by defining terms that are used in equation.

Equation can be defined as a mathematical statement which indicates that two algebraic expressions are equal. In other words, it is a mathematical statement that equates (relates with equal sign) two algebraic expressions. For example, y = 2x + 3, is an equation which equates y with 2x + y.

> ***Equation*** is a statement that two quantities or algebraic expressions are equal.
> ***Solving the equation*** is finding the value(s) of the variable(s) that make the equation true.
> Two equations are said to be **equivalent** if they have exactly the same solution set.

Equations are used to model or represent real world situations and are a convenient and concise way of representing relationship between quantities such as sales and advertising or sales commission, profit and time, cost and number of units manufactured, and so on.

An **algebraic expression** is a mathematical statement in which two numerical quantities are linked by mathematical operations (signs) such as + and – signs. For instance, 2x + 3 in the above equation is an algebraic expression. The letters in algebraic expressions are called **variables** (unknowns) and always represent a number. Thus, we can perform mathematical operations on a letter. In other words, it can be added, subtracted, multiplied, and divided. The values or variables separated by + or – signs are called **Terms**. Terms are often called monomials (mono = one) if an algebraic expression has one term. If an expression has more than one term, it is called polynomials (poly = many).

Linear equations are equations with a variable and a constant with degree one (first power). The terms of linear equations are a constant, or a constant times one variable to the first power. Linear equations are equations whose slope is constant throughout the line.

E.g. 2x - 3y = 7 - degree 1
- Constant 7
- Terms 2x &3y separated by the minus sign

However, 2x + 3xy = 7 is not a linear equation; b/c 3xy is a constant times the product of two variables. No X^2 terms, no $\sqrt{y}$ terms & no XY terms are allowed in a linear equation.

1.2. Parts of linear equation

The general form of a linear equation is, **Y = mx + b**,

Where, Y = the dependent variable its value depends on the value of x.

X = the independent variable its change affects the value of y.

m = slope

b = y-intercept, the value of y when x=0.

One of the important parts that that differentiate linear equations is the slope. So,lets discuss the slope and its interpretation.

Slope (m)

It is worth noting that this formula or notation holds true for all lines that are not parallel to the y-axis. A vertical line is represented by the equation X=a. In cost output relationship, b is the fixed cost, and m is the marginal cost. The cost is increases by the rate of the amount of the slope, m.

$$\text{Slope (M)} \quad \frac{\Delta y}{\Delta x} = \frac{(rise/\ fall)}{run} = \frac{Y_2 - y_1}{X_2 - x_1} = if \quad = X_1 \# X_2$$

Slope measures the steepness of a line. The larger the slope the steeper the line is both in value & in absolute value. A line's slope number tells us how much the line falls (or rises) for a stated change in x. Slopes can assume 4 different values: negative, positive, 0 and undefined.

⇒ The line that is parallel to the X- axis is the gentlest of all lines, m= 0

⇒ The line that is parallel to the Y - axis is the steepest of all lines, m = ∉

The slope of a line is defined as the change taking place along the vertical axis relative to the corresponding change taking place along the horizontal axis, or, the change in the value of y relative to a one - unit change in the value of x.

In linear equation of Y= mx + b, the coefficient of the independent variable is the slope of the line and the constant that stands alone is the vertical (y) intercept. That is, Dependent variable = (slope x independent Variable) + Intercept.

> ***Slope measures the amount by how much the dependent variable (y) changes for each unit change in the independent variable x.***

A positive value of m indicates that there is a direct relationship between the independent variable (x) and the dependent variable (y). which means that the value of y will increase by the value of m for each unit increases in the value of x. on the other hand, a negative value of m implies that there is an inverse (indirect) relationship between the value of X and the value of Y (the value of y decreases by the value of m for each unit increases in x).

Intercepts - Those points at which the graph of a line, L, crosses the axes are called intercepts. The X-intercept is the point at which the line crosses the X-axis and it is found at (X, 0) and the Y-intercept is the point at which the y-axis is crossed. Its coordinate is at (0, y).

1.3. Types of lines in linear equation

1.3.1. Horizontal and Vertical Lines

Horizontal lines are lines whose slope is zero. These lines are parallel to the X-axis. Vertical lines are lines whose slope is undefined. These lines are parallel to the Y-axis. When the equation of a line is to be determined from two given points, it is a good idea to compare corresponding coordinates because if the y values are the same the line is horizontal, and if the x values are the same the line is vertical.

Example

a) Given the points (3, 6) & (8, 6) - the line through them is horizontal because both y-coordinates are the same (6). The equation of the line becomes y=6.

b) Given the points (5, 2) and (5, 12), the line that passes through them is vertical, and its equation is x = 5. If we proceed to apply the point - slope procedure, we would obtain $\frac{12-2}{5-5} = \frac{10}{0} = \exists$, and if m = $\exists$ the line is vertical and the form of the equation is: x = constant.

1.3.2. Parallel and Perpendicular Lines

Two lines are parallel if the two lines have the same slope, and two lines are perpendicular to each other if the product of their slopes is - 1 or the slope of one is the negative reciprocal of the slope of the other. However, for vertical and horizontal lines, (they are perpendicular to each other), this rule of $m_1 . m_2 = -1$ doesn't hold true.

Example:

1. Y= 2x-10 and Y=2x+14 are parallel.
2. Y= $\frac{3}{2}$x +10 and Y = $\frac{-2}{3}$x +100 are perpendicular to each other.

1.3.3. Lines through the Origin

Any equation in the variables x and y that has no constant term other than zero will have a graph that passes through the origin. Or, a line that passes through the origin has an x-intercept and a y-intercept of (0,0). These lines are expressed in the form Y = mx.

Self-test 1.1.

Dear learners, *considering the following two equations; y = 3x+7, and Y= -3x +7, What do you understand about the relationship between the value of y and x for different values of X such as 4,5, and 6. Again considering the equation y = 3x+7 and y = $3x^{2+7}$, what do you understand about the change in y for the given value of x in the two equations*

1.4. Developing linear equation

Depending on the situation to be modeled, one of the three methods can be used to formulate a linear equation. These are:

1. The slope - intercept form
2. The slope - point form
3. Two-points form.

1.4.1. The slope-intercept form

This way of developing the equation of a line involves the use of the slope and the intercept to formulate the equation. Often the slope and y-intercept for a specific linear function are obtained directly from the description of the situation we wish to model.

E.g. Slope=10

Y-intercept=20

A line that has a slope of 10 and a y-intercept of 20 has the following equation:

$$Y=10x+20$$

Example: A Salesman has a fixed base salary of Br 200 a week. In addition, he receives a sales commission that is 20 percent of his total Birr values of sales. State the relationship between the salesman's total weekly salary and his sales for the week. Total weekly salary depends on total sales volume, thus,

Answer: $Y = 0.2x + 200$

1.4.2. The slope - point form

The equation of a non-vertical line, L, with slope, m that passes through the point (x, y is: **$Y-Y_1 = m(X-X_1)$.**

Slope = 4, point (1, 2), the equation becomes Y = 4X-2

Example: Assuming a linear relationship between total cost (y) and the number of units produced (x, and cost increases by Br. 7 for each additional unit produced, and the total cost of producing 10 units is Br. 180, formulate the equation that relates total cost (Y) and number of units produced (x).

<u>**Solution**</u>

The total cost depends on the number of units produced, then

Given = Slope = Br. 7

points (10 , 180) using , $Y-Y_1 = m(X-X_1)$

$= y- 180 = 7(x-10)$

$= y = 7x-70 + 180$

$= y = 7x + 110$

1.4.3. Two - Points Form

Two points completely determine a straight line and, of course, they determine the slope of the line. Hence we can first compute the slope, and then use this value of slope (m) together with either point in the point-slope form: $Y-Y_1 = m(X-X_1)$ to generate the equation of a line.

The slope can be computed using: $(Y-Y_1) = \frac{y_2 - y_1}{x_{x2} - x_1}(X - X_1)$

E.g. given the points (1, 10) and (6, 0)

First, find the slope $= \frac{0-10}{6-1} = \frac{-10}{5} = -2$, then use the Slope-point form.

$Y-Y_1 = M(X-X_1) =$ $Y-10 = -2(X-1)$

$Y-10 = -2X + 2$

$Y = -2X + 2$

Example: It costs a company Br. 400 to produce 20 units of a product and 30 units are produced for br. 500, assuming a linear relation between total cost and number of units of a product produced, assuming a linear relationship between total costs and number of units produced, formulate a linear equation which relates total costs with number of units produced.

Solution

Since the total cost depends on the number of units produced;

Given = points (20, 400) and (30, 500).

The slope (m) $= \frac{y_2 - y_1}{x_{x2} - x_1}$

= 400 - 300/ 30- 20 = 100/10 = br.10 , Indicating that it costs br. 10 to produce one additional unit of a product.

Once the slope is computed, the equation is formulated by using the slope and either of the points in the sloe point form;

$Y-Y_1 = m(X-X_1)$,using the first point (20,400)

= y- 400 = 10 (x- 20)

= y- 400 = 10x -200

= y= 10x+200

Alternatively using the points (30, 500), y-500 = 10(x-30) = y-500 = 10x-300

y = 10x +200

Self-test 1.2. Dear learners answer the following questions

1. *A salesman earns a weekly basic salary plus a sales commission of 20% of his total weekly sales. When his total weekly sales total Birr 1000, his total salary for the week is Birr 400. Derive the formula describing the relationship between total salary and sales.*
2. *Suppose the fixed cost for producing product x be Birr 2,000 and it costs Birr 10 per x produced. If the total cost is represented by y:*
   A. *Write the equation of this relationship in slope intercept form.*
   B. *State the slope of the line and interpret this number.*
   C. *State the y-intercept of the line and interpret this number.*
3. *A sales man has a base salary and, in addition, receives a commission, which is a fixed percentage of his sales volume. When his weekly sales are Birr 1000, his total salary is Birr 400. When his weekly sales are Birr 500, his total salary is Birr 300. Determine his base salary and his commission percentage and express the relationship between sales and salary in equation form. Answer: Y = 0.2x + 200*
4. *It costs Birr 1,400 for printing 100 copies of a report and Birr 3000 for printing 500 copies. Assuming a linear relationship what would be the price for printing 300 copies.*

1.5. Applications of linear equations in Cost – Volume – Profit (CVP) analysis

One of the areas of accounting and finance in which linear equation can be applied is in cvp analysis.

Cost –Volume –Profit (CVP) analysis is the examination of the relationship between costs production and sales volume and profits.

Depending on their behavior costs can be classified as fixed and variable costs.

A) **Fixed costs** – are costs that remain constant in total regardless of the change in the level of activity within the relevant range of activities.
   In the total cost equation fixed costs represent the intercept
B) **Variable cost** – are costs that vary in total in proportion to the change in the level of activity. However, variable cost per unit of a product remains constant. Thus, unit variable cost (v) represents the slope in the total cost equation. In other words, it is the amount by how much total cost will change for each additional units produced.

Total variable cost (TVC) = unit vu x units produced

$$TVC = V(Q)$$

Total cost = TVC + TFC

Total cost= V(Q) + TFC …. the total cost equation

Revenue also called sales is the amount of income generated by selling goods or providing services to customers. Selling price is the slope of revenue equation indicating that total revenue increases by the amount of selling prices for each unit of a product sold.

Total revenue = selling prices x number of units sold

TR = PQ ……..The revenue equation

Profit is the difference between revenues and total costs**.**

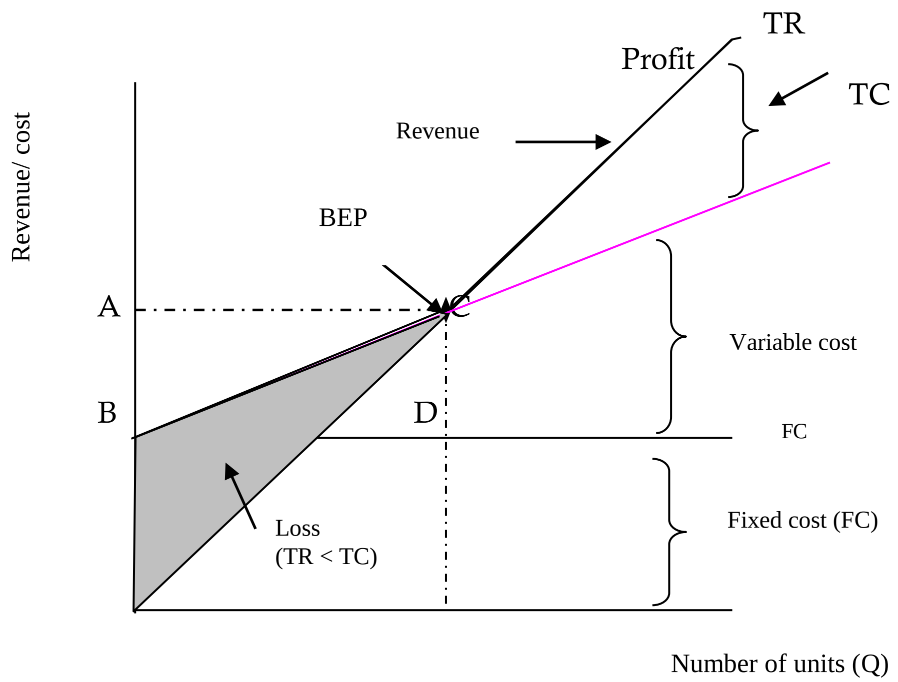
Profit = TR – TC, substituting the revenue and cost equations

= PQ – (V (Q) + TFC)

= PQ-VQ –TFC = since Q is common

P = (P-V) Q –TFC…..The profit equation

The relationship between costs, revenues and profit is depicted in the following graph.



Graph 1.1. The relationship between costs, revenues and profit

Interpretation of the graph

1. The vertical distance AB and CD is the same because fixed cost is the same at any level of output.
2. There is no revenue without sales. Hence, total revenue passes through the origin, but there is cost without production (because of total fixed cost) and the total cost function starts form A and doesn't pass through the origin.
3. Up to point F, total cost is greater than total revenue and results in loss while at point F, TR = TC = Breakeven (zero profit), and above point F, TR >TC and results in profit.
4. TFC remains constant regardless of the number of units produced, given that there is no any difference in scale of production. That is there is no either expansion or contraction of the business.
5. As production increases, TVC increases at the same rate and MC = V only in linear equations.
6. As production increases TC increases by the rate equal to the V = MC.
7. Unit variable cost V is the same throughout any level of production; however, AFC decreases when Q increases and ultimately ATC decreases when Q increases because of the effect of the decrease in AFC.
8. As Q increases TR increases at a rate of P and AR remains constant.

   AR = $\frac{TR}{Q} = \frac{p.Q}{Q} \Rightarrow AR = P =$ in linear functions.

1.5.1. Breakeven Analysis

Breakeven point (BEP) is the point at which total revenue equals total costs. In other words, there is no loss or profit to the company at BEP. It can be expressed either in terms of production quantity or revenue level depending on how the company states its cost equation. Manufacturing companies usually state their cost equation in terms of quantity (because they produce and sell) whereas retail business state their cost equation in terms of revenue (because they purchase and sell).

Case 1. Manufacturing Companies

Consider a company with equation TR = PQ

TC= VQ + FC

At BEP, TR = TC

$PQ_e = VQ_e + FC$

$PQ_e - VQ_e = FC$

$Q_e (P-V) = FC$

where:

Qe = break even quantity

FC = fixed cost

P = unit selling price

$Q_e = \frac{FC}{P-V}$ V= unit variable cost

Assumptions of Breakeven Analysis

1. Selling price is constant throughout the entire relevant range [relevant range – is the limit of cost-driver activity within a specified relationship between costs and the cost driver is valid].
2. Costs are linear over the relevant range.
3. In multi-product companies, the sales mix is constant.
4. In manufacturing firms, inventories do not change (Units produced = Units sold).
5. Expenses may be classified in to variable and fixed categories. Total variable expenses vary directly with activity level. Total fixed expenses do not change with activity level.
6. Efficiency and productivity will be unchanged.

Example: A manufacturing company has a fixed cost of 10,000 and a unit variable cost of Birr 5. If the company can sell what it produces at a price of Birr 10?

a. Formulate the revenue, cost and profit equations and interpret the result.

b. Find the breakeven point in terms of quantity and sales volume.

Solution

a) Revenue equation , TR = PQ = 10Q

Which implies that revenue increases by br.10 for each additional unit of a product sold.

Total cost equation, TC = VQ+ TFC = 5Q + 10,000

Indicating that total cost increases by br. 5 for each additional units of product produced.

Profit equation, P = (P-V) Q – TFC = (10-5) Q – 10,000 = 5Q - 10,000

Which implies that the total profit will increase by br.5 for each additional units of a product sold and the company will incurr a cost of br.10,000 if sales (production) is zero.

b) BEP in Quantity = TFC/P-V = 10,000/ 10-5 = 10,000/5 = Br.2,000.

BEP in revenue = BEPq x p = 2,000 x Br. 10 = Br. 20,000.

1.5.2. The Effect of Changing One Variable Keeping Others Constant on BEP

Changing one or more of the elements of CPV will have effects on the BEP.

Case 1: Fixed Cost

Assume for the above problem FC is decreased by Br 5,000, citrus paribus.

TC = 5Q + 5,000 $\quad Qe_1 = \frac{5,000}{5}$ = 1,000 units

TR = 10Q

Therefore, $\left.\begin{cases} FC\downarrow \quad Qe\downarrow \\ FC\uparrow \rightarrow Qe\uparrow \end{cases}\right\} FC\ \&\ Qe\ have\ Direct$ relationship

Case 2 - Unit variable cost

Assume for the above problem unit variable cost decreased by Birr 1, citrus paribus

TC = 4Q + 10,000 $\quad Qe_2 = \frac{10,000}{6}$ = 1,667 units

TR=10Q

Therefore, $\left.\begin{cases} V\downarrow\rightarrow Qc\downarrow \\ V\uparrow\rightarrow Qe\uparrow \end{cases}\right\}$ V & Qe have direct relationship

Case 3- Selling Price

Assume for the above problem selling price is decreased by Birr 1, Citrus Paribus.

TC = 5Q + 10,000 $\quad \Rightarrow Qe_3 = \frac{10,000}{4} = 2,500$ units

TR = 9Q

Therefore, $\left.\begin{matrix} P\downarrow \rightarrow Qe\uparrow \\ P\uparrow \rightarrow Qe\downarrow \end{matrix}\right\}$ P & Qe have indirect relationship

In the above example if a company can't produce and sell 5,000 units it has the following options:

a) Decreasing FC
b) Decreasing unit variable cost
c) Increasing the unit selling price

If the organization is faced between cases two and three, it is preferable to decrease the unit variable cost because if we increase the selling price, the organization may loose its customers; and also decreasing the FC is advisable.

1.5.3. Finding the Quantity level that involves profit or loss.

BEP = $\frac{FC+0}{P-v}$, Any Q is related to the cost, profit, ---

$$\Pi = TR - TC$$
$$= PQ - (VQ + FC)$$
$$= Q(P-V) - FC$$
$$\Pi = Q(P-V) - FC$$
$$FC \pm \Pi = Q(P-V)$$
$$\frac{FC \pm \Pi}{P-V} = Q \Rightarrow \text{ For any quantity level.}$$

Example: For the above manufacturing company, if it wants to make a profit of Birr 25,000, what should be the quantity level?

Solution

$\frac{FC \pm \Pi}{P-V} = Q \Rightarrow$ thus, Q= 10,000 + 25,000/5 =35,000/5 = 7,000 units.

This tells us when there is a profit; the quantities produced and sold have to be greater than the break-even quantity.

1.5.4. Applications of linear equations in CVP analysis for merchandising companies

Merchandisers (retailers) are business that acquire goods to resale it at a price above the purchase price. The difference between the selling price and purchase price is called a markup which is used to cover other selling expenses and to provide profit.

Markup can be expressed as a function of cost and sales as follows;
Markup as a function of cost is called **margin** = markup/ cost
Markup as a function of sales = markup /selling price (sales)

Example : Assume a business firm with product A has the following cost and revenue items.
Purchase cost of A = 100 Br
Selling price = 150 Br

Markup = Selling price - Cost = 150-100=50.

Mark up can be expressed:

As a function of cost, the mark-up percentage is 50/100 = 50%

As a function of retail price, the mark up is $50/150 = 33.3\%$, it is also called margin.

$$\underline{100\% - 33.3\% = 66.6\% = 67\%}$$

↓ ↓ ↓

Selling price *margin* *Cost of goods sold*

The cost of goods sold =

Selling expense = 1 percent of the selling price = 0.01x

So, the total cost equation becomes:

Y=0.68x + FC; Where X = sales revenue, and
Y = total cost.

The above 68% is interpreted as, Out of the 100% selling price 68% is the variable cost of goods purchased and sold.

To get the break-even sales volume level, we equate the total cost, Y with the sales volume level, X as X_e = Y = X, Y = mx + b

X = mx + b

X – mx = b

X (1 – m) = b

$$X_e = \frac{b}{1 - m}$$; Where m= unit variable cost /Birr of sales.

Example: Suppose a retail business sells its commodities at a margin of 25% of sales and the company uses a 5% commission as selling expense and Birr 12,000 as a fixed cost. Find the break-even revenue for the retail business after developing the TC equation.

<u>Solution:</u>

Let: x= sales revenue
Y= Total cost

Margin = 25% (0.25)

Purchase price = 1- margin = 1-.25 = 0.75

The vc = 0.75x + .05x (sales commission) , then the total cost equation is

Y= 0.8x + 12,000

The breakeven revenue = Total fc / 1-m = 12,000/1-0.2 = Br. 60,000.

The break-even revenue method is useful, because we can use a single formula for different goods so far as the company uses the same amount of profit margin for all goods. However, in breakeven quantity method it is not possible and hence we have to use deferent formula for different items.

When the break-even revenue equation is for more than one item it is impossible to find the break-even quantity. It is only possible for one item. By Qe = $\frac{Xe}{P}$ where

Xe = break-even revenue.

P= Selling price.

Qe = break-even quantity**.**

Given that the company purchases and sells single product, to change the cost equation in terms of revenue in to a cost equation in terms quantity we have to multiply price by the coefficient of X that is m. To change the cost equation in terms of quantity in to a cost equation in terms revenue we divide the unit variable cost, V, by the corresponding unit selling price.

Self-test 1.3. Dear learners, Answer the following questions

1. *ABC' company's cost function for the next four months is C= 500,000 + 5q.*

Required:

d) *The break-even dollar volume of sales if the selling price is Birr 6 per unit.*
e) *What would be the company's cost if it decided to shut down operations for the next four months?*
f) *If, because of a strike, the most the company can produce is 100,000 units, should it shut down? Why or why not?*

2. *In its first year, A Company had the following data.*

Sales = 25,000 units *Selling price = Birr 100*

Total variable cost = Birr 1,500,000 *TFC = Birr 350,000*

Required:

a) *Develop revenues, cost, and profit functions for the company in terms of quantity.*
b) *The intercept of the revenue equation is zero, what do you think is the reason?*
c) *Find the break-even point in terms of quantity.*
d) *Convert the cost equation in terms of quantity in to a cost equation in terms of revenue.*

e) *Find the break-even revenue.*

f) *If profit had been Birr 500,000 what would have been the sales volume (revenue) and the quantity of sales.*

g) *What would have been the profit if sales were Birr 2,000,000?*

3. *A retail co plans to work on a margin of 44% of retail price and to incur other variable costs of 4%. If it expects fixed cost of Birr 20,000,*

a) *Find the equation relating total cost to sales.*

b) *Find the profit it sales are Birr 60,000.*

c) *Find the breakeven revenue.*

d) *If profit is Birr 15,000, what should be the revenue level?*

e) *If the company has only one item at a price of Birr 15 per unit, how do you convert the cost equation in terms of revenue in to a cost equation in terms of quantity?*

1.6. Summary

Dear student, with confidence, you have already acquired knowledge about the concepts and the interpretative applications of linear equations, functions, and graphs in business. In this unit, we have considered the managerial applications of linear algebra and geometry so far. In so, we have considered that linear equations are mathematical expressions written in the form of

$$\mathrm{y = m\,x + b}$$

The graph of such equation on coordinate plane is a straight line. As a result, the slope of the line is constant for any given points on the line. The slope of a straight-line m, given two points on the line with coordinates of (x_1, y_1) and (x_2, y_2) is expressed by the equation.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Further, we have considered how to compute the distance between two points on a coordinate plane. Subsequently, approaches of developing equation of a line are discussed in the present unit. Above all, we have seen the interpretive applications of linear equations: analysis of linear cost-output relations, break-even analysis, and market supply and demand equilibrium analysis. In the next section, we will advance with the study of the matrix algebra and its application in solving business problems and backing management decisions that further organizational interests.

1.7. Review Questions

Part I: multiple choices

Instruction: Choose the best answer from the given alternatives

1. From the following which one is linear equation?
   - A. $6x^2 + 2y = 7$
   - B. $6x + 4y – 6 = 2$
   - C. $2yx + 7y = 0$
   - D. $\sqrt{y} + 2x = 5$
2. Suppose travelling cost of one person to A.A from Dessie by airline have fixed airline ticket of Birr 1,250. But the person should add 20 Birr for one Kilo gram materials it take with him. What is the marginal cost of adding one kilo gram material which transported with him?
   - A. 1,250
   - B. 12
   - C. 20
   - D. 1,270
3. From question # 3, what is the total cost travelling to A.A for one person if the materials of the person are 10 kilo gram?
   - A. 1,450
   - B. 1,270
   - C. 1,250
   - D. 20
4. Suppose the cost of college for accounting courses are linear relationship with its credit hours. To take 48 credit hours it costs 6000 Birr and for 64 credit hours 8000 Birr. On the first semester one student required to take 28 credit hours. The equation for this relationship is:-
   - A. Y = 250x
   - B. Y = 122x + 5
   - C. Y = 123x
   - D. Y = 125x
5. Assume that for question # 4, BS wants to study accounting in college. How much birr it costs him for one semester.
   - A. 6,000
   - B. 3,500
   - C. 8,000
   - D. 4,000
6. X company produces and sales a commodity for $10. The company's cost function is given by the equation (y) =5x + 250 where x is units of products produced and sold. what profit is earned if the annual sale are 1000
   - A. 250
   - B. 5250
   - C. 4750
   - D. None

7. A company expects fixed cost of Br. 100,000. It plans to work on a margin of 45% of retail price and to incur variable costs of Br. 0.05 per Br. of sales. What is the breakeven level of sales volume?
   - A. 250,000
   - B. 280,000
   - C. 180,000
   - D. None
8. For the function y = 3x + 2 the average rate of change of y when x increases from 1.5 to 1.7 is
   - A. 1
   - B. 0.5
   - C. 0.6
   - D. 0.3
9. What is the correct slope and y-intercept for the equation y = -3x?
   - A. slope: 0, y-intercept: -3
   - B. slope: -3, y-intercept: 0
   - C. slope: 3, doesn't have a y-intercept
   - D. slope: -3, doesn't have a y-intercept
10. If a line is horizontal, its slope is
    - A. 1
    - B. 0
    - C. Undefined
    - D. Negative
11. The following equations lines are parallel each other except one equations line, which equations line is it?
    - A. 2x + y = 20
    - B. 8y + 16x – 8 = 0
    - C. 3y + 9x + 3 = 0
    - D. 24x + 12y = 7
12. The equation of the line that pass through the origin, parallel to the line y=3x -14 is:
    - A. Y=-1/3x
    - B. y= 3x
    - C. y=3x + 14
    - D. None
13. On the line passing through (4, 6) and (2, 10), what is the y-coordinate of the point where x = 20
    - A. -10
    - B. 26
    - C. 24
    - D. None
14. break even quantity have:
    - A. A direct relationship with fixed cost
    - B. Indirect relationship with variable cost
    - C. A direct relationship with price
    - D. All
    - E. None

15. One of the following statement is **false**:-
    A. As production increases TC increases by the rate equal to the V = MC.
    B. Average fixed cost decreases when quantity increases and ultimately average total cost decreases when quantity increases.
    C. To change the cost equation in terms of revenue in to a cost equation in terms quantity we have to dived price by the coefficient of X that is “m”
    D. To change the cost equation in terms of quantity in to a cost equation in terms revenue we divide the unit variable cost by the price.

<u>Part II: work out</u>

Dear learners you are required to attempt the following questions by showing all the necessary steps.

1) A company expects fixed cost of birr 400,000. Margin estimated to be 52% of retail, and variable cost in addition to cost of goods sold is estimated to be birr 0.007 per sales

   <u>Required</u>

   a) The total cost function
   b) The total variable cost if sales is estimated to be 200,000
   c) Break even sales point
   d) Profit if sales is estimated to be 800,000
2) A manufacturer of cassette tapes has a fixed cost of Birr 75,000 and a variable cost of Birr 7 per cassette produced. Selling price is Birr 10 per cassette.
   a) Write the revenue, cost and profit functions
   b) At what number of unit will break even occur?
   c) At what sales volume (revenue) will break even occur?
   d) If profit is Birr 15,000, what should be the revenue level?
   e) Compute profit if;
      i) 20,000 units are made and sold.
      ii) 50,000 units are made and sold.

Answer for chapter review Questions

Part one: Choice

1.B

2.C

3.A

4.D

5.B

6.C

7.A

8.C

9.B

10.B

11.C

12.B

13.D

14.A

15.C

Chapter Two: Matrix Algebra and its Applications

Arthur Cayley (1821-1895) of England was the first Mathematician to introduce the term "matrix" in the year 1858. But in the present day applied Mathematics in overwhelmingly large majority of cases it is used, as a notation to represent a large number of simultaneous equations in a compact and convenient manner. Matrix Theory has its applications in Operations Research, Economics and Psychology. Apart from the above, matrices are now indispensable in all branches of Engineering, Physical and Social Sciences, Business and Statistics.

Chapter Objectives:

Dear students! at the end of this chapter you should be able to:

- Define matrix and identify the different types of matrices
- Perform a mathematical operation on the matrices including matrix addition, subtraction and matrix multiplication
- Understand and demonstrate the application of matrices in solving linear equations using inverse method.
- Understand the Gaussian method to solve word problems in the area of accounting and finance.
- Understand the Cramer's rule to solve word problems in the area of accounting and finance

2.1. Meaning and importance of matrix

A **matrix** is a rectangular array of numbers or elements of a ring. One of the principal uses of matrices is in representing systems of equations of the first degree in several unknowns. Each matrix row represents one equation, and the entries in a row are the coefficients of the variables in the equations, in some fixed order.

Matrix is a rectangular array of real numbers arranged in m rows and n columns. like sets, it is symbolized by a bold face capital letter enclosed by brackets or parentheses as:

$$A = \begin{bmatrix} a_{11} a_{12} --- a_{1n} \\ a_{21} a_{22} --- a_{2n} \\ a_{m1} a_{m2} --- a_{mn} \end{bmatrix}$$ in which a_{ij} are real numbers.

Each number appearing in the array is called an **element or component** of the matrix. Elements of a matrix are designated using a lower case form of the same letter used to symbolize the matrix itself. These letters are subscript, as a_{ij}, to give the row and column location of the element with in the array. The first subscript always refers to the row location of the element; the second subscript always refers to its column location. Thus, component a_{ij} is the component located at the intersection of the i^{th} row and the j^{th} column.

The number of rows, m, and the number of columns, n, of the array give its order, or its dimensions, mxn (read "m by n") = a mxn or $[a_{ij}]$ (mxn).

Example: The following are examples of matrices

$$A = \begin{bmatrix} 1 & 7 \\ 5 & 3 \\ 4 & 2 \end{bmatrix}$$ This is a 3 x 2 matrix

The size of a matrix is given by the number of rows and columns, so that M_1, M_2, M_3, and M_4 are, in that order, of sizes 3 × 3 (3 by 3), 3 × 3, 3 × 2, and 2 × 3. The general matrix of size $m \times n$ is frequently represented in double-subscript notation, with the first subscript i indicating the row number, and the second subscript j indicating the column number; a_{23} is the element in the second row, third column. This general matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

may be abbreviated to A = $[a_{ij}]$, in which the ranges $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ should be explicitly given if they are not implied by the text. If $m = n$, the matrix is square, and the number of rows (or columns) is the order of the matrix. Two matrices, A = $[a_{ij}]$ and B = $[b_{ij}]$, are equal if and only if they are of the same size and if, for every i and j, $a_{ij} = b_{ij}$. The elements a_{11}, a_{22}, a_{33}, ... constitute the main or principal diagonal of the matrix A = $[a_{ij}]$, if it is square. The transpose A^T of a matrix A is the matrix in which the ith row is the ith column of A and in which the jth column is the jth row of A.

Some of the above matrix elements are the followings: $a_{11} = 1$, $a_{12}= 7$, $a_{21} = 5$, $a_{22} = 3$, $a_{31}=4$ and $a_{32}=2$

$$\begin{bmatrix} 1 & 5 & 9 & 15 \\ 2 & 6 & 10 & 20 \\ 3 & 7 & 11 & 30 \\ 4 & 8 & 12 & 45 \end{bmatrix}$$ This is a 4 x 4 matrix Elements $X_{44} = 45$ $x_{32} = 7$

Importance of matrices

Matrices provide a most convenient vehicle for organizing and storing large quantities of data. Because the basic idea is to organize the data, we cannot over emphasize the importance of the location of each number with in the matrix. It is not simply a matter of putting numbers in to rows and columns; each row-column location within each matrix carries with it special interpretation; a matrix is, in essence, a tool for organizing vast quantities of data. Matrices are used to represent complex systems and operations by compact entities.

Matrix representations are possible

- ⇒ Transportation matrix
- ⇒ Distance matrix
- ⇒ Cost matrix
- ⇒ Brand switching

In general, matrices are used:

- ➢ To handle large linear systems
- ➢ To solve complex linear equations
- ➢ An effective means for summarizing and organizing voluminous data

Self-test 2.1. Dear learners, given the following matrix A;

$$\begin{bmatrix} 11 & 12 & 5 \\ 12 & 10 & 7 \\ 2 & 4 & 9 \\ 6 & 3 & 8 \end{bmatrix}$$

A) Determine its dimension

B) Find the element a_{12}, a_{23}, a_{31}, and a_{44}

2.2. Types of matrices

Based on their dimension (order) or size, matrices are classified in to different ways. The following are some of the matrices types:

1. **Vector matrix** - is a matrix which consists of either one row or one column. That is, it is an mx1 or a 1 x n matrix.
   a. **Row vector** is a vector matrix which has only one row and two or more columns. It is a 1 x n matrix.
      E.g. W = [-1, 0, 6] is 1x3 row vector
   b. **Column Vector** is a vector matrix which has only one column and two or more rows. It is a mx1 matrix.

E.g. B = $\begin{bmatrix} 2 \\ 5 \\ 7 \\ 0 \end{bmatrix}$ is a 4x1 column vector

The transpose of an mxn matrix denoted A^{-t} is an nxm matrix whose rows are the columns in A (in the same order) and whose columns are the rows in A (in the same order).

If $A = \begin{bmatrix} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{bmatrix}$ then A^{-t} = $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \\ 10 & 11 & 12 \end{bmatrix}$

Note that $a_{ij}{}^{t} = a_{ij}$

The transpose of a row vector is a column vector and the transpose of a column vector is a row vector.

2. **Square Matrix** is a matrix that has the same number of rows and columns. It is also called an n^{th} order matrix.

   E.g. 2x2, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or 4x4, B= $\begin{bmatrix} 1 & 5 & 9 & 15 \\ 2 & 6 & 10 & 20 \\ 3 & 7 & 11 & 30 \\ 4 & 8 & 12 & 45 \end{bmatrix}$ are square matrices

3. **Null (zero) matrix** - is a matrix that has zero for every entry. It is generally denoted by O_{mn}. In matrix operations it is used in much the same way that the number zero is used in regular algebra. Thus, the sum of a zero matrix and any matrix gives that given matrix and the product of a zero matrix and any matrix equals that zero matrix.

4. **Identity matrix** - a square matrix in which all of the primary diagonal entries are ones and all of the off diagonal entries are zeros. Generally, it is denoted as I_n. Primary diagonal represents: a_{11}, a_{22}, a_{33}, a_{44}, --- a_{nn} entries.

   I_2 = $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, I_4 = $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   The product of any given matrix and the identity matrix is the given matrix it self. That is, A x I = A and I.A = A. Thus, the identity matrix behaves in matrix multiplication like the number 1 in an ordinary arithmetic.

5. **Scalar matrix** - is a square matrix where elements on the primary diagonal are the same and the rest zeros.
6. **Diagonal matrix**- a square matrix where elements on the primary diagonal are consecutive and others zeros.
7. **Equal matrices** -two matrices A & B, are said to be equal only if they are of the same dimensions and if each element in A is identical to its corresponding element in B; that is, if and only if a_{ij} = bij for every pair of subscripts i and j. If A = B, then B = A; or if A≠B, then B ≠A.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ Is equal to } \quad B = A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\text{However; } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ is not equal to } C = A = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

Even though they contain the same set of numerical values, A and C are not equal because their corresponding elements are not equal; that is, $a_{11} \neq C_{11}$ and so on.

Self-test 2.2. Dear learners state whether do you agree or disagree for the following statements and justify for your response.

1. An Identity matrix is a scalar matrix, but a scalar matrix may not be an identity matrix…………………………………………………………………………………… ………………………………………………………………………………………… …………………………………………………………………………………………
2. A scalar matrix is a diagonal matrix but a diagonal matrix may not be a scalar matrix…………………………………………………………………………………… ………………………………………………………………………………………… ………………………………………………………………………………………… …………..

2.3. Matrix Algebra

Algebra - is a part of mathematics that deals with operations (+, -, x÷). Thus, matrix algebra is mathematical operation performed on the matrix. As well, it is also called as matrix operation (the application of mathematical operations like addition, subtraction, multiplication and division in matrices).

2.3.1. Matrix Addition (subtraction)

Two matrices of the same dimensions are said to conformable for addition. The addition is performed by adding corresponding elements from the two matrices and entering the reset in the same row-column position of a new matrix [element-wise addition].

If A and B are two matrices, each of size mxn, then the SUM of A and B is the mxn matrix C whose elements are:

$C_{ij} = A_{ij} + b_{ij}$ for $i = 1, 2,$ ------- m

$j = 1, 2,$ -------- n.

2.3.2. **Laws** of Matrix Addition

The operation of adding two matrices that are conformable for addition has these two basic properties:

1. A + B = B + A ---- The commutative law of matrix addition.
2. (A+B) +C = A+ (B+C) -------- the associative law of matrix addition.

$$eg\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 7 & 9 \\ 8 & -10 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 10 & -6 \end{bmatrix}$$

Given that two matrices do have the same dimension; the way we subtract a matrix from another matrix is the same as the way we add two matrices.

2.3.3. Matrix Multiplication

There are types of matrix multiplications. These are;

a. Scalar multiplication
b. Vector-by-Vector multiplication
c. Matrix by matrix multiplication

a. Matrix Multiplication by a Constant (Scalar Multiplication)

A matrix can be multiplied by a constant by multiplying each component in the matrix by a constant. The result is a new matrix of the same dimensions as the original matrix.

If K is any real number and A is an mxn matrix, then the product KA is diffident to be the matrix whose components are given by k times the corresponding component of A; that is, $KA= [Ka_{ij}]$ (mxn).

Example: If X = [6 5 7], then 2X = [(2x6) (2x5) (2x7)]

2X = [12 10 14]

Laws of Scalar Multiplication

The operation of multiplying a matrix by a constant (a SCALAR) has the following basic properties. If x and y are real numbers and A and B are mxn matrices, conformable for addition, then:

1. XA = AX
2. (X+Y)A = XA+YA
3. X (A+B) = XA + XB
4. X (YA) = XY (A)

b. Vector-by-Vector multiplication

In multiplying two vectors always a row vector is written in the first position and the column vector in the second position. Each component of a row vector is multiplied by the corresponding component of the column vector to obtain a result known as PARTIAL PRODUCT. The sum of all partial products is called INNER/DOT PRODUCT of two vectors, and this is a number not a vector. In other words, Vector- by- Vector results in a real number rather than a matrix.

Example: Consider the product (AB) of the following row and column vectors.

$$A = \begin{bmatrix} 3 & 4 & -2 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ 5 \\ 7 \\ 0 \end{bmatrix}$$

3 x 2 = 6
4 x 5 = 20
-2 x 7 = -14
6 x 0 = 0 } partial products
12 ← Inner/Dot Product

c. Matrix by Matrix Multiplication

If A and B are two matrices, the product AB is defined if and only if the number of columns in A is equal to the number of rows in B, i.e., if A is an m x n matrix, B should be an n x b. If this requirement is met, A is said to be conformable to b for multiplication. The matrix resulting from the multiplication has dimensions equivalent to the number of rows in A and the number of columns in B.

Matrix by matrix multiplication indicates a row by column multiplication, where the entry in the i^{th} row and j^{th} column of the product AB is obtained by multiplying the entries in the i^{th} row of A by the corresponding entries in the j^{th} column of B and then adding the results. That is, to obtain the entry in the i^{th} row and j^{th} column of the product AB, use the i^{th} raw of A and the j^{th} column of B in the following form:

> The first element in the raw is multiplied by the first element in the column; the second element in the row is multiplied by the second element in the column and so on until the n^{th} row element is multiplied by n^{th} column element. These products are then summed up to obtain the single number that is the product of the two vectors.

If A is a matrix of dimension n x m (which has m columns) and B is a matrix of dimension p x q (which has p rows) and if m is different from p, the product AB is not defined. That is, multiplication of matrices is possible only if the number of columns of the first equals the number of rows of the second.

If A is of dimension n x m and if B is of dimension m x p, then the product A.B is of dimension n x p.

Example

$A = \begin{bmatrix} 2 & 3 & 4 \\ 6 & 9 & 7 \end{bmatrix}$ B = $\begin{bmatrix} -1 & 7 \\ 0 & 8 \\ 5 & 1 \end{bmatrix}$

A.B = (2x-1) + (3x0) + (4x5) (2x7) + (3x8) + (4x1)
= 18 42

= (6x-1) + (9x0) + (7x5) (6x7) + (9x8) + (7x1)
= 29 121

AB = $\begin{bmatrix} 18 & 42 \\ 29 & 121 \end{bmatrix}$

The result for BA is different:

B.A = (-1x2) + (7x6) = 40 (-1×3) + (7×9) = 60 (-1×4) + (7×7) = 45
= (0×2) + (8×6) = 48 (0×3) + (8×9) = 72 (0×4) + (8×7) = 56
= (5×2) + (1×6) = 16 (5×3) + (1×9) = 24 (5×4) + (1×7) = 27

$$BA = \begin{bmatrix} 40 & 60 & 45 \\ 48 & 72 & 56 \\ 16 & 24 & 27 \end{bmatrix}$$

Special Properties of Matrix Multiplication

1. The Associative and distributive laws of ordinary algebra apply to matrix multiplication. Given three matrices A, B and C, which are conformable for multiplication?
   - A (BC) = (AB) C -------------------- Associative law, not C (AB).
   - A (B+C) = AB + AC -------------- Distributive law
   - (A+B) C = AC + BC -------------- Distributive law

2. The commutative law of multiplication does not apply to matrix multiplication. For any two real numbers X and Y, the product XY is always identical to the product YX. But for two matrices A and B, it is not generally true that AB equals BA. (In the product AB, we say that B is pre multiplied by A and that A is post multiplied by B). In many instances for two matrices A and B, the product AB may be defined while the product BA is not defined, or vice versa. In some special cases, AB does equal BA. In such special cases A and B are said to Commute.

3. The product of two matrices can be the zero matrixes even though neither of the two matrices themselves is zero matrix! We cannot conclude from the result AB = 0 that at least one of the matrices A or B is a zero matrix.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 7 & -10 & 4 \\ 8 & 3 & 2 \end{bmatrix}, AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4. In matrix Algebra, necessarily conclude from the results AB = AC that B = C, even if matrix A is not equal to a zero matrix. Thus the cancellation law does not hold, in general, in matrix multiplication.

$$A\begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}, B\begin{bmatrix} 4 & -1 \\ 2 & 5 \end{bmatrix}, C\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$AB = AC = \begin{bmatrix} 10 & 14 \\ -20 & -28 \end{bmatrix} \text{ but } B \neq C.$$

Self-test 2.3. Dear learners, *given the following three matrices A, B and C, try to answer the following questions.*

$$A = \begin{bmatrix} 2 & x \\ 4 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 7 \\ 4 & 5 \end{bmatrix} \quad C = \begin{bmatrix} -9 & 3 \\ 1 & 6 \end{bmatrix}$$

Required:

A) Assuming, 2A = 4B, find the value of x.

B) Compute A+B and B+A

C) Calculate A-B and B-A

D) Is matrix subtraction commutative? Why or why not……………………………………

………………………………………………………………………………………………………

………………….

E) is the product AB defined? Why

………………………………………………………………………………………………………

………………………………………………………………………………………………………

………………

F) Are A and B matrices commute? Justify for your response………………………………………………………………………………………

………………………………………………………………………………………………………

……………

G) Proof the following;

- *A (BC) = (AB) C*
- *A (B+C) = AB + AC*
- *(A+B) C = AC + BC*

2.4. The Multiplicative Inverse of a Matrix

If A is a square matrix of order n, then a square matrix of its inverse (A^{-1}) of the same order n is said to be the inverse of A, if and only if $AA^{-1} = I = A^{-1}A$.

Two square matrices are inverse of each other if their product is the identity matrix: $I = AA^{-1} = A^{-1}A$.

Not all matrices have an inverse. In order for a matrix to have an inverse, the matrix must, first of all, be a square matrix. Still not all square matrices have inverse. If a matrix has an inverse, it is said to be INEVITABLE or NON-SINGULAR. A matrix that doesn't have an inverse is said to be SINGULAR. An inevitable matrix will have only one inverse; that is, if a matrix does have an inverse and that inverse is unique.

In short:

- Inverse of a matrix is defined only for square matrices
- If B is an inverse of A, then A is also an inverse of B.
- Inverse of a matrix is unique.
- If matrix A has an inverse, A is said to be inevitable and not all square matrices are inevitable.

$$eg\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Finding the Inverse of a Matrix

Let us begin by considering a tabular format where the square matrix. A is augmented with an identity matrix of the same order, as [A/I]. This process is called adjoining. Now, if the inverse matrix A^{-1} were known, we could multiply the matrices on each side of the vertical line by A^{-1}, as $[AA^{-1}/A^{-1}I]$. Then, because $AA^{-1} = I$ and $A^{-1}\, I = A^{-1}$, we would have $[I/A^{-1}]$. We do not follow this procedure, because the inverse is not known at this juncture; we are trying to determine the inverse. We instead employee a set of permissible row operations on the augmented matrix [A/I] to transform A on the left side of the vertical line in to an identity matrix (I). As the identity matrix is formed on the left of the vertical line, the inverse of A is formed on the right side. The allowable manipulations are called ELEMENTARY ROW OPERATIONS. These Elementary Row Operations are operations permitted on the row of a matrix.

In a matrix Algebra there are 3 types of row operations.

i. Any pair of row in a matrix may be interchanged /Exchange operations/. **Interchanging rows.**

ii. A row can be multiplied by any non-zero real number /Multiple operations/. **The multiplication of any row by a non-zero number.**

iii. A multiple of any row can be added to any other row /Add-A-Multiple operations/. **The addition /subtraction of (a multiple of) one row to/from) another row.**

Example. 1. $A\begin{bmatrix} 4 & 3 & 2 \\ -2 & 6 & 7 \end{bmatrix}$, B $\begin{bmatrix} -2 & 6 & 7 \\ 4 & 3 & 2 \end{bmatrix}$ = interchanging rows

2. $A\begin{bmatrix} 4 & 3 & 2 \\ -2 & 6 & 7 \end{bmatrix}$ B = $\begin{bmatrix} 8 & 6 & 4 \\ -2 & 6 & 7 \end{bmatrix}$ multiplying the first row by 2

3. $A\begin{bmatrix} 4 & 3 & 2 \\ -2 & 6 & 7 \end{bmatrix}$ B = $\begin{bmatrix} 4 & 3 & 2 \\ 6 & 12 & 11 \end{bmatrix}$ = Multiplying the first row by 2 and add to 2nd row.

Theorem on row operations

A row operation performed on product of two matrices is equivalent to row operation performed on the pre-factor.

Consider the following AB = C

$A\begin{bmatrix} -1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ B = $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$ C, = $\begin{bmatrix} 9 & 13 \\ 13 & 19 \end{bmatrix}$

Interchange R_1 with R_2

$A\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$ B = $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$ C, = $\begin{bmatrix} 13 & 19 \\ 9 & 13 \end{bmatrix}$

Basic Procedures to Find the Inverse of a Square Matrix

1. To get ones first in a column and next zeros (within a given column)
2. To get zeros first in a matrix and next ones.

One's First: Try to set ones first in a column and then zeros of the same column.

Go from left to right

Zeros First: Find the off diagonal zeros first, and following this obtain ones on the main diagonal. It can simplify the work involved in hand calculation by avoiding fractions until the last step.

Self-test 2.4. Dear learners, find the inverse of the following matrices:

A) $\begin{bmatrix} 4 & 6 \\ 5 & 4 \end{bmatrix}$

B) $\begin{bmatrix} 4 & 8 & 9 \\ 8 & 3 & 15 \\ 12 & 11 & 10 \end{bmatrix}$

C) $\begin{bmatrix} 11 & 12 & 5 \\ 12 & 10 & 7 \\ 2 & 4 & 9 \\ 6 & 3 & 8 \end{bmatrix}$

2.5. Matrix Applications

The inverse of the matrix is an important instrument in order to solve linear equations and find the values of variables in the equation. The following two examples illustrates how to solve linear equations using the inverse of a matrix.

2.5.1. Solving Systems of Linear Equations

n by n systems

Systems of linear equations can be solved using different methods. Some are:

- Elimination method for 2 variable problems (equations).
- Matrix method
  - i. Inverse method
  - ii. Gaussian Method.
  - iii. Cramer's rule – using determinants (independent study)

Elimination method

The elimination of variable procedure is generally satisfactory for solving systems of two equations in two variables. However, for systems containing more than two equations and involving more than two variables, this procedure is not as efficient as we would like; so we find ourselves turning to another solution technique. The alternative solution procedure we shall use employs a **MATRIX FORMAT** to organize the data.

The two linear equations of the system

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

can be represented by the **MATRIX EQUATION AX = B** where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is called the **COEFFICIENT MATRIX,**

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is the **SOLUTION VECTOR,** or **VECTOR OF UNKNOWNS,**

$$\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

is called the **RIGHT-HAND-SIDE VECTOR,** or **VECTOR OF CONSTANTS.**

The vector **X =** $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$ represents a solution of the system if both equations are satisfied when we substitute $x_1 = n_1$ and $x_2 = n_2$.

Matrix Method

i. Inverse Method

The inverse of the matrix is an important instrument in order to solve linear equations and find the values of variables in the equation. The following two examples illustrates how to solve linear equations using the inverse of a matrix.

Example 1: Find the value of X, Y, determinant method

$$x - 3y = 3$$
$$x + 2y = 8$$

Solution

After we get the inverse of the matrix, if we multiply the inverse with the product matrix we will get the values of x and y.

$$x - 3y = 3$$
$$x + 2y = 8$$

The matrix form of the above equations is

$$\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

Here our intension is to change the left matrix in to identity matrix

R1:	1	-3	|	3	1	0
R2:	1	2	|	8	0	1

Goal 1: change row one column one in to 1

Step 1: just put it as it is, because it is already 1.

R1:	1	-3	|	3	1	0
R2:	1	2	|	8	0	1

Goal 2: Change row two column one in to 0

Step 2: Subtract row one from row two R2→ R2 – R1

R1:	1	-3	|	3	1	0
R2:	0	5	|	5	-1	1

Goal 3: Change row two column two in to 1

Step 3: divide row two by 5 $R2 \rightarrow \frac{R2}{5}$

R1: 1 -3 | 3 | 1 0

R2: 0 1 | 1 | $\frac{-1}{5}$ $\frac{1}{5}$

Goal 4: Change row one column two in to 0

Step 4: multiply row two by 3 and add from row one R1→ R1 + 3R2

R1: 1 0 | 6 | $\frac{2}{5}$ $\frac{3}{5}$ → Inverse of a matrix

R2: 0 1 | 1 | $\frac{-1}{5}$ $\frac{1}{5}$

Now the left side is changed to identity matrix then our result is x = 6 and y = 1. This value is correct because we have already proved above.

Matrix Determinant

It is also possible to find the invers of a matrix using determinate method. ***Determinant of a matrix*** is a number which determines either the matrix has an inverse or not.

2 x 2 matrix determinant

For any matrix which has 2 x 2 dimension, the determinant is calculated as follows. First, we have to assign a, b, c, d for each element in the matrix as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then, determinant of the matrix is denoted by **det (A) or /A/** and calculated by **ad – bc**

Example 1: Find the determinant of the matrix A and B, if

$$A = \begin{bmatrix} 1 & 5 \\ -2 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & -2 \\ 12 & 6 \end{bmatrix}$$

Solution: A = $\begin{bmatrix} 1 & a & 5 & b \\ -2 & c & 8 & d \end{bmatrix}$

Det (A) = /A/ = (ad) – (bc)

= (1x 8) – (5 x -2)

= 8 - - 10

= 8 + 10 = 18

Det (A) = 18. Since the determinant of this matrix is different from zero this matrix has an inverse.

Solution: B = $\begin{bmatrix} -4 & a & -2 & b \\ 12 & c & 6 & d \end{bmatrix}$

$$\textbf{Det (B)} = /B/ = (ad) - (bc)$$
$$= (-4 \times 6) - (-2 \times 12)$$
$$= -24 - -24$$
$$= -24 + 24 = 0$$

Det (B) = 0 . Since the determinant of this matrix is zero this matrix has ***no*** inverse.

3 x 3 matrix determinant

Before we discuss the determinant and inverse of a 3 x 3 matrix, we need to introduce an additional concept known as a ***cofactor***. Corresponding to each element *aij* of a matrix **A**, there is a cofactor, *Aij*. A 3 x 3 matrix has nine elements, so there are nine cofactors to be computed.

The cofactor, *Aij*, is defined to be the determinant of the 2 × 2 matrices obtained by deleting row *i* and column *j* of **A**, prefixed by a '+' or '−' sign according to the following pattern

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

For example, suppose we wish to calculate *A*22, which is the cofactor associated with *a*22 in the matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

The element *a*22 lies in the second row and second column. Consequently, we delete the second row and second column to produce the 2 × 2 matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The cofactor, *A*22, is the determinant of this 2 × 2 matrices prefixed by a '+' sign because from the pattern $\begin{bmatrix} + & - & + \\ - & \textcircled{+} & - \\ + & - & + \end{bmatrix}$

We see that *a* 22 is in a plus position. In other words, the cofactor for

$$\mathbf{A}\ 22 = \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}$$

$$= (a_{11} \times a_{33} - a_{13} \times a_{31})$$

And the process continues until we get the cofactor for all elements of the matrix.

Example:1 Find the co-factor of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 4 \\ 4 & 6 & 5 \end{bmatrix}$$

$$\mathbf{a_{11}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 4 \\ 4 & 6 & 5 \end{bmatrix} \quad + \begin{bmatrix} 7 & 4 \\ 6 & 5 \end{bmatrix} = \quad +((7 \text{ x } 5) - (4 \text{ x } 6)) \quad = 35 - 24 = 11$$

$$\mathbf{a_{12}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 4 \\ 4 & 6 & 5 \end{bmatrix} \quad - \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix} = \quad -((2 \text{ x } 5) - (4 \text{ x } 4)) \quad = -10 + 16 = 6$$

$$\mathbf{a_{13}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 4 \\ 4 & 6 & 5 \end{bmatrix} \quad + \begin{bmatrix} 2 & 7 \\ 4 & 6 \end{bmatrix} = \quad +((2 \text{ x } 6) - (7 \text{ x } 4)) \quad = 12 - 28 = -16$$

$$\mathbf{a_{21}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 4 \\ 4 & 6 & 5 \end{bmatrix} \quad - \begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix} = \quad -((2 \text{ x } 5) - (3 \text{ x } 6)) \quad = -10 + 18 = 8$$

$$\mathbf{a_{22}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 4 \\ 4 & 6 & 5 \end{bmatrix} \quad + \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} = \quad +((1 \text{ x } 5) - (3 \text{ x } 4)) \quad = 5 - 12 = -7$$

$$\mathbf{a_{23}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 4 \\ 4 & 6 & 5 \end{bmatrix} \quad - \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} = \quad -((1 \text{ x } 6) - (2 \text{ x } 4)) \quad = -6 + 8 = 2$$

$$\mathbf{a_{31}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 4 \\ 4 & 6 & 5 \end{bmatrix} \quad + \begin{bmatrix} 2 & 3 \\ 7 & 4 \end{bmatrix} = \quad +((2 \text{ x } 4) - (3 \text{ x } 7)) \quad = 8 - 21 = -13$$

$$\mathbf{a_{32}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 4 \\ 4 & 6 & 5 \end{bmatrix} \quad - \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \quad -((1 \text{ x } 4) - (3 \text{ x } 2)) \quad = -4 + 6 = 2$$

$$\mathbf{a_{33}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 4 \\ 4 & 6 & 5 \end{bmatrix} \quad + \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \quad +((1 \text{ x } 7) - (2 \text{ x } 2)) \quad = 7 - 4 = 3$$

We are now in a position to describe how to calculate the determinant of a 3 x 3 matrix. The determinant is found by multiplying the elements in any one row or column by their corresponding cofactors and adding together. It does not matter which row or column is chosen; exactly the same answer is obtained in each case. If we expand along the first row of the matrix,

$$\mathbf{a} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

We get det (**A**) = a_{11} x A_{11} + a_{12} xA_{12} + a_{13} x A_{13}

Similarly, if we expand down the third column, we get

Det (**A**) = a_{13} x A_{13} + a_{23} x A_{23} + a_{33} x A_{33}

From our previous example matrix

Co-factor of Matrix a = $\begin{bmatrix} 11 & 6 & -16 \\ 8 & -7 & 2 \\ -13 & 2 & 3 \end{bmatrix}$ $\quad Matrix$ A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 4 \\ 4 & 6 & 5 \end{bmatrix}$

Here in order to find the determinant of a matrix, we can select any row or column. Let us select the ***first row*** and the **third column** for a checkup. Hopefully, the result will be the same.

If we select first row   Det A = a_{11} x A_{11} + a_{12} xA_{12} + a_{13} x A_{13}

= 11x 1 + 6 x 2 + -16 x 3

= 11 + 12- 48

= -25

If we select third column   Det (**A**) = a_{13} x A_{13} + a_{23} x A_{23} + a_{33} x A_{33}

= - 16 x 3 + 2 x 4 + 3 x 5

= - 48 + 8+ 15

= - 25

If we select the second row? Det (**A**) = a_{21} x A_{21} + a_{22} x A_{22} + a_{23} x A_{23}

= 8 x 2 + (-7 x 7) + (2 x 4)

= 16 - 49 + 8

= **- 25**

Wow we got the same result. From the result we can also understand that this matrix is *not singular* because its determinant is different from ***zero***. As a result, we can find an inverse for the given matrix.

ii. The Gaussian Method

One "matrix" procedure that can be used to solve systems of linear equations is known as the **Gaussian Method.** This procedure was develoed by the mathematician Karl F. Gauss (1777-1855). We begin with the augmented matrix representation [**A|B**] of the original system of equations. Then we systematically reduce the augmented matrix to equivalent augmented matrices in simpler form. We continue until we have reached the simplest possible augmented matrix representation, one from which the solution of the sysem can be obtained by inspection. Indeed, we seek to transform the augmented matrix [**A**| **B**] into the augmented matrix [**I**|**X**], where **I** is the idemtity matrix and **X** is the solution vector.

The operations that may be performed on the augmented matrices to convert them to simpler form are the **ELEMENTARY ROW OPERATIONS (EROs).** The elementary row operation may consist of the following:

> **Operations permitted on the rows of a matrix** are called **ELEMENTARY ROW OPERATIONS.** They are as follows:
>
> **Type I.** Any pair of rows in a matrix may be interchanged. (**EXCHANGE** operation)
> **Type II.** A row can be multiplied by any nonzero real number. (**MULTIPLY** operation)
> **Type III.** A multiple of any row can be added to any other row. (**ADD A-MULTIPLE** operation)

Again, it is recommended that we follow the same step-by-step procedure for obtaining the identity matrix format that we used when calculating an inverse. That is, working one column at a time, obtain a one in the correct cell using **ERO** Type **II**, Then, obtain zeros in all other cells of the column using **ERO** Type **III** and multiples of the row which has the one.

Solving systems of linear equations using the Gaussian method involves the following steps:

1. Write all equations in a matrix form.
2. Change coefficient matrix in to identity matrix and apply the same elementary row operations on the vector of constants
3. The resulting value (of the RHS vector) will be the solution.

Example: Using the Gaussian procedure, find the solution to the system of equations

$$2x + y = 60$$
$$x + 3y = 105$$

We set up the augmented matrix tableau

$$\left[\begin{array}{cc|c} 2 & 1 & 60 \\ 1 & 3 & 105 \end{array}\right]$$

To obtain a 1 in the row one-column one cell, we multiply row one by ½, to obtain

$$\left[\begin{array}{cc|c} 1 & ½ & 30 \\ 1 & 3 & 105 \end{array}\right] \quad R_1 \leftarrow (½)R_1$$

Now, a zero is required in the row two-column one cell. Employing **ERO** Type III, we subtract row one from row two, to obtain

$$\left[\begin{array}{cc|c} 1 & ½ & 30 \\ 0 & 5/2 & 75 \end{array}\right] \quad R_2 \leftarrow R_2 - R_1$$

Column one is now in the identity matrix format, and we turn our attention to column two.

To obtain a 1 in the row two-column two cell, we multiply row two by , as

$$\left[\begin{array}{cc|c} 1 & ½ & 30 \\ 0 & 1 & 30 \end{array}\right] \quad R_2 \leftarrow (2/5)\, R_2$$

Finally, to obtain a 0 in the row one-column two cell, we subtract ½ times row two from row one, as

$$\left[\begin{array}{cc|c} 1 & 0 & 15 \\ 0 & 1 & 30 \end{array}\right] \quad R_1 \leftarrow R_{1\_+} (-½)\, R_2$$

The solution vector is

$$\mathbf{X} = \begin{bmatrix} 15 \\ 30 \end{bmatrix}$$

which is interpreted as x = 15, y = 30.

Therefore, Gaussian method makes a distinction between no solution and infinite solution, unlike the inverse method. Summarizing our results for solving an "n" by "n" system, we start with matrix (A/B), and attempt to transform it in to the matrix (I/C). One of the three things will result:

1. An n by n matrix with the unique solution; e.g.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

2. A row that is all zeros except in the constant column, indicating that there are no solutions; e.g.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 7 \end{array}\right)$$

3. A matrix in a form different from (1) and (2), indicating that there are an unlimited number of solutions. Note that for an n by n system, this case occurs when there is a row with all zeros, including the constant column; e.g.

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Using the inverse to solve a system of equations

Given an n × n system of linear equations, represented in matrix notation as **AX** = **B**, we may be able to find the solution vector using the inverse of **A.** If the coefficient matrix **A** is invertible, the system has a unique solution which is given by

$$X = A^{-1}B$$

The result stems from the fact that, if **A** has an inverse, both sides of the matrix equation can be multiplied by that inverse, as

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$(A^{-1}A)X = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$

To solve systems of linear equations using the inverse method the coefficient matrix should be inevitable, and it involves the following steps:

1. Put all equations in a matrix form (square matrix form).
2. Find the inverse of the coefficient matrix.
3. Multiply the inverse with right hand side values (vector of constants)

Example $x_1 + 3x_2 = 16$

$2x_1 + 2x_2 = 16$

The inverse of the coefficient matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

is found by using elementary row operations to covert the augmented matrix [**A** | **I**] into the format [**I** | $\mathbf{A}^{-1}$]. This inverse is

$$\mathbf{A}^{-1} = \begin{bmatrix} -½ & ¾ \\ ½ & -¼ \end{bmatrix}$$

This inverse can be used to determine the solution to the system. Accordingly the solution is given as follows:

$$\begin{bmatrix} -½ & ¾ \\ ½ & -¼ \end{bmatrix} \cdot \begin{bmatrix} 16 \\ 16 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} (-½ \times 16) + (¾ \times 16) \\ (½ \times 16) + (-¼ \times 16) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

When the information becomes that 18 for the right hand side value for the first equation and 16 as it is for the second one, the solution can easily be determined by introducing tha new change to the solution procedure as given below.

$$\begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} 18 \\ 16 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

The inverse could be used in this manner to determine new solution. The inverse method provides us with unique solution, or no solution and infinite solution (without separating them).

m by n linear systems

The mxn linear systems are those systems where the number of rows (m) and number of columns (n) are unequal or it is the case where the number of equations (m) and the number of variables (n) are unequal. And it may appear as m>n or m<n.

A. Linear equations where m>n

To solve an m by n systems of equations with m>n, we start with the matrix (A/B), and attempt to transform it in to the matrix (I/C). One of the three things will result:

1. An n by n identity matrix above m-n bottom rows that are all zeros, giving the unique solution
2. A row that all zeros except in the constant column, indicating that there are no solutions.
3. A matrix in a form different from (1) and (2), indicating that there are an unlimited number of solutions.

B. **Linear equations where m<n**

Our attempt transform (A/B) in to (I/C) in the case where m<n will result in either of the two: A row that is all zeros except in the constant column, indicating that there are no solutions. Or a matrix in a form different from (2), indicating that there are an unlimited number of solutions. Generally, every system of linear equations has either no solution, exactly one solution or infinitely many solutions."

Self-test 2.5. Dear learners, find the value of the variables for the following linear equations using the inverse and Gaussian methods;

1. 3x – 2y = 14
   X + 3y = 1

2. -2x + y = -3
   X – 4y = -2

3. 2X- 3Y = 6
   X + 5Y = 29
   3X – 4Y = 11

4. The weights of six people before taking a weight reduction program were 350, 249, 260, 195, 275, and 295. The weights of these same people after the weight reduction program are 345, 200, 220, 140, 200, and 230, respectively. Summarize this information in a (6 by 2) matrix.

5. LLM produces three grades of commercial fertilizers. A 100-lb bag of grade-A fertilizer contains 18 lb of nitrogen, 4 lb of phosphate, and 5 lb of potassium. A 100-lb bag of grade-B fertilizer contains 20 lb of nitrogen and 4 lb each of phosphate and potassium. A 100-lb bag of grade-C fertilizer contains 24 lb of nitrogen, 3 lb of phosphate, and 6 lb of potassium. How many 100-lb bags of each of the three grades of fertilizers should Lawnco produce if 26,400 lb of nitrogen, 4900 lb of phosphate, and 6200 lb of potassium are available and all the nutrients are used?

6. A private investment club has $200,000 earmarked for investment in stocks. To arrive at an acceptable overall level of risk, the stocks that management is considering have been classified into three categories: high-risk, medium-risk, and low-risk. Management estimates that high-risk stocks will have a rate of return of 15%/year; medium-risk stocks, 10%/year; and low-risk stocks, 6%/year. The members have decided that the investment in low-risk stocks should be equal to the sum of the investments in the stocks of the other two categories. Determine how much the club should invest in each type of stock if the investment goal is to have a return of $20,000/year on the total investment. (Assume that all the money available for investment is invested.)

iii. Cramer's rule

In business decision making from many of the variables, only a few of the variables are actually needed. For instance, if we have the following equations

$$x + y + z = 8$$

$$- x + 2y + z = 8$$

$$x - 4y + 2z = 15$$

The variable *x* might be the only variable which is needed to make a decision. Under these circumstances, it is clearly wasteful expending a large amount of effort calculating the inverse matrix, particularly since the values of the remaining variables, *y* and z are not required.

In this section, we describe an alternative method that finds the value of one variable at a time. This new method requires less effort if only a selection of the variables is required. It is known as ***Cramer's rule*** and makes use of matrix determinants. Cramer's rule for solving any *n x n* system, **Ax** = **b**, states that the *i*th variable, *i*, can be found from

$$i = \frac{\text{Det}(A_i)}{\text{Det}\ (A)}$$

Where A*i* is the *n x n* matrix found by replacing the *i*th column of **A** by the right-hand-side vector **b**.

To understand this, let us find the value of X from our previous example which is 2 x 2,

Example 1:

$x - 3y = 3$

$x + 2y = 8$

$$\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} \text{ x } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

Our interest is the first variable i. e *X =? We are not interested in y. Hence,*

$$x = \frac{\text{Det}(Ax)}{\text{Det}\ (A)} \qquad A = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} \qquad Ax = \begin{bmatrix} 3 & -3 \\ 8 & 2 \end{bmatrix}$$

Here, A is the original coefficient matrix **A**, while Ax is the matrix found from **A** by replacing the first column (since we are trying to find the first variable x) by the right hand-side vector $\begin{bmatrix} 3 \\ 8 \end{bmatrix}$.

As a result, the determinants of A and Ax are calculated as follows.

Det /Ax/ = (ad) – (bc)

= (3 x 2) – (-3 x 8)

= 6+24

= 30

Det /A/ = (ad) – (bc)
= (1 x 2) – (-3 x 1)
= 2+3
= 5

$$x = \frac{\text{Det}(Ax)}{Det\ (A)} = \frac{30}{5} = 6$$, yes it is correct.

What if our interest was y?

$y = \frac{\text{Det}(Ay)}{Det\ (A)}$ $A = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$ $Ay = \begin{bmatrix} 1 & 3 \\ 1 & 8 \end{bmatrix}$

Det /Ay/ = (ad) – (bc)
= (1 x 8) – (3 x 1) = 8-3 = 5

Det /A/ = (ad) – (bc)
= (1 x 2) – (-3 x 1) = 2+3 = 5

$y = \frac{\text{Det}(Ay)}{Det\ (A)} = \frac{5}{5}$ = 1, yes it is correct. Wow! This method is time saving.

3x3 or above

The procedure for Cramer's rule is the same for any n x n matrix.

2.5.2. Markov Chains: Concept, Model and Solutions

This model is a forecasting model. It is **probabilistic/ stochastic model**. A Russian mathematician called Andrew Markov around 1907 developed this model. Markov chains are models which are useful in studying the evolution of certain system over repeated trials. These repeated trails are often successive time periods where the state (outcome, condition) of the system in any particular time period cannot be determined with certainty. Therefore, a set of transition probabilities is used to describe the manner in which the system makes transition from one period to the next. Hence, we can predict the probability of the system being in a particular state at a given time period. We can also talk about the long run/equilibrium, steady state.

- **System** - which we want to study, machine, and person
- **Trials** - successive time period any convenient length of time day, week, month, year, etc.
- **State/outcome, condition -** the system can have various number of outcomes.
- **Transition probabilities** - set of input data, and are assumed to be constant.
- **Long/stead state** - the system cannot change any more. There is the same probability between n and n + 1 period after the long period.

The necessary assumptions of the chain are:

1. The system has a finite number of states - the outcomes of the system should be finite.
2. The system condition/outcome, state in any given period depends on its state in the preceding period and on the transition probabilities
3. The transition probabilities are constant over time.
4. Changes in the system will occur once and only once each period.
5. The transition period occurs with regularities.
6. The states are both mutually exclusive and collectively exhaustive.
7. The system is a closed one, that is, there will be no arrival or exits from the system.

Information flow in the analysis

The Markov model is based on two sets of input data

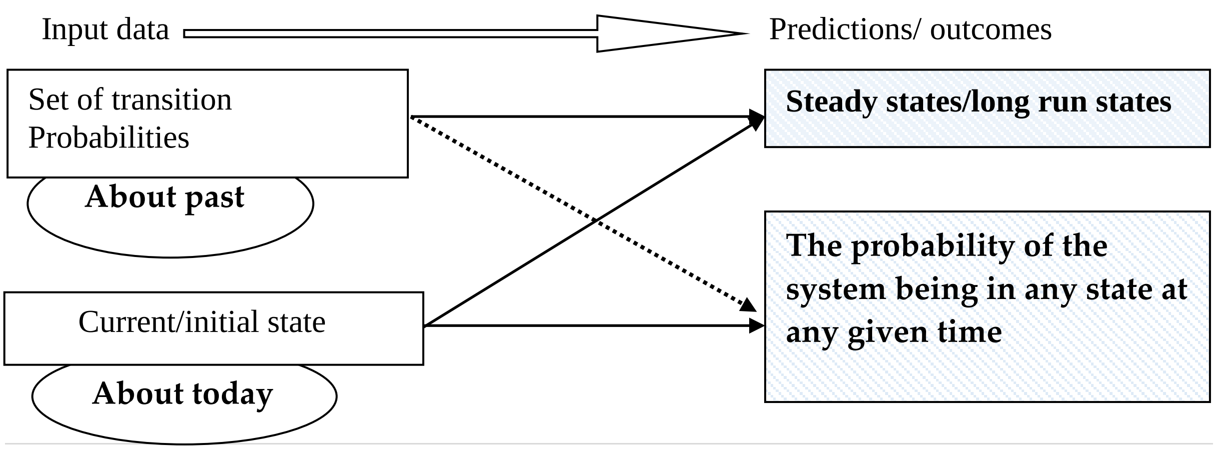
- The set of transition probabilities.
- The existing or initial or current conditions or states.

The Markov process, therefore, describes the movement of a system from a certain state in the current state/ time period to one of n possible states in the next stage. The system moves in an uncertain environment all that is known is **the probability associated with any possible move or transition**. This probability is known as **transition probability symbolized by P_{ij}**. It is the likelihood that the system which is currently in state **i** will Smoke to state **j** in the next period.

From these inputs the model makes two predictions usually expressed as vectors:

1. The probabilities of the system being in any state at any given future time period.
2. The long run / equilibrium, steady state probabilities.

The set of transition probabilities are necessary for both predictions (time period n, and steady state), but the initial state is needed for only the first prediction.



Example: Currently it is known that 80% of customers shop at store 1 and 20% shop at store 2. In reviewing a past data suppose we find that out of all customers who shopped at store 1 in a given week 90% remain loyal for the next week (store one again), 10% switch to store 2. Out of all customers who shopped at store 2, in a given week 80% remain loyal for the next week (store 2 again), 20% switch to store 1. What will be the proportion of customers shopping at store 1 and 2

a) In each of the next two weeks?
b) In the long run?

<u>Solution</u>

a)In each of the next two weeks

Let's denote Store 1 by 1 and Store 2 by 2.

V_{12}= (.8 .2) - initial state/ current state probability matrix.

<u>To next weekly shopping period</u>

From one week		S_1	S_2
	S_1	0.9	0.1
	S_2	0.2	0.8

⇒ The sum of rows in the transition matrices should be one.

⇒ We have to be consistent in writing the elements.

P_{11}, P_{22}, P_{33}, P_{44} ---------------------P_{nn} that represent the primary diagonal show loyalty. Others switching.

Markov Chain Formula

nth state of a Markov Chain.

V_{ij} (n) = V_{ij} (n-1) x p, or V_{ij} (n) = V_{ij} (0) x $(P)^n$.

Or

V_{ij} (n) = V_{ij} (0) x $(P)^n$.

Where: P = transition matrix

V_{ij} (n) = Vector for period n.

V_{ij} (n-1) = vector for period n-1.

$V_{12}(0) = (.8\ .2)$
$V_{12}(1) = V_{12}(0) \times P$

$$= (.8 \quad .2)\begin{pmatrix} .9 & .1 \\ .2 & .8 \end{pmatrix}$$

$= (.8 \times .9) + (.2 \times .2) \qquad (.8 \times .1) + (.2 \times .8)$
$= .72 + .04 \qquad .08 + .16$
$= 0.76 \qquad .24$

$V_{12}(1) = (.76 \quad .24)$

$V_{12}(2) = V_{12}(1) \times P$
$= V_{12}(1) \times P$
$= (.76 \quad .24)$

$$= (.76 \quad .24)\begin{pmatrix} .9 & .1 \\ .2 & .8 \end{pmatrix} = (0.732 \quad .268)$$

b) In the long run $(V_1\ V_2)(n)*(p) = (V_1\ V_2)(n+1)$
n p n+1

$$(V_1 \quad V_2)\begin{pmatrix} .9 & .1 \\ .2 & .8 \end{pmatrix} = (V_1 \quad V_2)$$

$0.9V_1 + .2V_2 = V_1$
$.1V_1 + .8V_2 = V_2$
$V_1 + V_2 = 1$

$$\left.\begin{array}{l} -0.1V1 + .2V2 = 0 \\ 0.1V1 + -.2V2 = 0 \end{array}\right\} \text{one is the -ve of the other.}$$

$.9V_1 + .2(1 - V_1) = V_1$
$.9V_1 + .2 - .2V_1 = V_1$
$.7V_1 + .2 = V_1$
$.2 = .3V_1$
$V_1 = 0.2/0.3 = 2/3$
$V_2 = 1 - V_1$
$= 1 - 2/3$
$V_2 = 1/3$

In short, the switching over the sum of the switching gives us the long run state.

		To	
		S_1	S_2
From	S_1	.9	.1
	S_2	.2	.8

$V_1 = \dfrac{Switch\ to\ state\ 1}{Switch\ to\ state\ 1 + switch\ to\ state\ 2}$ $V_2 = \dfrac{Switch\ to\ state\ 2}{Switch\ to\ state\ 1 + switch\ to\ state\ 2}$

$= \dfrac{.2}{.2+.1} = \dfrac{2}{3}$ $= \dfrac{.1}{.2+.1} = \dfrac{1}{3}$

$(V_1\ V_2) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

In the long run 67 of the customer will shop in store 1 and 33% in store 2.

Prediction: Long run - only the transition matrix.

At specified time - the transition matrix and state vector.

Hence, unless the transition matrix is affected, the long run state will not be affected. Moreover, we cannot know the number of years, weeks, or periods to attain the long run state, point but we can know the share.

Absorbing Markov Chain

It is a special type of Markov chain in which at least one of the states eventually doesn't lose members. We call such a state absorbing because it can absorb members from other states, but doesn't give up any of its members.

For example, if we take the above example and change the transition matrix

	S_1	S_2
S_1	1	0
S_2	.2	.8

The state S_1 (store 1) in absorbing

In short:

Consider a Markov chain with n different states $\{S_1, S_2$, and S_3 --- $S_n\}$.

The i^{th} state S_i is called absorbing if $P_{ii} = 1$. Moreover, the Markov chain is called absorbing if it has at least one absorbing state, and it is possible for a member of population to move from any non-absorbing state to an absorbing one in a finite number of transitions.

Remark: Note that for an absorbing state Si, the entry on the main diagonal p must be $P_{ii} = 1$ and all other entries in the i^{th} row must be 0.

Example a.

$$\begin{array}{c} \overline{\quad To \quad} \\ S_1 \quad S_2 \quad S_3 \end{array}$$

$$from\left\{\begin{Bmatrix} S_1 & 0.4 & 0 & 0.6 \\ S_2 & 0 & 1 & 0 \\ S_3 & 0 & 0.5 & 0.5 \end{Bmatrix}\right\} Absorbing\ \ Markov\ \ Chain$$

Example b.

$$\begin{array}{c} \overline{\quad To \quad} \\ S_1 \quad S_2 \quad S_3 \end{array}$$

$$from\left\{\begin{Bmatrix} S_1 & 0.4 & 0 & 0.6 \\ S_2 & .5 & 0.5 & 0 \\ S_3 & 0 & 0.5 & .5 \end{Bmatrix}\right\} has\ \ no\ \ absorbing\ \ states.$$

$$\begin{array}{c} \overline{\quad To \quad} \\ S_1 \quad S_2 \quad S_3 \quad S_4 \end{array}$$

$$from\begin{pmatrix} S_1 & .5 & .5 & 0 & 0 \\ S_2 & 0 & 1 & 0 & 0 \\ S_3 & 0 & 0 & .4 & 6 \\ S_4 & 0 & 0 & 5 & .5 \end{pmatrix}$$

The second state is absorbing. However the corresponding Markov chain is not observing. Because there is no way to move from state 3 or state 4 to state 2.

A Markov chain is absorbing it has at least one absorbing state, and if from every state it is possible to go to an absorbing state (not necessarily in one step).

Example: A division of the ministry of public health has conducted a sample survey on the public attitudes towards the use of condoms. From the results of the survey the department concluded that currently only 20% of the population uses condoms and every month 10% of non-users become users, where as 5% of users discontinue using.

Required

a. Write the current transition matrices.
b. What will be the percentage of users from total population just after two months?
c. What will be the proportion of the non-users and users in the long run?

Solution

Let. U - Stands for users, and N- stands for nonuser

1. Initial state $V_{UN}{}^{(0)} = 0.2 \quad 0.8$

		To the next month	
From one month		Users (U)	Non Users (N)
	Users (U)	.95	.05
	Non Users (N)	.10	.90

2. $V^{(1)}{}_{UN} = V^{(0)}{}_{UN} \times p$

$= 0.2 \quad 0.8$

$(0.27 \quad 0.73)$

$V^{(2)}{}_{UN} = V^{(1)}{}_{UN} \times p$

$$= 0.27 \quad 0.73 \begin{pmatrix} .95 & .05 \\ .10 & .90 \end{pmatrix}$$

$= (.3295 \quad 0.6705)$

3. $V_U\ V_N = (? \quad ?)$

$$V_U = \quad V_U = \frac{switchtoN}{Switchtou + SwitchtoU}$$

$$= \frac{.05}{.15} = 0.33$$

$V_U\ V_N = 0.67 \quad 0.33$

$V_{UN}{}^{(n)} = 0.67 \ \ 0.33$

Example: A city has two suburbs: suburb x and suburb y. Over the past several years, the city has experienced a population shift from the city to the suburbs, as shown in the table below.

To the next year

From one year		City (C)	Suburb x (X)	Suburb y (Y)
	City (C)	.85	.07	.08
	Suburb x (X)	.01	.96	.03
	Suburb y (Y)	.01	.02	.97

In $20x_o$, the city had a population of 120,000, suburb x had a population of 80,000, and suburb by had a population of 50,000. Assuming that the population in the metropolitan area remains constant at 250,000 people,

a. How many people will live in each of the three areas in $20X_2$?
b. How many people will live in each of the three areas in the long run

Solution

Let C stands for the city
X stands for the suburb X.
Y stands for the Suburb y.

C	120,000	120,000/250,000	0.48
X	80,000	80,000/250,000	0.32
Y	50, 000	50,000/250,000	0.20
Total	250,000		1.00

Initial state $V^{(0)}c\ x\ y$ (0.48 0.32 0.20)
The transition matrix. From one year

$$P = \begin{pmatrix} & C & X & Y \\ C & .85 & .07 & .08 \\ X & .01 & .96 & .03 \\ Y & .01 & .02 & .97 \end{pmatrix}$$

$$V^{(1)}cxy = V^{(0)}cxy * p = (.48\ .32\quad .20) \begin{pmatrix} .85 & .07 & .08 \\ .01 & .96 & .03 \\ .01 & .02 & .97 \end{pmatrix}$$

$V^{(1)}cxy =$ <u>(.4132 .3448 .2420)</u>

$$V^{(2)}cxy = (.4132\quad .3448\quad .2420) \begin{pmatrix} .85 & .07 & .08 \\ .01 & .96 & .03 \\ .01 & .02 & .97 \end{pmatrix}$$

$V^{(2)}cxy = (.3571\quad .3648\quad .2781)$

Thus, in $20X_2$, 89,275, 91,200 and 69,525 people will live in the city, suburb x and suburb y respectively.

long run

b. n p n+1

$$(V_c\ \ V_x\ \ V_y) \begin{pmatrix} .85 & .07 & .08 \\ .01 & .96 & .03 \\ .01 & .02 & .97 \end{pmatrix} (V_c\ \ V_x\ \ V_y)$$

.85C + .01x + .01y = C

.07C + .096x + .02y = x

.08C + .03X + .97y=y

c + x + y =1 $V_c\ \ V_x\ \ V_y = 1$

-.15C + .01x + .01y = 0

.07c - .04x + .02y = 0

.08C + .03x - .03Y = 0

X = 1-C-Y

.07c - .04 (1-c-y) + .02y = 0

.07c - .04 + .04c + .04y + .02y = 0

(.07c + .04c) - 04 + (04y + .02y) = 0

.11c+.06y - .04 = 0 --------------(1)

.08c + .03 (1-c-y) - .03y = 0

.08c + .03 - .03c - .03y - .03y = 0

(.08c - .03c) + .03 - .03y - .03y = 0

.05C +.03 - .06y = 0 --- (2)

$$+\begin{cases} .11c + .06y - .04 = 0 \\ .05C - .06Y + .03 = 0 \end{cases}$$

.16C - .01 = 0

.16C = .01

$$C = \frac{.01}{.16}$$

C = 0.0625

.11 (.0625) + .06y - .04 = 0

.006875 + .06y - .04 = 0

.06y = .033125

y = 0.5521

C+X+y = 1

.0625+x+.5521=1

0.6146+x=1

X = .3854

$(V_c \ V_x \ V_y) = (.0625 \ \ .3854 \ \ .5521)$

In the long run 15,625, 96,350 and 138,025 people will live in the city suburban X and suburban respectively.

Example: A vigorous television advertising campaign is conducted during the football season to promote a well-known brand X shaving cream. For each of several weeks, a survey is made and it is found that each week 100% of those using brand X continue to use it. It is also found that of those not using brand x, 20% switch to brand X while the other 80% continue using another band.

a. Write the transition matrix, assuming the transition percentages continue hold for succeeding weeks.
b. If 20% of the people are using brand X at the start of the advertising campaign, what percentage will be using brand X one week later? Two weeks later?
c. What portion of the market will be using brand X area the end of the season, assuming the transition matrix remains the same? Find the Steady-state matrix)

solution.

A.

	X	*nonx*
X	1	0
nonX	.2	.8

B . $V^{(n)}xx^1 = V^{(0)}{}_{xx}{}^1 p = (.2 \quad .8) \begin{pmatrix} 1 & 0 \\ .2 & .8 \end{pmatrix}$

$V^{(1)}xx^1 = (.36 .64)$

$V^{(2)}xx^1 = V^{(1)}{}_{xx}{}^1 xp = (.36 \qquad .64) \begin{pmatrix} 1 & 0 \\ .2 & .8 \end{pmatrix}$

$= (.488 \qquad .512)$

C. $V_x = \dfrac{0.2}{0.2+0} = 1$

$V_{x'} = \dfrac{0}{0.2+0} = 0$

Self-test 2.6. Dear learners, *check your progress using the following question.*

In a certain college class, 70% of the students who receive an "A" on the current examination will receive an "A" on the next examination. Moreover, 10% of the students who do not receive an "A" on the current examination will receive an "A" on the next examination. Assuming that this pattern continues, what is the stable matrix?

Answer: VA VA1 = (.25 .75)

2.6. Summary

Dear learner! We have seen about types of matrices, matrix operations, inverse of a matrix, ways of finding an inverse and applications of matrix algebra. The following gives the summary of major points. The equality of matrices is assured by equality of corresponding elements of the same dimension. Matrix addition and subtraction is defined for matrices of the same dimension but matrix multiplication is defined by considering the equality of inner dimensions. Inverse of a matrix is defined only for square matrices. Inverse of a matrix is unique. If matrix B is the inverse of matrix A, then matrix A is the inverse of matrix B. Every square matrix may not have an inverse. If a matrix has no inverse, then it is said to be singular and if a matrix has an inverse, it is said to be invertible or non- singular. Matrix algebra is applied in solving system of linear equations.

2.7. Review Questions

Dear learners, the following questions are selected to assess your progress for this specific chapter. Thus, attempt to answer all of the following questions.

Part -I: Multiple Choices (*choose the correct answer among the given alternatives.)*

1. Which one of the following is correct about the inner/dot product?
   A. Inner/Dot product is always an mx1 matrix
   B. Inner/ Dot product are the product of every Vector matrix.
   C. Inner/Dot product is the product of column vector and row vector
   D. Inner/Dot product is the product of row vector and column vector
   E. All
2. From the following statement one is **correct** about the transpose matrix?
   A. The transpose of the 2x3 dimension matrix is 3x2 dimension matrix
   B. The transpose of the square matrix is always the same and equal as its original square matrix
   C. The transpose of the identity matrix is non-identity matrix
   D. The off diagonal element of the transpose matrix is the same as the original matrix.
   E. None
3. All of the following statement is **incorrectly** stated **except** one:
   A. Scalar matrix is always identity matrix
   B. The product of any given matrix and the identity matrix is the given matrix itself.
   C. The sum of a zero matrix and any matrix gives that zero matrix
   D. Diagonal matrix is always Identity matrix
4. Which one of the following is **false** about special property of matrix multiplication?
   A. Associative law and distributive law works for matrix multiplication if the matrixes are conformable for multiplication
   B. Commutative law works for matrix multiplication two if two matrix are conformable for multiplication
   C. The cancellation law does not hold in matrix multiplication.
   D. All
   E. None

5. If matrix A have dimension of m by n and matrix B have dimension of z by j and conformable for multiplication then the dimension of their product is
   A. n by z
   B. m by z
   C. m by j
   D. m by n
6. From the following one is **true** about the inverse matrix?
   A. Inverse matrix only works for square matrix
   B. All square matrix have an inverse
   C. The product of inverse matrix and it original matrix is original matrix
   D. The matrix which is inevitable is said to be singular matrix
7. The mathematics test of three students before taking tutorial is 2, 3 & 5. But after taking tutorial and long study the results of each student changed in to 12, 13 & 14 respectively. If the teacher puts their result for comparison as matrix each student in column, it should be
   A. $\begin{bmatrix} 2 & 12 \\ 3 & 13 \\ 5 & 14 \end{bmatrix}$
   B. $\begin{bmatrix} 12 & 14 & 13 \\ 2 & 3 & 5 \end{bmatrix}$
   C. $\begin{bmatrix} 2 & 3 & 5 \\ 12 & 13 & 14 \end{bmatrix}$
   D. It is not matrix type.

Part-II: Workout Questions

1. A manufacturing company produces two types of boats: one-person and two-person models. The company has two plants x and y at different parts of the country. In both plants, there are two departments, fabricating and finishing. A one-person boat requires 4-labor hr in the fabricating department and 1 labor hr in the finishing department. The two-person boat requires 6-labor hr in the fabricating department and 1.5-labor hr in the finishing department. Suppose the hourly rates of labor cost in the fabricating and finishing departments be Br 8 and Br 6 respectively at plant x, and Br 7 and Br 4 at plant Y. Using matrix algebra, find the labor cost of making one unit of each product at each of the two plants. Interpret the results.
2. ABC Carpet Company has in inventory 1,500 square yards of wool and 1,800 square yards of nylon for the manufacture of carpeting. Two grades of carpeting are produced. Each roll of superior grade carpeting requires 20 sq. yards of wool and 40 square yards of nylon. Each roll of quality-grade carpeting requires 30 square yards of wool and 30 square yard of nylon. If Asrat would like to use all the material in inventory, how many rolls of superior and how many rows of quality carpeting should be manufactured?
3. A manufacturer is costing out one product line which consists of three different models, A, B, and C. These models are assembled from three types of parts, 1,2 and 3. The manufacturer would like to produce such quantities of the three models as to completely deplete the inventory of parts of hand, in the final production run. Each model a uses one unit of part 1, three units of part 2, and two units of part 3. Each unit of model B uses one unit of part 1, two units of part 2, and one unit of part 3. Each model C uses two units of part 1, and three units of part 3. Inventory records show that there are on hand 1,500 units of part 1 and 1,900 units each of part 2 and 3. How many of each model should the manufacturer plan to produce?
4. Alemayehu invested a total of 10,000 in three different savings accounts. The accounts paid simple interest at an annual rate of 8 percent, 9 percent and 7.5 percent respectively. Total interest earned for the year was Br 845. The amount in the 9 percent account was twice the amount invested in the 7.5 percent account. How much did Alemayehu invest in each account?
5. Attendance records indicate that 40, 000 people attended the 12^{th} African Youth championship at its opening ceremony at the Addis Ababa Stadium. Total ticket receipts were Br 1, 750,000. Admission prices were Br 37.5 for the second class and Br 62.5 for

the first class. Determine the number of people who attended the opening ceremony at first class and second class.

6. A population of 100,000 consumers makes the following purchases during a particular week: 20,000 purchases Brand A, 35,000 Brand purchase B and 45,000 purchase neither brand. From a market study, it in estimated that of those who purchase Brand A, 80% will purchase it again next week, 15% will purchase Brand B next week, and 5% will purchase neither brand. Of those who purchase B, 85% will purchase it again next week, 12% will purchase brand A next week, and 3% will purchase neither band. Of those who purchased neither brand, 20% will purchase as A next week, 15% will purchase Brand B next week, and 65% will purchase neither band next week. If this purchasing pattern continues, will the market stabilize? What will the stable distribution be?

7. A manufacturing firm which manufactures office furniture finds that it has the following variable costs in dollars.

$$\begin{array}{cccc} \text{Desks} & \text{Chairs} & \text{Tables} & \text{Cabinets} \\ \end{array}$$
$$\begin{bmatrix} 50 & 20 & 15 & 25 \\ 30 & 15 & 12 & 15 \\ 30 & 15 & 8 & 20 \end{bmatrix}$$

Assume that an order of 5 desks, 6 chairs, 4 tables and 12 cabinets has just been received. What are the total material, labor and overhead costs associated with the production of ordered items?

8. A person invests in A, B and C rated bonds. The average yield is 8% on A bonds, 6% on B bonds, and 7% on C bonds. Twice as much is invested in C bonds as B bonds. Moreover, the total annual return for all three types of bonds in Br. 2800. How much is invested in each type of bond if the total investment is

a. 37,500? b. 40,000?

Answer for chapter review questions

Part one: Choices

1.D	3.B
2.A	4.B
5.C	7.A
6.A	

Part two: Workout Questions

1.**Answer**

Plant	Product: One person	Two person
X	38	57
Y	32	48

2. **Answer: 15 and 40.**
3. **Answer: 100, 800, and 300.**
4. **Answer: Br 1,000, Br 6,000 and Br 3,000**
5. **Answer: 30,000 and 10,000.**
6. **Answer: Yes Y_A V_b V_c = (.4 .5 .1)**
7. **Answer: Birr 1,710.**
8. **Answer: a. Answer: Br 22,500, 5,000 and 10,000**
   b. Br 10,000, 10,000 and 20,000

Chapter Three: Introduction to Linear Programming

Chapter objectives

Dear students! after completing this chapter, you are expected to:

- Internalize linear programming and linear programming models.
- Identify and understand the components of linear programming models.
- Understand the assumption of linear programming models.
- Demonstrate the formulation of linear programming models.
- Understand and demonstrate the graphical approach to solve linear programming models.
- Demonstrate how to solve linear programming models using the Simplex Solution method.

3.1. Meaning of Linear Programming

Linear Programming is an optimization method, which shows how to allocate scarce resources such as money, materials or time and how to do such allocation in the best possible way subject to more than one limiting condition expressed in the form of inequalities and/or equations. It enables users to find optimal solution to certain problems in which the solution must satisfy a given set of requirements or constraints.

Optimization in linear programming implies either maximization (such as profit, revenue, sales, and market share) or minimization (such as cost, time, and distance) a certain objective function. It implies that in LP we cannot max/min two quantities in one model. It involves linearly related multivariate functions, i.e., functions with more than one independent variable. The goal in linear programming is to find the best solution given the constraints imposed by the problem; hence the term constrained optimization.

> **Linear Programming,** is a mathematical and operations-research technique, used in administrative and economic planning to maximize the linear functions of a large number of variables, subject to certain constraints.

3.2. Linear programming Models (LPM)

Linear Programming models (LPM) are mathematical representations of linear programming problems. Some LPM has a specialized format, whereas others have a more generalized format. Despite this, LPM has certain characteristics in common. Knowledge of these characteristics enables us to recognize problems that are amenable to a solution using LP models and to correctly formulate an LP model. The characteristics can be grouped into two categories: ***Components*** and ***Assumptions***. The *components* relate to the *structure* of a model, whereas the *assumptions* reveal the *conditions under which the model is valid*.

Components		**Assumptions**	
1. Objective function	Model Structure	1. Linearity	Model Validity
2. Decision variables		2. Divisibility	
3. Constraints		3. Certainty	
4. Parameters & RHSV		4. Non-negativity	

3.2.1. Components of LP model

Linear programming model has four different components.

1. **The Objective Function** - is the mathematical or quantitative expression of the objective of the company/model. The objective in problem solving is the criterion by which all decisions are evaluated. In LPMs a *single* quantifiable objective must be specified by the decision maker. For example, the objective might relate to profits, or costs, or market share, but to only one of these. Moreover, because we are dealing with optimization, the objective will be either maximization or minimization, but not both at a time.
2. **The Decision Variables** - represent unknown quantities to be resolved for. These decision variables may represent such things as the number of units of different products to be sold, the amount of Birr to be invested in various projects, the number of ads to be placed with different media. Since the decision maker has freedom of choice among actions, these decision variables are ***controllable variables*.**
3. **The constraints** - are restrictions which define or limit the feasibility of a proposed course of action. They limit the degree to which the objective can be pursued.

   Atypical restriction embodies *scarce resources* (such as labor supply, raw materials, production capacity, machine time, storage space), l*egal or contractual requirements* (e.g. Product standards, work standards), or they may reflect other limits based on forecasts, customer orders, company policies etc.

4. **Parameters** - are fixed values that specify the impact that one unit of each decision variable will have on the objective and on any constraint it pertains to as well as to the numerical value of each constraint.

The components are the building blocks of an LP model. We can better understand their meaning by examining a simple LP model as follows.

Example:

Maximize: $4X_1 + 7X_2 + 5X_3$ (Profit) ________________ objective function

Subject to:

$2X_1 + 3X_2 + 6X_3 \leq 300$ labor hrs

$5X_1 + X_2 + 2X_3 \leq 200$ lb raw material A } System constraints

$3X_1 + 5X_2 + 2X_3 \leq 360$

$X_1 = 30$

$X_2 \geq 40$ } Individual constraints

$X_1, X_2, X_3 \geq 0 \rightarrow$ Non-negativity constraints.

System constraints involve more than one decision variables whereas, individual constraints involve only one decision variable. None-negativity constraints specify that no variable will be allowed to take on a negative value. The non-negativity constraints typically apply in an LP model, whether they are explicitly stated or not.

3.2.2. Assumption of LP Models

Linear programming models are formulated based on four assumptions.

1. **Linearity** - the linearity requirement is that each decision variable has a linear impact on the objective function and in each constraint in which it appears. Following the above example, producing one more unit of product 1 adds Br. 4to the total profit. This is true over the entire range of possible values of X_1. The same applies to each of the constraints. It is required that the same coefficient (fro example, 2 lb. per unit) apply over the entire range of possible value so the decision variable.
2. **Divisibility** - the divisibility requirement pertains to potential values of decision variables. It is assumed that non-integer values are acceptable. For example: 3.5TV sets/hr would be acceptable $\rightarrow$ 7 TV sets/2hrs.
3. **Certainty** - the certainty requirement involves two aspects of LP models.

i) With respect to model parameters (i.e., the numerical values) – It in assumed that these values are known and constant e.g. in the above example each unit of product 1 requires 2lab his is known and remain constant, and also the 300 lab/hr available is deemed to be known and constant.

ii) All the relevant constraints identified and represented in the model are as they are.

Non-negativity - the non-negativity constraint is that negative values of variables are unrealistic and, therefore, will not be considered in any potential solution; only positive values and zero will be allowed.

3.2.3. Formulating LP Models

Once a problem has been defined, the attention of the analyst shifts to formulating a model. Just as it is important to carefully define a problem, it is important to carefully formulate the model that will be used to solve the problem. If the LP model is ill formulated, ill-structured, it can easily lead to poor decisions.

Formulation of linear programming models involves the following steps:

1. Define the problem/problem definition

   * To determine the # of type 1 and type 2 products to be produced per mother so as to maximize the monthly profit given the restrictions.
2. Identify the decision variables or represent unknown quantities

   * Let X_1 and X_2 be the monthly qualities of Type 1 and type 2 products
3. Determine the objective function

   * Once the variables have been identified, the objective function can be specified. It is necessary to decide if the problem is maximization or a minimization problem and the coefficients of each decision variable.

Note that: a. The units of all the coefficients in the objective function must be the same. For example, if the contribution of type 1 is in terms of Birr so does for type 2.

b. All terms in the objective function must include a variable each term has to have 1 variable.

c. All decision variables must be represented in the objective function.

4. Identifying the constraints

- System constraints - more than one variable
- Individual constraints - one variable
- Non-negative constraints

Example: A firm that assembles computer and computer equipment is about to start production of two new microcomputers. Each type of micro-computer will require assembly time, inspection time and storage space. The amount of each of these resources that can be devoted to the production of microcomputers is limited. The manager of the firm would like to determine the quantity of each microcomputer to produce in order to maximize the profit generated by sales of these microcomputers.

Additional information

In order to develop a suitable model of the problem, the manager has met with design and manufacturing personnel. As a result of these meetings, the manager has obtained the following information:

	Type 1	Type 2
Profit per unit	Birr 60	Birr 50
Assembly time per unit	4hrs	10hrs
Inspection time per unit	2hrs	1hr
Storage space per unit	3cubic ft	3cubic ft

The manager also has acquired information on the availability of company resources. These weekly amounts are:

Resource	Resource available
Assembly time	100hrs
Inspection time	22hrs
Storage space	39 cubic feet

The manger also meet with the firm's marketing manager and learned that demand for the microcomputers was such that whatever combination of these two types of microcomputer is produced, all of the output can be sold.

Required: Formulate the Linear programming model.

Solution:

Step 1: Problem Definition

- To determine the number of two types of microcomputers to be produced (and sold) per week so as to maximize the weekly profit given the restriction.

Step 2: Variable Representation

- Let X_1 and X_2 be the weekly quantities of type 1 and type 2 microcomputers, respectively.

Step 3: Develop the Objective Function

Maximize or $Zmax = 60X_1 + 50X_2$

Step 4: Constraint Identification

System constraints:	$4X_1 + 10X_2 \leq 100$hrs	Assembly time
	$2X_1 + X_2 \leq 22$hrs	Inspector time
	$3X_1 + 3X_2 \leq 39$ cubic feet	Storage space
Individual constraint	No	
Non-negativity constraint	$X_1, X_2 \geq 0$	

In summary, the mathematical model for the microcomputer problem is:

$Zmax = 60X_1 + 50X_2$

Subject to: $4X_1 + 10X_2 \leq 100$

$2X_1 + X_2 \leq 22$

$X_1 + 3X_2 \leq 39$

$X_1, X_2 \geq 0$

Example: An electronics firm produces three types of switching devices. Each type involves a two-step assembly operation. The assembly times are shown in the following table:

Assembly time per Unit (in minutes)

	Section #1	Section #2
Model A	2.5	3.0
Model B	1.8	1.6
Model C	2.0	2.2

Each workstation has a daily working time of 7.5 hrs. The manager wants to obtain the greatest possible profit during the next five working days. Model A yields a profit of Birr 8.25 per unit, Model B a profit of Birr 7.50 per unit and Model C a profit of Birr 7.80 per unit. Assume that the firm can sell all it produces during this time, but it must fill outstanding orders for 20 units of each model type.

Required: Formulate the linear programming model of this problem.

Solution:

Step 1: Problem definition

To determine the number of three types of switching devices to be produced and sold for the next 5 working days so as to maximize the 5 days profit.

Step 2: Variable representation

Let X_1, X_2 and X_3 be the number of Model A, B and C switching devices respectively, to be produced and sold.

Step 3: Develop objective function

Zmax: $8.25X_1 + 7.50X_2 + 7.80X_3$

Step 4: Constraint identification

$2.5X_1 + 1.8X_2 + 2.0X_3 \leq 2250$ minutes Ass. time station 1 } System constraint
$3.0X_1 + 1.6X_2 + 2.2X_3 \leq 2250$ minutes Ass. time station 2 }

$X_1 \geq 20$ Model A }
$X_2 \geq 20$ Model B } Individual constraint
$X_3 \geq 20$ Model C }

$X_1, X_2, X_3 \geq 0$ Non negativity

In summary:

Zmax: $8.25X_1 + 7.50X_2 + 7.80X_3$

Subject to : $2.5X_1 + 1.8X_2 + 2.0X_3 \leq 2250$ minutes
$3.0X_1 + 1.6X_2 + 2.2X_3 \leq 2250$ minutes
$X_1 \geq 20$ model A
$X_2 \geq 20$ model B
$X_3 \geq 20$ model C
$X_1, X_2, X_3 \geq 0$ non negativity

Self-test 3.1: Dear learns, answer the following question.

A farm consists of 600 hectares of land of which 500 hectares will be planted with corn, barley and wheat, according to these conditions.

(1) At least half of the planted hectare should be in corn.

(2) No more than 200 hectares should be barley.

(3) The ratio of corn to wheat planted should be 2:1

It costs Birr 20 per hectare to plant corn, Birr 15 per hectare to plant barley and Birr 12 per hectare to plant wheat.

Required: *Formulate this problem as an LP model that will minimize planting cost while achieving the specified conditions.*

3.3. Approaches to linear programming

There are two solution approaches to linear programming. These are;

1. The graphic solution and
2. The simplex approach/ Algebraic solution

1. The Graphic Solution Method

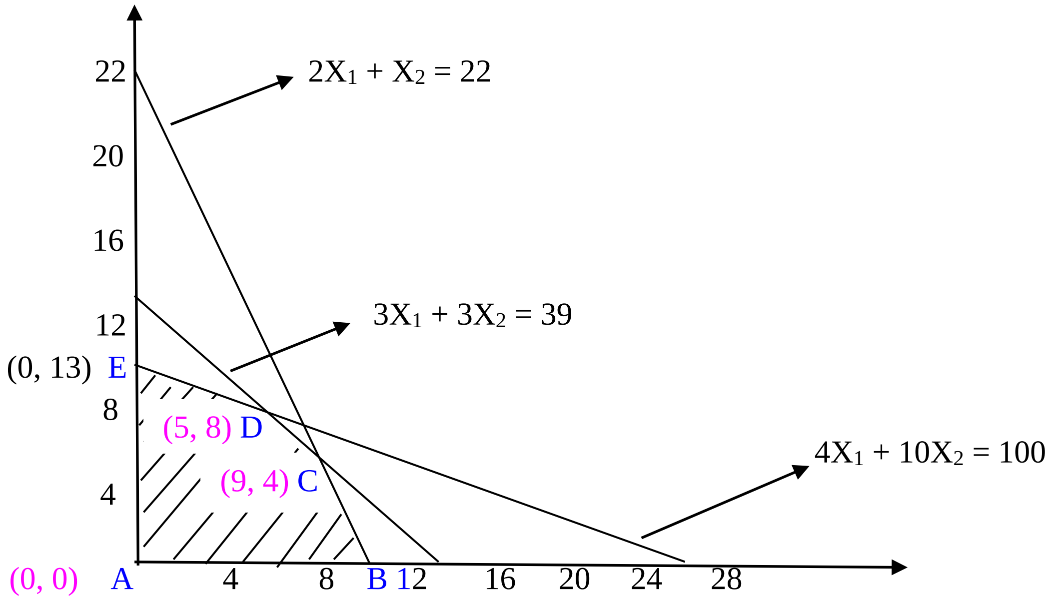
It is a relatively straightforward method for determining the optimal solution to certain linear programming problems. It gives as a clear picture. This method can be used only to solve problems that involve two decision variables. However, most linear programming applications involve situations that have more than two decision variables, so the graphic approach s not used to solve them.

Example: Solving the micro-computer problem with graphic approach

$$Zmax = 60X_1 + 50X_2$$
$$: 4X1 + 10X_2 \leq 100$$
$$2X_1 + X_2 \leq 22$$
$$3X_1 + 3X_2 \leq 39$$
$$X_1, X_2 \geq 0$$

Steps:

1. Plot each of the constraints and identify its region – make linear inequalities linear equations.
2. Identify the common region, which is an area that contains all of the points that satisfy the entire set of constraints.
3. Determine the Optimal solution- identify the point which leads to maximum benefit or minimum cost.



To identify the maximum (minimum) value we use the corner point approach or the extreme point approach. The corner point/extreme point approach has one theorem: It states that; for problems that have optimal solutions, a solution will occur at an extreme, or corner point. Thus, if a problem has a single optimal solution, it will occur at a corner point. If it has multiple optimal solutions, at least one will occur at a corner point. Consequently, in searching for an optimal solution to a problem, we need only consider the extreme points because one of those must be optimal. Further, by determining the value of the objective function at each corner point, we could identify the optimal solution by selecting the corner point that has the best value (i.e., maximum or minimum, depending on the optimization case) at the objective function.

3.1 Determine the values of the decision variables at each corner point. Sometimes, this can be done by inspection (observation) and sometimes by simultaneous equation.

3.2 Substitute the value of the decision variables at each corner point.

3.3 After all corner points have been so evaluated, select the one with the highest or lowest value depending on the optimization case.

Points	Coordinates X_1	X_2	How Determined	Value of Objective function $Z = 60X_1 + 50X_2$
A	0	0	Observation	Birr 0
B	11	0	Observation	Birr 660
C	**9**	**4**	**Simultaneous equations**	**Birr 740**
D	5	8	Simultaneous equations	Birr 700
E	0	10	Observation	Birr 500

Basic solution

$X_1 = 9$

$X_2 = 4$

Z = Birr 740

After we have got the optimal solution, we have to substitute the value of the decision variables into the constraints and check whether all the resources available were used or not. If there is an unused resource, we can use it for any other purpose. The amount of unused resources is known as *SLACK*-the amount of the scarce resource that is unused by a given solution.

The slack can range from zero, for a case in which all of a particular resource is used, to the original amount of the resource that was available (i.e., none of it is used).

Computing the amount of slack

Constraint	Amount used with X1 = 9 and X2 = 4	Originally available	Amount of slack (available – Used)
Assembly time	4(9) + 10(4) = 76	100 hrs	100 – 76 = 24 hrs
Inspection time	2(9) = 1 (4) = 22	22 hrs	22 – 22 = 0 hr
Storage space	3(9) + 3(4) = 39	39 cubic ft	39 – 39 = 0 cubic ft

Constraints that have no slack are sometime referred to as binding constraints since they limit or bind the solution. In the above case, inspection time and storage space are binding constraints; while assembly time has slack.

Knowledge of unused capacity can be useful for planning. A manager may be able to use the assembly time for other products, or, perhaps to schedule equipment maintenance, safety seminars, training sessions or other activities.

Interpretation: The Company is advised to produce 9 units of type 1 microcomputers and 4 units of type 2 microcomputers per week to maximize his weekly profit to Birr 740; and in do so the company would be left with unused resource of 24-assembly hours that can be used for other purposes.

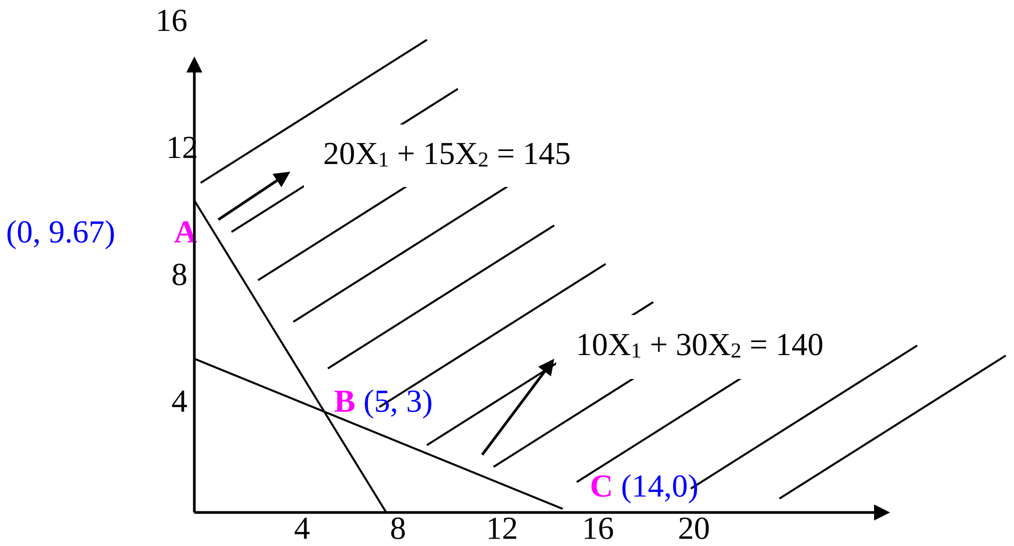
Example: Solving the diet problem with graphic approach

Cmin: $5X_1 + 8X_2$

$10X_1 + 30X_2 \geq 140$

$20X_1 + 15X_2 \geq 145$

$X_1, X_2 \quad \geq 0$



Points	Coordinates		How Determined	Value of the objective function
	X_1	X_2		$Z = 5X_1 + 8X_2$
A	0	9.67	Observation	Birr 77.30
B	**5**	**3**	Simultaneous equations	Birr 49
C	14	0	Observation	Birr 70

Basic solution: X1 = 5 pounds

X_2 = 3 pounds

C = Birr 49

Interpretation: To make the diet at the minimum cost of Birr 49 we have to purchase 5 pounds of Type1 food and 3 pounds Type 2 food.

If there is a difference between the minimum required amount and the optimal solution, we call the difference *surplus*: That is, Surplus is the amount by which the optimal solution causes a ≥ constraint to exceed the required minimum amount. It can be determined in the same way that slack can: substitute the optimal values of the decision variables into the left side of the constraint and solve. The difference between the resulting value and the original right-hand side amount is the amount of surplus. Surplus can potentially occur in a ≥ constraint.

3.4. The Simplex Algorithm/Algebraic Solution Method

The simplex method is an iterative technique that begins with a feasible solution that is not optimal, but serves as a starting point. Through algebraic manipulation, the solution is improved until no further improvement is possible (i.e., until the optimal solution has been identified). Each iteration moves one step closer to the optimal solution. In each iteration, one variable that is not in the solution is added to the solution and one variable that is in the solution is removed from the solution in order to keep the number of variables in the basis equal to the number of constraints.

The optimal solution to a linear programming model will occur at an extreme point of the feasible solution space. This is true even if a model involves more than two variables; optimal solutions will occur at these points. Extreme points represent intersections of constraints. Of course, not every solution will result is an extreme point of the feasible solution space; some will be outside of the feasible solution space. Hence, not every solution will be a feasible solution. Solutions which represent *intersections of constraints* are called **basic solutions***;* those which also satisfy all of the constraints, including the non-negativity constraints, are called **basic feasible solutions**. The simplex method is an algebraic procedure for systematically examining basic feasible solutions. If an optimal solution exists, the simplex method will identify it. The simplex procedure for a maximization problem with all ≤ constraints consists of the following steps.

Write the LPM in a standard form: when all of the constraints are written as equalities, the linear program is said to be in standard form. We convert the LPM in to a standard form by applying the slack variables, S, which carries a subscript that denotes which constraint it applies to. For example, S_1 refers to the amount of slack in the first constraint, S_2 to the amount of slack in the second constraint, and so on. When slack variables are introduced to the constraints, they are no longer inequalities because the slack variable accounts for any difference between the left and right-hand sides of an expression. Hence, once slack variables are added to the constraints, they become equalities. Furthermore, every variable in a model must be represented in the objective function. However, since slack does not provide any real contribution to the objective, each slack variable is assigned a coefficient of zero in the objective function.

Slack = Requirement – Production, surplus = Production – Requirement

Taking the microcomputer problem its standard form is as follows:

$Z\max = 60X_1 + 50X_2$	$Z\max = 60X_1 + 50X_2 + 0S1 + 0S2 + 0S3$
S.t. : $4X1 + 10X_2 \leq 100$	S.t. : $4X1 + 10X_2 + S_1 = 100$
$2X_1 + X_2 \leq 22$	$2X_1 + X_2 + S_2 = 22$
$3X_1 + 3X_2 \leq 39$	$3X_1 + 3X_2 + S_3 = 39$
$X_1, X_2 \geq 0$	$X_1, X_{2,} S_1, S_2, S_3 \geq 0$

1. Develop the initial tableau: the initial tableau always represents the "Do Nothing" strategy, so that the decision variables are initially non-basic.
   a) List the variables across the top of the table and write the objective function coefficient of each variable jut above it.
   b) There should be one row in the body of the table for each constraint. List the slack variables in the basis column, one per raw.
   c) In the Cj column, enter the objective function coefficient of zero for each slack variable. (Cj - coefficient of variable j in the objective function)

   Compute values for row Zj

   Compute values for Cj – Zj

Sol/n basis	Cj	60 X_1	50 X_2	0 S_1	0 S_2	0 S_3	RHSV	Øj = bj/xj (aij)
S1	0	4	10	1	0	0	100	100/4 = 25
S2	**0**	**2***	**1**	**0**	**1**	**0**	**22**	**22/2 = 11**
S3	0	3	3	0	0	1	39	39/3 = 13
	Zj	0	0	0	0	0	0	
	Cj-Zj	60	50	0	0	0	0	

Leaving variable

Entering variable   Pivot column   Pivot row

*** Pivot Element**

2. Develop subsequent tableaus
   2.1. Identify the entering variable - a variable that has a largest positive value is the Cj – Zj raw.
   2.2. Identify the leaving variable - Using the constraint coefficients or substitution rates in the entering variable column divide each one into the corresponding

quantity value. However do not divide by a zero or negative value. The smallest non-negative ratio that results indicate which variable will leave the solution.

3. Find unique vectors for the new basic variable using row operations on the pivot element.

Sol/n basis	Cj	60 X_1	**50 X_2**	0 S_1	0 S_2	0 S_3	RHSV	Øj = bj/xj (aij)
S1	0	0	**8**	1	-2	0	56	56/8 = 7
X1	60	1	**1/2**	0	½	0	11	11/. 5 = 22
S3	**0**	**0**	**3/2**	**0**	**-3/2**	**1**	**6**	**6/1.5 = 4**
	Zj	60	**30**	0	30	0	660	
	Cj-Zj	0	**20**	0	-30	0	0	

Leaving variable

Entering Variable

Sol/n basis	Cj	60 X_1	50 X_2	0 S_1	0 S_2	0 S_3	RHSV	Øj = bj/xj (aij)
S1	0	0	0	1	6	-16/3	24	
X1	60	1	0	0	1	-1/3	9	
X2	50	0	1	0	-1	2/3	4	
	Zj	60	50	0	10	40/3	740	
	Cj-Zj	0	0	0	-10	-40/3		

Optimal solution: X1 = 9
X2 = 4
S1 = 24 hrs
Z = Birr 740

4. Compute the Cj – Zj raw
5. If all Cj – Zj values are zeros and negatives you have reached optimality.
6. If this is not the case (step 6), rehear step 2to5 until you get optimal solution.

"A simplex solution is a maximization problem is optimal if the Cj – Zj row consists entirely of zeros and negative numbers (i.e., there are no positive values in the bottom row)."

Note: The variables in solution all have unit vectors in their respective columns for the constraint equations. Further, note that a zero appears is raw c - z in every column whose variable is in solution, indicating that its maximum contribution to the objective function has been realized.

Example: A manufacturer of lawn and garden equipment makes two basic types of lawn mowers: a push-type and a self-propelled model. The push-type requires 9 minutes to assemble and 2 minutes to package; the self-propelled mower requires 12 minutes to assemble and 6 minutes to package. Each type has an engine. The company has 12 hrs of assembly time available, 75 engines, and 5hrs of packing time. Profits are Birr 70 for the self-propelled models and Birr 45 for the push-type mower per unit.

Required:

1. Formulate the linear programming models for this problem.
2. Determined how many mower of each type to make in order to maximize the total profit (use the simplex procedure).

Solution:

1.

a) To determine ho many units of each types of mowers to produce so as to maximize profit.
b) Let X_1 - be push type mower.
   X_2 - be self-propelled mower.
c) Determine the objective function
   $Zmax = 45X_1 + 70X_2$
d) Identify constraints

$9X_1 + 12X_2 \leq 720$ minutes	Assembly time
$2X_1 + 6X_2 \leq 300$ minutes	packing time
$X_1 + X_2 \leq 75$ engines	Engines
$X1, X2 \geq 0$	

In summary:

$$Zmax = 45X_1 + 70X_2$$
$$: 9X_1 + 12X_2 \leq 720$$
$$2X_1 + 6X_2 \leq 300$$
$$X_1 + X_2 \leq 75$$
$$X_1, X2 \geq 0$$

2.

a. Write the LPM in a standard form

$$Zmax = 45X_1 + 70X_2 + OS_1 + OS_1 + OS_3$$
$$: 9X_1 + 12X_2 + S_1 = 720$$
$$2X_1 + 6X_2 + S_2 = 300$$
$$X_1 + X_2 + S_3 = 75$$
$$X_1, X_2, S_1, S_2, S_3 \geq o$$

b. Develop the initial tableau – in LP matrices are commonly called tableaus

Sol/n basis	Cj	45 X_1	**70 X_2**	0 S_1	0 S_2	0 S_3	RHSV	Øj = bj/xj (aij)
S1	0	9	**12**	1	0	0	720	720/12 =60
S2	**0**	**2**	**6**	**0**	**1**	**0**	**300**	**300/6 =50** → Leaving variable
S3	0	1	**1**	0	0	1	75	75/1 = 75
	Zj	0	**0**	0	0	0	0	
	Cj-Zj	45	**70**	0	0	0		

↑ Entering variable

c. Develop the subsequent tableaus

- Identify the entering variable
- Identify the leaving variable

Sol/n basis	Cj	**45 X_1**	70 X_2	0 S_1	0 S_2	0 S_3	RHSV	Øj = bj/xj (aij)
S1	**0**	**5**	**0**	**1**	**-2**	**0**	**120**	**120/5 = 24** → Leaving variable
X2	70	**1/3**	1	0	1/6	0	50	50/. 333 =150
S3	0	**2/3**	1	0	-1/6	1	25	25/.666 = 75
	Zj	**70/3**	70	0	70/6	0	3500	
	Cj-Zj	**65/3**	0	0	-70/6	0		

↑ Entering variable

Sol/n basis	Cj	45 X_1	70 X_2	0 S_1	0 S_2	0 S_3	RHSV	Øj = bj/xj (aij)
X1	45	1	0	1/5	-2/5	0	24	
X2	70	0	1	-1/15	3/10	0	42	
S3	0	0	0	-2/15	1/10	1	9	
	Zj	45	70	13/3	3	0	4020	
	Cj-Zj	0	0	-13/3	-3	0		

Optimal solutions: X1 = 24 units
X2 = 42 units
S3 = 9 engines
Z = Birr 4020

Interpretation: The Company is advised to produce 24 units of push type mowers and 42 units of self-propelled mowers so as to realize a profit of Birr 4020. In doing so the company would be left with unused resource of 9 engines which can be used for other purposes.

3.5. Special Issues in LP

1. Unbounded solutions

A solution is unbounded if the objective function can be improved without limit. The solution is unbounded if there are no positive ratios in determining the leaving variable. A negative ratio means that increasing a basic variable would increase resources! A zero ratio means that increasing a basic variable would not use any resources. This condition generally arises because the problem is incorrectly formulated. For example, if the objective function is stated as maximization when it should be a minimization, if a constraint is stated ≥ when it should be ≤ or vice versa.

2. Multiple optimal solutions

The same maximum value of the objective function might be possible with a number of different combinations of values of the decision variables. This occurs because the objective function is parallel to a binding constraint. With simplex method this condition can be detected by examining the Cj – Zj row of the final tableau. If a zero appears in the column of a non-basic variable (i.e., a variable that is not in solution), it can be concluded that an alternate solution exists.

E.g. $Z = 60X_1 + 30X_2$

$4X_1 + 10X_2 \leq 100$

$2X_1 + X_2 \leq 22$

$3X_1 + 3X_2 \leq 39$

$X1, X2 \geq 0$

The other optimal corner point can be determined by entering the non-basic variable with the C - Z equal to zero and, then, finding the leaving variable in the usual way.

3. Degeneracy

In the process of developing the next simplex tableau for a tableau that is not optimal, the leaving variable must be identified. This is normally done by computing the ratios of values in the quantity column and the corresponding row values in the entering variable column, and selecting the variable whose row has the smallest non-negative ratio. Such an occurrence is referred to degeneracy, because it is theoretically possible for subsequent solutions to cycle (i.e., to return to previous solutions). There are ways of dealing with ties in a specific fashion; however, it will usually suffice to simply select one row (variable) arbitrarily and proceed with the computations.

3.6. Limitations of linear programming

1. In linear programming uncertainty is not allowed, i.e., LP methods are applicable only when values for costs, constraints, etc. are known, but in real life such factors may be unknown.
2. According to the LP problem, the solution variables can have any value, whereas sometimes it happens that some of the variables can have only integral values. For example, in finding how may machines to be produced; only integral values of decision variables are meaningful. Except when the variables have large values, rounding the solution to the nearest integer will not yield an optimal solution. Such situations justify the use of Integer Programming.
3. Many times, it is not possible to express both the objective function and constraints in linear form.

3.7. Summary

The standard form of LP problem should have the characteristics of(1)All the constraints should be expressed as equations by slack or surplus and/or artificial variables (2)The right hand side of each constraint should be made non-negative; if it is not, this should be done by multiplying both sides of the resulting constraint by -1 and (3) Three types of additional variables, namely (1) **Slack Variable**(**S**)(2) **Surplus variable** (**-S**), and (3) **Artificial variables** (**A**) are added in the given LP problem to convert it into standard form for two reasons: the extra variables needed to add in the given LP problem to convert it into standard form is given below:

Types of constraint	**Extra variables to be added**	**Coefficient of extra variables <u>in the objective function</u> MaxZ MinZ**	**Presence of variables in the initial solution mix**
≤	**Add only slack variable**	**0** **0**	**Yes**
≥	**Subtract surplus variable**	**0** **0**	**No**
	Add artificial variable	**-M** **+M**	**Yes**
=	**Add artificial variable**	**-M** **+M**	**Yes**

2. Test of optimality

- If all $C_j - Z_j \leq 0$, then the basic feasible solution is optimal (Maximization case).
- If all $C_j - Z_j \geq 0$, then the basic feasible solution is optimal (Minimization case).

3. Variable to enter the basis

- A variable that has the most positive value in the $C_j - Z_j$ row (Maximization case)
- A variable that has the highest negative value in the $C_j - Z_j$ row (Minimization case)

  N:B-'Highest negative' values in this case refers to the negative value which is far from zero on the number line!

4. Variable to leave the basis

- The row with the non-negative and minimum replacement ratio (For both maximization and minimization cases

  i.e: **RR > 0**

3.8. Review Questions

Dear learners, the following questions are selected to assess your progress for this specific chapter. Thus, attempt to answer all of the following questions.

1. A diet is to include at least 140 mgs of vitamin A and at least 145 Mgs of vitamin B. These requirements are to be obtained from two types of foods: Type 1 and Type 2. Type 1 food contains 10Mgs of vitamin A and 20mgs of vitamin B per pound. Type 2 food contains 30mgs of vitamin A and 15 mgs of vitamin B per pound. If type 1 and 2 foods cost Birr 5 and Birr 8 per pound respectively, how many pounds of each type should be purchased to satisfy the requirements at a minimum cost?
2. A firm produces products A, B, and C, each of which passes through assembly and inspection departments. The number of person hours required by a unit of each product in each department is given in the following table.

   Person hours per unit of product

	Product A	Product B	Product C
Assembly	2	4	2
Inspection	3	2	1

During a given week, the assembly and inspection departments have available at most 1500 and 1200 person-hours, respectively. if the unit profits for products A, B, and C are Birr 50, Birr 40, and Birr 60, respectively, determine the number of units of each product that should be produced in order to maximize the total profit and satisfy the constraints of the problem.

3. The state chairman of a political party must allocate an advertising budget of birr 3,000,000 among three media: radio, television, and newspapers. The expected number of votes gained per birr spent on each advertising medium is given below.

Expected votes per Birr spent

Radio	Television	Newspapers
3	5	2

Since these data are valid with in the limited amounts spent on each medium, the chairman has imposed the following restrictions:

- ✍ No more than Birr 500,000 may be spent on television ads.
- ✍ No more than Birr 1,200,000 may be spent on radio ads.
- ✍ No more than Birr 2,400,000 may be spent on television and newspaper ads combined.

How much should be spent on each medium in order to maximize the expected number of votes gained?

Answer for chapter review questions

1. Answer

	Vitamins	
Foods	A	B
Type 1	10	20
Type 2	30	15

2. **Answer**: 0 unit of product A, 0 unit of product B, 750 units of product C, unused inspection time of 450 hours, and a maximum profit, Z ,of Birr 45,000.
3. **Answer:** Birr 500,000 should be spent on radio ads.
   Birr 1,200,000 should be spent on television ads.
   Birr 1,200,000 should be spent on newspaper ads.
   Slack in the budget constraint is Birr 100,000.
   Z = 9,900,000 is the maximum expected number of votes gained.

Chapter Four: Mathematics of Finance

Chapter objectives

Dear learners, after successfully completed this chapter, you are expected to:

- Understand and compute of simple and compound interest
- Understand and compute Present Value of compound interest
- Understand and compute equivalent rates
- Understand and compute effective rate
- Understand and compute Annuities
- Understand and compute Ordinary annuity

4.1. Introduction

Dear student! What do you know about mathematics of finance? Why we need to learn mathematics of finance?

____

____

Mathematics of finance is concerned with the analysis of time-value of money. The fundamental premise behind such analysis is the concept that entails the value of money changes overtime. Putting it in simple terms, the value of one birr today is not the same after a year. Mathematics of finance has an important implication in organizations as transactions and business dealings are mostly pecuniary. Such matters as lending and borrowing money for various purposes, leasing materials, accumulating funds for future use, sell of bonds are some of the cases that involve the concept of time value of money. Likewise, finance mathematics is equally important in our personal affairs. For example, we might be interested in owning a house, in financing

> Money has a time value i.e. a Birr today is worth more than a Birr tomorrow, which is expressed in terms of interest charges. Since the use of money bears the cost of interest, management must optimize the use the employment of investable money (funds); it must choose a wide array of investment opportunities and choose the one which is most profitable.

our educational fees, having a car, having enough retirement funds etc. All these cases and others involve financial matter. Cognizant of this fact, we proceed to the study of mathematics of finance in this section.

4.2. Terminologies

1) **Principal amount (P):** this is the amount of money that is initially being considered. It might be an amount about to be invested or loaned (borrowed) or it may be the initial value or cost of plant asset or machinery.
2) **Number of time periods (n)** the number of time periods over which amount of money is being invested or borrowed is normally denoted by the symbol "n" is usually a number of years, it could represent other time periods, such as a number of quarters or months.
3) **Rate of interest (i)** is the name given to appropriate amount of money which is added to some principal amount (invested or borrowed). It is normally denoted by "r" and expressed as a percentage rate per annum. For example, if Birr 100 is invested at interest rate 10% per year (annum), it will accumulate to Birr 100+ (10% of 100) which is at the end of one year.
4) **Accrued amount (accumulated amount A):** is the amount of money after some time has elapsed for which interest has been calculated and added.

Today, businesses and individuals are faced with a bewildering array of loan facilities and investment opportunities. In this section we explain how these financial calculations are carried out to enable an informed choice to be made between the various possibilities available.

We begin by considering what happens when a single lump sum is invested and show how to calculate the amount accumulated over a period of time.

Assume that someone gives you the option of receiving Birr 500 now or Birr 500 in three years' time. Which of these alternatives would you accept? Most people would take the money now, partly because they may have an immediate need for it, but also because they recognize that Birr 500 is worth more today than in 3 years' time. Even if we ignore the effects of inflation, it is still better to take the money now, since it can be invested and will increase in value over the 3-year period. In order to work out this value we need to know the rate of interest and the basis on which it is calculated. Let us begin by assuming that the Birr 500 is invested for 3 years at 10% interest compounded annually. What exactly do we mean by '10% interest compounded annually'? Well, at the end of each year, the interest is calculated and is added on to the amount currently invested. If the original amount is Birr 500 then after 1 year the interest is 10% of Birr 500, which is: $\frac{10}{100}$ * Birr 500 = 0.1 * Birr 500 = Birr 50. So the amount rises by Birr 50 to Birr 550.

What happens to this amount at the end of the second year? Is the interest also Birr 50? This would actually be the case with ***simple interest***, when the amount of interest received is the same for all years. However, with ***compound interest***, we get 'interest on the interest', which means an interest generated in last period would be added to the principal and generate interest in the next period.

4.3. Simple interest and discounts

Simple Interest is interest that is paid solely on the amount of the principal P is called simple interest. Simple interest formula:

$$I = p\,i\,n$$

Where, I = Simple interest (in dollars or birr)
P = Principal (in dollar, or birr) and it is the amount
i = Rate of interest per period (the annual simple interest rate)
n = Number of years or fraction of one year

In computing simple interest, any stated time period such as months, weeks or days should be expressed in terms of years. Accordingly, if the time period is given in terms of,

i. Months, then

$$n = \frac{\text{Number of months}}{12}$$

ii. Weeks, then

$$n = \frac{\text{Number of Weeks}}{52}$$

iii. Days, then

a. Exact interest

$$n = \frac{\text{Number of days}}{365}$$

b. Ordinary simple interest

$$n = \frac{\text{Number of days}}{360}$$

Maturity value (future value) represents the accumulated amount or value at the end of the time periods given. Thus,

Future value (F) = Principal (P) + Interest (I)

Example: A credit union has issued a 3 year loan of Birr 5000. Simple interest is charged at a rate of 10% per year. The principal plus interest is to be repaid at the end of the third year.

a. Compute the interest for the 3-year period.
b. What amount will be repaid at the end of the third year?

Solution:

Given values in the problem

3 – Years loan = Principal = Birr 5000
Interest rate = i = 10% = 0.1
Number of years (n) = 3 years

a. I = p i n
   I = 5000 x 0.1 x 3
   I = Birr 1500
b. The amount to be repaid at the end of the third year is the maturity (future) value of the specified money (Birr 5000). Accordingly, F = P + I
   F = 5000 + 1500
   F = Birr 6500

   Or, using alternative approach,
   F = P + I
   Then, substitute I = P i n in the expression to obtain
   F = P + Pin
   F = P (1 + in)

Consequently, using this formula we can obtain

F = 5000 (1+ (0.1x3)
F = 5000 x 1.3
F = Birr 1500

4.3.1. Ordinary and Exact Interest

In computing simple interest, the number of years or time, n, can be measured in days. In such case, there are two ways of computing the interest.

i. The Exact Method: if a year is considered as 365 days, the interest is called exact simple interest. If the exact method is used to calculate interest, then the time is

   n = number of days / 365

ii. The Ordinary Method (Banker's Rule): if a year is considered as 360 days, the interest is called ordinary simple interest. The time n, is calculated as

n = number of days/ 360

Example: Find the interest on Birr 1460 for 72 days at 10% interest using,

a. The exact method
b. The ordinary method

Solution:

Given	a)	b)
P = Birr 1460	P=1460	P=1460
n = 72 days	n=72/*365*	n= 72/*360*
i = 10% = 0.1	i= 0.1	i=0.1
	I=Pin =1460*72/365*0.1=<u>28.8</u>	I=Pin= 1460*72/365*0.1=<u>29.2</u>

4.3.2. Simple Discount: Present Value

The principal that must be invested at a given rate for a given time in order to produce a definite amount or accumulated value is called present value. The present value is analogous to a principal P. It involves discounting the maturity or future value of a sum of money to a present time. Hence, the simple present value formula is derived from the future value (F) formula as follows.

Future Value = Principal + Interest

F = P + I but I = Pin

Thus, F = P + Pin

F = P (1+ in)

Then from this, solve for P.

$$P = F/(1+in)$$

If P is found by the above formula, we say that F has been discounted. The difference between F and P is called the simple discount and is the same as the simple interest on P.

Example: 90 days after borrowing money a person repaid exactly Birr 870.19. How much money was borrowed if the payment includes principal and arch nary simple interest at 9 ½ %?

Solution:

Given values in the problem,

n in ordinary method = Number of days / 360

= 90 /360

n = 0.25

F = the amount repaid = Birr 870.19

i = 9 ½% = 9.5% = 0.095

Required:

The amount borrowed which is the same as simple present value, P.

$$P = F/_{1+in}$$

$$= 870.19 / (1+ (0.095 \times 0.25))$$

$$P = 870.19 \div 1.024$$

$$P = \text{Birr } 849.795$$

4.3.3. Promissory Notes and Bank Discount

A promissory note is a promise to pay a certain sum of money on a specified date. It is also considered as a written contract containing an unconditional promise by the debtor called the maker of the note to pay a certain sum of money to the creditor called the payee of the note, under terms clearly specified in the contract. Promissory note is unconditional in a sense that it gives the maker of the note an exclusive right either to sell, borrow, or discount it against the value of the note.

A bank discount is the amount of money received or collected after discounting a note before its due date. It is not unusual when borrowing money from a bank that one is required to pay a charge based on the total amount that is to be repaid (maturity value), instead of the principal used. If the maturity value is used in determining the charge for use of money, we say that the promissory note (or simply the note) is discounted. Consequently, a charge of loan computed in this manner is called 'Bank Discount' and it is always computed based on the maturity value. Bank discount is the amount that is charged on maturity value. Hence, the amount of money payable to the debtor or the amount that the borrower receives is called 'Proceed.' The amount that the borrower is going to pay to the creditor (lender) is called 'maturity value.' To further our understanding of this concept, let's develop mathematical expressions (formula) for computation of the variables at stake.

Proceed = Maturity Value – Bank Discount

Symbolically,

$$P = F - D, \text{ and } \quad D = Fdt$$

Where,
P = Proceed
F = Maturity value
D = Bank discount
d = Rate of discount
t = Time of discount

Now we can further elaborate the above formula for proceed. To begin with,

$$P = F - D, \quad \text{but} \quad D = Fdt$$

Therefore, $$P = F - Fdt = F(1 - dt)$$

In sum, proceeds can be calculated by

$$P = F(1 - dt)$$

For example, if Birr 1000 is borrowed at 12% for 6 months, the borrower receives the proceeds, P, and pays back F = Birr 1000. The proceeds will be Birr 1000 minus the interest on Birr 1000. This will be:

$$P = 1000 - (1000 \times 0.12 \times 6/12) = \text{Birr } 940$$

Or, $$P = 1000\,(1 - (0.12 \times 6/12) \quad P = \text{Birr } 940$$

i. Proceeds are an amount received now for payment in the future. Therefore, they are analogous to present value. Yet, proceeds are not equal to present value because the proceeds from a futures obligation to pay are always less than the present value of that obligation if, of course, the same rate of interest is used in both adulations.

ii. Proceeds should be completed when the interest rate is stated by the qualifier word as discount rate or a bank discount or interest deducted – in – advance, and present value should be computed where the interest is given without such qualifiers, discount.

iii. The computation of simple interest and bank discount is the same except in the former case principal and in the later case the maturity values are used for between trimmings the amount discount.

Having the idea of promissory notes and bank discounts, we may now progress to consider some illustrative problems.

Example: Find the bank discount and proceeds on a note whose maturity value is Birr 480 which is discounted at 4% ninety days before it is due.

Solution: Given values in the problem

F = Birr 480

d = 4% = 0.04

t = 90 days or 3 months = 3/12 = 90/360 = 0.25

D = ? and P = ?

To find the value of the bank discount, we use the formula D = Fdt. Accordingly,

$$D = 480 \times 0.04 \times 3/12$$

$$D = \text{Birr } 4.8$$ is the amount of bank discount.

In the same manner, the proceed can be obtained as follows.

$P = F - D$ or $P = F(1 - d\,t)$

$P = 480 - 4.8$ or $P = 480(1 - (0.04 \times 0.25)$

$P = \text{Birr } 475.2$ or $P = 480(0.99) = \text{Birr } 475.2$

Self-test 4.1. Dear learners, the following questions concerns simple interest, attempt to answer.

Dear learner, we have seen above the concepts of promissory notes and bank discount with examples now, would you try to do this question. A borrower signed a note promising to pay a bank Birr 5000 ten months from now.

a. How much will the borrower receive if the discount rate is 6%?
b. How much would the borrower have to repay in order to receive Birr 5000 now?

4.3.4. The Compound Interest:

If an amount of money, P, earns interest compounded at a rate of I percent per period it will grow after n periods to the compound amount F, and it is computed by the formula:

Compound amount formula: $F_n = P(1 + i)^n$

Where,
- P = Principal
- i = Interest rate per compounding periods
- n = Number of compounding periods (number of periods in which the principal earn interest)
- F = Compound amount

A period, for this purpose, can be any unit of time. If interest is compounded annually, a year is the appropriate compounding or conversion or interest period. If it is compounded monthly, a month is the appropriate period. It is important to know that the number of compounding period/s within a year is/are used in order to find the interest rate per compounding periods and it is denoted by i in the above formula. Consequently, when the interest rate is stated as annual interest rate and is compounded more than once a year, the interest rate per compounding period is computed by the formula:

i = j / m, where j is annual quoted or nominal interest rate
m number of conversation periods per year or the compounding periods per year

n = m x t, where t is the number of years

Example: Assume that we have deposited Birr 6000 at commercial Bank of Ethiopia which pays interest of 6% per year compounded yearly. Assume that we want to determine the amount of money we will have on deposit (our account) at the end of 2 years (the first and second year) if all interest is left in the savings account.

Solution:

Give values in the problem, P = Birr 6000, j = 6% = 0.06, t = 2 years

m = compounded annually = i.e. only once

n = m x t = 1 x 2 = 2

i = j / m = 0.06 / 1 = 0.06

Then, the required value is the maturity or future value

$F = P(1 + i)^n$

$= 6000 (1 + 0.006)^2$

$=$ Birr $6000 (1.06)^2 =$ Birr 6741.6

Example: An individual accumulated Birr 30,000 ten years before his retirement in order to buy a house after he is retried. If the person invests this money at 12% compounded monthly, how much will be the balance immediately after his retirement?

Solution:

Given values, P = Birr 30,000 , t = 10 years , i = 12% = 0.12

m = compounded monthly = 12

i = j / m = 0.12 / 12 = 0.01

n = m x t = 12 x 10 = 120 and what is required is the Future Value F.

Then, $F = P (1 + i)^n$

$= 30,000 (1.01)^{120}$

$= 30,000 (1.01)^{120}$

F = Birr 99,011.61

Having the understanding of how compound interest works and computation of future value, in subsequent example, we will consider how to determine the number of periods it will take for P birr deposited now at *i* percent to grow to an amount of F birr.

4.3.5. Present Value of a Compound Amount

Future (maturity) value is the value of the present sum of money at some future date (time). Conversely, present value (or simply principal) is the current birr or dollar value equivalent of the future amount. It is the sum of money that is invested initially and that is expected to grow to some amount in the future at a specified rate. If we put the present and future (maturity) values on a continuum as shown below, we can observe that they are inverse to one another. And, future value is always greater than the present value or the principal since it adds/earns interest over specified time-period.

0 1 2 3 . . . n

Present Value (P) Future Value (Compound Amount)

Future value is obtained by compounding technique and the expression $(1 + i)^n$ is called compound factor. On the other hand, present value is obtained by discounting techniques and the expression $(1 + i)^{-n}$ is referred to as the compound discount factor.

$$P = \frac{F_n}{(1+i)^n} = F_n(1+i)^n \qquad\qquad F_n = P(1+i)^n$$

The formula for present value of compound amount is simply derived from compound amount formula by solving for P.

Examples:

1. What is the present value of
   a. Birr 5000 in 3 years at 12% compounded annually?
   b. Birr 8000 in 10 years at 10% compounded quarterly?
2. Suppose that a person can invest money in a saving account at a rate of 6% per year compounded quarterly. Assume that the person wishes to deposit a lump sum at the beginning of the year and have that some grow to Birr 20,000 over the next 10 years. How much money should be deposited at the beginning of the year?
3. A young man has recently received an inheritance of birr 200,000. He wants to make a portion of his inheritance and invest it for his late years. His goal is to accumulate Birr 300,000 in 15 years. How much of the inheritance should be invested if the

money will earn 8% per year compounded semi-annually? How much interest will be earned over the 15 years?

Solution:

1. (a) Given the values, $Fn = F_3$ = Birr 5000, t = 3 years m = 1 (compounded annually)

    n = t x m = 3 x 1 = 3

    j = 12 % = 0.12

    i = j / m = 0.12/1 = 0.12 and we are required to find Present Value P.

    Thus, $P = F_n (1 + i)^{-n}$

    $= 5000 (1 + 0.12)^{-3}$

    $= 5000 (1.12^{-3})$

    = 5000 (0.7118)

    P = Birr 3559

    (b) $F_n = F_{40}$= Birr 8000, t = 10 years , m = quarterly = 4

    n = t x m = 10 x 4 = 40 , i = 10% = 0.1 , i = j / m = 0.1 / 4 = 0.025

    p = ? but $P = F_n (1 + i)^{-n}$

    $= 8000 (1 + 0.025)^{-40} = 8000 (1.025)^{-40}$

    P = Birr 2979.5

2. Given the values, i = 6% = 0.06 , m = quarter = 4 times a year

    i = j ÷ m = 0.06 ÷ 4 = 0.015

    F = Birr 20,000 shall be accumulated

    t = 10 years

    n = m x t = 10 x 4 = 40 interest periods

    P = how much should be deposited now?

    $P = F_n(1 + i)^{-n}$

    $= 20,000 (1+0.015)^{-40} = 20,000(1.015^{-40})$

    P = Birr 11,025.25

3. Inheritance = Birr 200,000

    F_n = Birr 300,000 (the person's goal of deposit) , t = 15 years , j = 8% = 0.08

    m = semi-annual = 2 times a year

    i = j ÷ m = 0.08 ÷ 2 = 0.04

    n = t x m = 15 x 2 = 30 interest periods/semi-annuals

    P = how much of the inheritance should be invested now? $P = F_n(1 + i)^{-n}$

    I = Amount of interest?

    $= 300,000 (1+0.04)^{-30} = 300,000(1.04)^{-30} = 300,000(0.3083)$

    = Birr 92,490

The present value of Birr 300,000 after 15 years at 4% semi-annual interest rate is equal to Birr 92,490. Therefore, from the total inheritances received Birr 92,490 needs to be deposited now.

Amount of compound interest = Future Value – Preset Value = 300,000 – 92,490

Amount of compound interest = Birr 207,510

4.3.6. Annuities:

Annuity refers to a sequence or series of equal periodic payments, deposits, withdrawals, or receipts made at equal intervals for a specified number of periods. For instance, regular deposits to a saving account, monthly expenditures for car rent, insurance, house rent expenses, and periodic payments to a person from a retirement plan fund are some of particular examples of annuity. Payments of any type are considered as annuities if all of the following conditions are present:

i. The periodic payments are equal in amount
ii. The time between payments is constant such as a year, half a year, a quarter of a year, a month etc.
iii. The interest rate per period remains constant.
iv. The interest is compounded at the end of every time.

Annuities are classified according to the time the payment is made. Accordingly, we have two basic types of annuities.

i. **Ordinary annuity:** is a series of equal periodic payment is made at the end of each interval or period. In this case, the last payment does not earn interest.
ii. **Annuity due:** is a type of annuity for which a payment is made of the beginning of each interval or period.

It is only for ordinary annuity that we have a formula to compute the present as well as future values. Yet, for annuity due case, we may drive it from the ordinary annuity formula. To proceed, let us first consider some important terminologies that we are going to use in dealing with annuities.

i. Payment interval or period: it is the time between successive payments of an annuity.

ii. Term of annuity: it is the period or time interval between the beginning of the first payment period and the end of the last one.

iii. Conversion or interest period: it is the interval between consecutive conversions of interest.

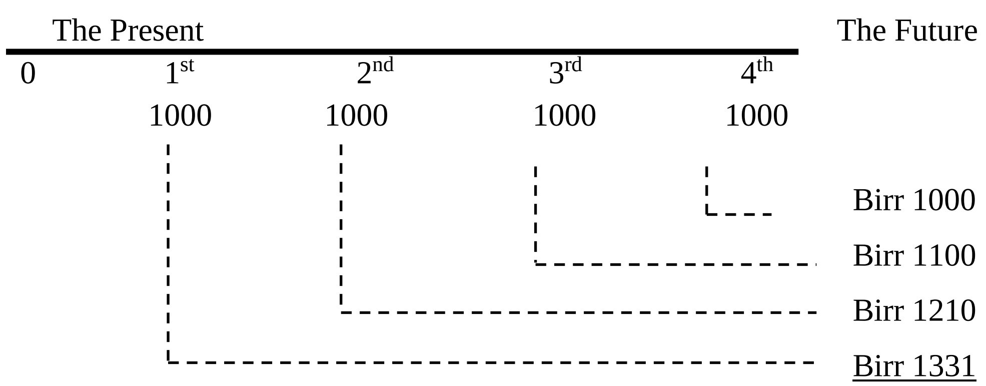
iv. Periodic payment/rent: it is the amount paid at the end or the beginning or each payment period.

v. Simple annuity: is the one in which the payment period and the conversion periods coincides each other.

4.3.7. Sum of Ordinary Annuity: Maturity Value

Maturity value of ordinary annuity is the sum of all payments made and all the interest earned there from. It is an accumulated value of a series of equal payments at some point of time in the future. Suppose you started to deposit Birr 1000 in to a saving account at the end of every year for four years. How much will be in the account immediately after the last deposit if interest is 10% compounded annually?

In attempting this problem, we should understand that the phrase at the end of every year implies an ordinary annuity case. Likewise, we are required to find out the accumulated money immediately after the last deposit which also indicates the type of annuity. Further, the term of the annuity is four years with annual interest rate of 10%. Thus, we can show the pattern of deposits diagrammatically as follows.



Total Future Value = Birr 4641

The first payment earns interest for the remaining 3 periods. Therefore, the compound amount of it at the end of the term of annuity is given by,

$$F = P(1+i)^n = 1000(1+0.1)^3 = \text{Birr } 1331$$

In the same manner, the second payment earns interest for two periods (years). So,

$$F = 1000(1+0.1)^2 = 1210$$

The 3rd payment earns interest for only one period. So,

$$F = 1000(1+0.1)^1 = 1100$$

No interest for the fourth payment since it is made at the end of the term. Thus, its value is 1000 itself. In total, the maturity value amounts to Birr 4641. This approach of computing future value of ordinary annuity is complex and may be tiresome in case the term is somewhat longer. Thus, in simple approach we can use the following formula for sum of ordinary annuity (Future Value).

$$Fn = R\left[\frac{(1+i)^n - 1}{i}\right]$$

Where, n = the number of payment periods

i = interest rate per period

R = payment per period

F_n = future value of the Annuity or sum of the annuity after n periods

Now, let us consider the above example. That is,

R = Birr 1000

i = 0.1 and n = 4

$$F_4 = 1000\left[\frac{(1+0.1)^4 - 1}{0.1}\right]$$

Future Value = Birr 4641

Example: A person plans to deposit 1000 birr in a savings account at the end of this year and an equal sum at the end of each following year. If interest is expected to be earned at the rate of 6% per year compound semi-annually, to what sum will the deposit (investment) grow at the time of the fourth deposit?

Solution:

The known values in the problem are,

R = 1000, j = 6% = 0.06 , m = semi-annual = twice a year

i = 0.06 ÷ 2 = 0.03

n = 4

$F_4 = ?$ $\qquad$ $F_4 = R\left[\frac{(1+i)^n - 1}{i}\right]$

$$= 1000\left[\frac{(1+0.03)^4 - 1}{0.03}\right]$$

$$= 1000\ \left[(1.03)^4 - 1/0.03\right]$$

$$= 1000 \text{ x } 4.\ 183627$$

$$F = \text{Birr } 4183.63$$

Self-test 4.2. Dear learners, try to solve yourself.

A 12 years old student wants to begin saving for college. She plans to deposit Birr 50 in a saving account at the end of each quarter for the next 6 years. Interest is earned at a rate of 6% per year compounded quarterly. What should be her account balance 6 years from now? How much interest will she earn?

4.3.8. Ordinary Annuities: Sinking Fund Payments

A sinking fund is a fund into which periodic payments or deposits are made at regular interval to accumulate a specified amount (sum) of money in the future to meet financial goals and/or obligations. The equal periodic payment to be made constitute an ordinary annuity and our interest is to determine the equal periodic payments that should be made to meet future obligations. Accordingly, we will be given the Future Amount, F, in n period and our interest is to determine the periodic payment, R. Then we can drive the formula for R as follows.

$$F_n = R\left[\frac{(1+i)^n - 1}{i}\right]$$

Multiplied both sides by $\frac{i}{(1+i)^n - 1}$

That is

$$F_n \times \frac{i}{(1+i)^n - 1} = R\left[\frac{(1+i)^n - 1}{i}\right] \times \frac{i}{(1+i)^n - 1}$$

Then,

$$R = F_n\left[\frac{i}{(1+i)^n - 1}\right]$$ is the sinking found formula.

Where, R = Periodic payment amount of an annuity

i = Interest per period which is given by j ÷ m

j = Annual nominal interest rate

m = Number of conversion periods per year

n = Number of annuity payment or deposits (also, the number of compounding

periods)

F = Future value of ordinary annuity

In general, a sinking fund can be established for expanding business, buying a new building, vehicles, settling mortgage payment, financing educational expenses etc.

Example:

A corporation wants to establish a sinking fund beginning at the end of this year. Annual deposits will be made at the end of this year and for the following 9 years. If deposits earn interest at the rate of 8% per year compounded annually, how much money must be deposited each year in order to have 12 million Birr at the time of the tenth deposit? How much interest will be earned?

Solution:

1. Future level of deposit desired = F_n = Birr 12 million
   Term of the annuity = t = 10 years
   Conversion periods = m = annual = 1
   n = t x m = 10 x 1 = 10 annuals
   j = 0.08
   i = j ÷ m = 0.08 ÷ 1= 0.08

R = the amount to be deposited each year to have 12 million at the end of the 10th year = ? Then to obtain the value of R, we shall use the formula for sinking fund.

$$R = F_n\left[\frac{i}{(1+i)^n - 1}\right]$$

R = Birr 828,353.86

On the other hand, the amount of interest, I, is obtained by computing the difference between the maturity value (F_n = 12,000,000) and the sum of all periodic payments made.

$$R = 12{,}000{,}000\left[\frac{0.08}{(1+0.08)^{10}-1}\right]$$

$$R = 12{,}000{,}000\left[\frac{0.08}{(1.08)^{10}-1}\right] \qquad R = 12{,}000{,}000\left[\frac{0.08}{1.158925}\right]$$

Thus,

$$\begin{aligned} I &= F_n - R(10) \\ &= 12{,}000{,}000 - 823{,}353.86\,(10) \\ &= 12{,}000{,}000 - 8{,}283{,}538.6 \\ &= \text{Birr } 3{,}716{,}461.4 \end{aligned}$$

4.3.9. Present Value of Ordinary Annuity:

The present value of annuity is an amount of money today, which is equivalent to a series of equal payments in the future. It is the value at the beginning of the term of the annuity. The present value of annuity calculation arise when we wish to determine what lump sum must be deposited in an account now if this sum and the interest it earns will provide equal periodic payment over a defined period of time, with the last payment making the balance in account zero. Present value of ordinary annuity is given by the formula:

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

Where, R= Periodic amount of an annuity

i = Interest per period which is given by j ÷ m

j = Annual nominal interest rate

m = Interest/ conversion periods per year

n = Number of annuity payments / deposits (also, the number of compounding periods)

P = Present value of ordinary annuity

Example: A person recently won a state lottery. The term of the lottery is that the winner will receive annual payments of birr 18,000 at the end of this year and each of the following 4 years. If the winner could invest money today at the rate of 6% per year compounded annually, what is the present value of the five payments?

Solution:

R = Annual payments of Birr 18,000

Term of the annuity = t = this year and the following 4 years = 5 years

i = 6% = 0.06 (since the conversion period per year is annual)

n = 5

Present value of payments = P = ?

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$P = 18{,}000\left[\frac{1-(1+0.06)^{-5}}{0.06}\right] = 18{,}000\left[\frac{1-(1.06)^{-5}}{0.06}\right]$$

P = Birr 75,822.55

4.3.10. Mortgage Payments and Amortization

Another main area of application of annuities in to real world business situations in general and financial management practices in particular is mortgage amortization or payment. Mortgage payment is an arrangement whereby regular payments are made in order to settle an initial sum of money borrowed from any source of finance. Such payments are made until the outstanding debt gets down to zero. An individual or a firm, for instance, may borrow a given sum of money from a bank to construct a building or undertake something else. Then the borrower (debtor) may repay the loan by effecting (making) a monthly payment to the lender (creditor) with the last payment settling the debt totally.

In mortgage payment, initial sum of money borrowed and regular payments made to settle the respective debt relate to the idea of present value of an ordinary annuity. Along this line, the expression for mortgage payment computation is derived from the present value of ordinary annuity formula. Our intention in this case is to determine the periodic payments to be made in order to settle the debt over a specified time – period.

Hence, we know that

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

Now, we progress to isolate R on one side. It involves solving for R in the above present value of ordinary annuity formula. Hence, multiply both sides by the interest rate i to obtain:

$$P\, i = R\, [1 - (1 + i)^{-n}]$$

Further, we divide both sides by $[1 - (1 + i)^{-n}]$ and the result will be the mathematical expression or formula for computing mortgage periodic payments as follows.

$$R = P\left[\frac{i}{1-(1+i)^{-n}}\right]$$

Where, R = Periodic amount of an annuity
i = Interest per conversion period which is given by j ÷ m
j = Annual nominal interest rate
m = Interest or conversion periods per year
n = the number of annuity payments/deposits (number of compounding periods)
P = Present value of an ordinary annuity

Example: Emmanuel purchased a house for Birr 115,000. He made a 20% down payment with the remaining balance amortized in 30 years mortgage at annual interest rate of 11% compounded monthly.

a. Find the monthly mortgage payment?
b. Compute the total interest.

Solution:

1. Total cost of purchase = Birr 115,000

Amount paid at the beginning (Amount of down payment) = 20% of the total cost
= 0.2 x 115,000 = Birr 23,000

Amount Unpaid or Mortgage or Outstanding Debt = 115,000 – 23,000
= Birr 92,000

t = 30 years

j = 11% = 0.11 , m = 12 , i = 0.11 ÷ 12 = 0.00916

n = t x m = 30 x 12 = 360 months

The periodic payment R = ?

$$R = P\left[\frac{i}{1-(1+i)^{-n}}\right]$$

= 92,000 (0.009523233)

R = Birr 876.14

$$R = 92{,}000\left[\frac{0.00916}{1-(1+0.00916)^{-360}}\right]$$

b. Total Interest = (R x n) – P

= 876.14 x 360 – 92,000

= Birr 223,409.49

Over the 30 years period Emmanuel is going to pay a total interest of Birr 223,409.49, which is well more than double of the initial amount of loan. Nonetheless, the high interest can be justified by the fact that value of a real estate is usually tend to increase overtime. Therefore, by the end of the term of the loan the value of the real estate (house) could be well higher than its purchase cost in addition to owning a house to live in for the 30 years and more.

Summary

The following are some of important formula that deal with mathematics of finance.

- Simple Interest

  = P i n
- Future Value Of A Simple Interest

  = P (1 + i n)
- Compound Amount

  = $P (1 + i)^n$
- Present Value Of A Compound Amount

  = $F (1 + i)^{-n}$
- Maturity Value Of Ordinary Annuity (F_n)

$$Fn = R\left[\frac{(1+i)^n - 1}{i}\right]$$

- Sinking Fund Payment Formula

$$R = F_n\left[\frac{i}{(1+i)^n - 1}\right]$$

- Present Value Of Ordinary Annuity

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

- Mortgage Payments and Amortization

$$R = P\left[\frac{i}{1-(1+i)^{-n}}\right]$$

***Self-test 4.3. Dear learners*, the following questions concerns annuities, try to solve yourself.**

1. A person deposits Br. 400 a month for four years into an account that pays 7% compounded monthly. After the four years, the person leaves the account untouched for an additional six years. What is the balance after the 10 year period?
2. How much should you deposit in an account paying 6% compounded quarterly in order to be able to withdraw Birr 1000 every 3 months for the next 3 years?
3. At the time of retirement, a person has Birr 200,000 in an account that pays 12% compounded monthly. If he decides to withdraw equal monthly payments for 10 years, at the end of which time the account will have a zero balance, how much should he withdraw each month?

4.4. Summary

The following are some of important formula that deal with mathematics of finance.

- Simple Interest

= P i n

- Future Value Of A Simple Interest

= P (1 + i n)

- Compound Amount

$$= P(1+i)^{n}$$

- Present Value Of A Compound Amount

$$= F(1+i)^{-n}$$

- Maturity Value Of Ordinary Annuity (F_n)

$$Fn = R\left[\frac{(1+\mathrm{i})^{\mathrm{n}} - 1}{\mathrm{i}}\right]$$

- Sinking Fund Payment Formula

$$R = F_n\left[\frac{i}{(1+i)^n - 1}\right]$$

- Present Value Of Ordinary Annuity

$$\boldsymbol{P = R\left[\frac{1-(1+i)^{-n}}{i}\right]}$$

- Mortgage Payments and Amortization

$$\boldsymbol{R = P\left[\frac{i}{1-(1+i)^{-n}}\right]}$$

4.5. Review questions

1. If money worth 14% compounded semi-annually, would it be better to discharge a debt by paying Birr 500 now or Birr 600 eighteen months from now?
2. A bank states that the effective interest on savings accounts that earn continuous interest is 10%. Find the nominal rate.
3. If you borrow Br. 1, 000 from Commercial Bank of Ethiopia for 1 year to pay at 6% interest rate your tuition fee. Find the simple interest and the maturity value of the loan.
4. What is the present value of a loan that will amount to Br. 5, 000 in 5 years if money is worth 3% compounded semi-annually? \\
5. How much should be deposited in an account paying 10% compounded quarterly in order to have a balance of Br. 10, 000 ten (10) years from now? What would be the amount of compound interest after 10 years?
6. A small boy at the age of 10 drops 0.25 cents into a Jar each day. At the end of each month (30 days months) he deposits this amount at Dashen Bank that pays 5% interest compounded quarterly. If he makes the deposit without interruption, how much will the boy have at the age of 20?
7. Hiwot deposits Br. 1, 000 at the end of every 3 months period in to an account for 5 years which earn 10% interest compounded quarterly and then her deposits are changed to Br. 500 monthly for the next 5 years which earn 12% interest compounded monthly. How much is the account by the end of the time period considered?
8. ABC Company purchased a delivery truck on credit from XYZ which requires a payment of Br. 400, 000 plus 5% interest compounded annually at the end of 5 years. The Company plans to set up a sinking fund to accumulate the amount required to settle the debt.

    Required:

    A. Find the total debt at the end of the 5 year.

    B. What should be the monthly deposit into the fund be if the account pays 15% interest, compounded monthly?
9. If Br. 10, 000 is invested at 8% compounded:

    A. Annually

    B. Semi Annually

    C. Quarterly

D. What can you observe from your answers in A, B and C
What is the amount after 5 years?

10. Assume you won a lottery and you want to deposit/ invest your money in the following to investment alternatives. Investment A which pays 15% compounded monthly and B that pays 14% compounded semiannually, which is the better investment, assuming other things are the same.
11. An investor has an opportunity to invest in two investment alternatives A and B which pays 15% compounded monthly, and 15.2% compounded semi-annually respectively. Which investment is better investment, assuming all else equal?
12. Mr. Kebede has a savings goal of Br. 100, 000 which he would like to reach 15 years from now. During the first 5 years he is financially able to deposit only Br. 1000 each quarter into the savings account. What must his quarterly deposit over the last 10 (ten) years be if he is to reach his goal? The account pays 10% interest, compounded quarterly.
13. What is the present value of an annuity that pays Br. 500 a month for the next five years if money is worth 12% compounded monthly?
14. If you have Br. 100,000 in an account that pays 6% compounded monthly and I you decide to withdraw equal monthly payments for 10 years at the end of which time the account will have a zero balance, how much should be withdrawn each month?
15. Andinet and Florence are looking to purchase a home. They found one that they like that costs Birr 150,000. They can get a 30-year mortgage at 9% and plan to make a down payment of 20% of the selling price.
    A. What will be their monthly mortgage payment?
    B. When Andinet and Florence go to the bank, they are offered an annual percent rate of 6% if they take a 15-year loan rather than one for 30 years. Andinet and Florence are skeptical because they can't afford to make twice the payment calculated for 30 years. In actual fact, how much would their payment be if they repaid the mortgage in 15 years?
    C. Andinet is 25 years old and wants to be a millionaire by the time he is 50. He is planning to put aside a sum of money at the end of each year sufficient to accumulate a million Birr in 25 years using an interest rate of 10%. How much must he put aside?
    D. Considering your answer in part c above, suppose Andinet can only put aside Birr 10,000 per year. How high a rate of return must he realize to achieve his goal?

Chapter Five: Elements and Applications of Calculus

Chapter Objectives

Dear learners, at the end of this chapter you should be able to:

- Understand Differential Calculus
- Understand Limits and Continuity of a function:
- Understand and compute Derivative
- Demonstrate how to apply Differential calculus in business
- Understand integral calculus and the basic rules for integration
- Demonstrate how to apply integral calculus in business

5.1. Introduction to Calculus

Dear learner! What do calculus mean? Why we study calculus? ____________________

____

It is a dried fact that the application of concepts of calculus in the business arena specially; in marginal analysis and optimization problems is paramount. In this part of the module, basic concepts in calculus to be seen include: concept of limit and continuity, derivatives, definite and indefinite integration, and their major application areas in business; typically, marginal analysis, optimization problems and area functions.

Calculus is branch of mathematics concerned with the study of such concepts as the rate of change of one variable quantity with respect to another, the slope of a curve at a prescribed point, the computation of the maximum and minimum values of functions, and the calculation of the area bounded by curves.

Calculus is the branch of mathematics that concerns itself with the rate of change of one quantity with respect to another quantity. **Calculus** is a mathematical tool to solve problems in business, economics and other areas which mainly rely about changes. There are two types of calculus: ***Differential calculus and Integral calculus.***

5.2. Differential Calculus

It is one aspect of calculus that measures the rate of change in one variables as another variable changes. It broadens the idea of slope. There are two core concepts which lie down the foundation for differential calculus. These are: Limits and Continuity.

A function: if for every value of a variable x, there corresponds exactly one and only one value of the variable y, we call y is a function of x, written as:

$Y = f(x)$.

Limits: the limit of f(x) as x ⟶ c is ℓ which is written as ; $\lim_{x \to c} (x) = \ell$, if and only if the functional value *f*(*x*) is close to the single real number ℓ, whenever x is close to but not equal to c (on either side of c).

Example-1:

For the function $f(x) = x^2 + 2$, find the limit of *f*(*x*) as x approaches 1.

Solution:

X	0.8	0.9	0.99	0.999	⟶	1	⟵	1.0001	1.001	1.01	1.1
$f(x)=x^2+2$	264	2.81	2.9801	2.998				3.0002	3.002	3.021	3.21

$\ell^- = 3$ $\ell^+ = 3$

$\lim_{x \to 1} f(x) = 3$

Example-2:

For the function $f(x) = \frac{/x/}{X}$, find;

a. $\lim_{x \to 2} f(x)$ b. $\lim_{x \to 0} f(x)$

Solution:

X	1.9	1.99	1.999	⟶	2	⟵	2.001	2.001	2.01
$f(x)=\frac{/x/}{x}$	1	1	1				1	1	1

$\ell^- = 1$ $\ell^+ = 1$

Thus; $\lim_{x \to 1} f(x) = 1$

b.

X	-0.999	-0.99	-0.9	⟶	0	⟵	0.0001	0.001	0.01	0.1
$f(x)=\frac{/x/}{x}$	-1	-1	-1				1	1	1	1

$\ell^- = -1$ $\ell^+ = 1$

$\ell^- \neq \ell^+ \to$ So, $\lim_{x \to 0} f(x)$ *doesn't exist.*

Limit Theorems:

1. If k is any constant , $\lim_{x \to a} k = k$

E.g. $\lim_{x \to 5} 10 = 10$

$\lim_{a \to b} c = c$

2. $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$

E.g. $\lim_{x \to a} 3x^2 = 3(\lim_{x \to a} x^2) = 3(a^2) = 3a^2$

3. $\lim_{x \to a} (f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

E.g. $\lim_{x \to a} (x^2\text{-}2\times\text{+}3) = \lim_{x \to a} x^2 - 2\lim_{x \to a} x + \lim_{x \to a} 3$

$= a^2\text{-}2a + 3$

4. $\lim_{x \to a} [f(x).g(X)] = (\lim_{x \to a} f(x)] (\lim_{x \to a} g(x)]$

E.g. $\lim_{x \to 2} (x+3)(X\text{-}2) = [\lim_{x \to 2} (x+3)] [\lim_{x \to 2} (x\text{-}2)$

$= [5] [0]$

$= 0$

5. $\lim_{x \to a} [f(x)^n] = [\lim_{x \to a} f(x)^n]$

E.g. $\lim_{x \to 3} (x\text{-}1)^5 = [\lim_{x \to 3} (x\text{-}1)]^5 = 2^5 = 32$

6. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = m$, then;

a, If $m \neq 0$, then $\lim [f(x)/g(x)] = L/m$

b. If m = 0 and L ≠ 0, the $\lim [f(x)/g(x)]$ = doesn't exist

c. If m = 0 and L = 0, then f(x) and g(X) have a common factor and the limit can be evaluated after employing the process of cancellation.

Continuity of a Function:

Definition: a function *f* is continuous at the point x = c if:

1. $\lim f(x) \to$ exists
2. $f(c) \longrightarrow$ is defined
3. $\lim f(x) = f(c)$

Example:

Using the definition of continuity, discuss the continuity of the function $f(x) = \frac{x^2-4}{x-2}$ at c =1 and c = 2

<u>Solution:</u>

At c =1, $\lim_{x\to 1} f(x)$

X	0.9	0.99	0.999	→	1	←	1.0001	1.001	1.01	1.1
$f(x) = \frac{x^2-4}{x-2}$	2.9	2.99	2.999				3.0001	3.001	3.01	3.1

$\ell^- = 3$ $\ell^+ = 3$

$\lim_{x\to 1} f(x) = 3$ ⟶ $\lim_{x\to 1} f(x)$ exists

$f(1) = \frac{1^2-4}{1-2} = 3$ → $f(c)$ is defined

$\lim_{x\to 1} f(x) = 3 = f(1)$

Therefore; $f(x) = \frac{x^2-4}{x-2}$ is continues at c =1

At c = 2

X	1.9	1.99	1.999	→	2	←	2.0001	2.001	2.01	2.1
F(x)	3.9	3.99	3.999				4.0001	4.001	4.01	4.1

$\ell^- = 4$ $\ell^+ = 4$

$\lim_{x\to 2} f(x) = 4$ ⟶ It exists

$f(2) = \frac{2^2-4}{2-2} = \frac{4-4}{0} = \frac{0}{0}$ = ¢

$\lim_{x\to 2} f(x) \neq f(c) \to 4 \neq$ ¢

Therefore; $f(x)$ is not continuous at c = 2.

Derivatives:

Definition: for $y = f(x)$ we define the derivative of f at x, denoted by $f''(x)$ to be;

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x\ 0 \to} \frac{f(x+\Delta x)- f(x)}{\Delta x}$$

Example-1

Find f′(x) for f(x) = 2x +4

$$f(x) = \lim_{\Delta x \to 0} \frac{f(x+ \Delta x) - f(x)}{\Delta x}$$

$1^{st} \rightarrow$ find $\Delta f(x)$ =

$\Delta f(x) = f(x+\Delta x)-f(x)$

$= [2(x+\Delta x) +4] - (2\times+4)]$

$= 2\times+2\Delta x+4-2x-4$

$\Delta f(x) = 2\Delta x$

$2^{nd} \longrightarrow$ find the limit: $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta X}$

$f'(x) = 2$

Example-2

For the function $f(x) = 4x-x^2$ find $f'(x)$

$f'(x)= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$1^{st} \rightarrow$ find $\frac{\Delta f(x)}{\Delta x} = \frac{4(x+\Delta x)-(x+\Delta x)^2 - (4x-x^2)}{\Delta x}$

$= \frac{4x+4\Delta x-x^2 - 2x\Delta x - \Delta x^2-4x+x^2}{\Delta x}$

$= \frac{\Delta x (4-2x-\Delta x)}{\Delta x}$

$= 4-2x-\Delta x$

$2^{nd} \rightarrow$ find the limit of the resulting function

$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x \to 0} 4-2x-\Delta x$

$f'(x) = 4-2x$

Rule of differentiation:

1. **A constant function rule**

   If $f(x) = c$ then $f'(x) = 0$

   E.g. If $f(x) = 5$ then $f'(x) = 0$

2. **The power rule**

   The derivative of the power function is the power times the function raised the power minus one.

   If $f(x) = ax^n$, then $f'(x) = anx^{n-1}$

   E.g. If $f(x) = x^5$, then $f'(x) = 5x^{5-1} = 5x^4$

   If $f(x) =3x^3$, $f'(x) =3\times3x^{3-1} = 9x^2$

3. **The sum and difference rule**

The derivative of the sum or difference of two functions is the derivative of the first function plus or minus the derivative of the second function.

If $f(x) = u(x) \pm v(x)$, then $f'(x) = u'(x) \pm v'(x)$

E.g. $f(x) = 3x+8$, $f'(x) = 3+0 = 3$

$f(x) = 4x-x^2$; $f'(x) = 4-2x$

4. **The product rule**

The derivative of the product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first.

If $f(x) = u(x).(v(x))$ then;

$$f'(x) = u(x).\, v'(x) + v(x).\, u'(x)$$

E.g. $f(x) = 3x^2 (4x-1)$, the $f'(x) =$

$$3x2(4) + (4x-1)(6x)$$
$$= 12x^2 + 24x^2 - 6x$$
$$= 36x^2 - 6x$$

5. **The quotient rule**;

The derivative of the division of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator over the denominator square.

If $f(x) = \frac{u(x)}{V(x)}$, then;

$$f'(x) = \frac{v(x).u'(x) - u(x).v'(x)}{[V(x)]^2}$$

If $f(x) = x^2/2x-1$

$$f'(x) = \frac{(2x-1)(2x) - (x^2)(2)}{(2x-1)^2}$$
$$= \frac{4x^2-2x-2x^2}{4x^2-4x+1}$$
$$= \frac{2x^2-2x}{(2x-1)^2}$$

Application of Differential Calculus to Marginal Analysis:

The word marginal refers to rate o f change→ that is a derivative.

Let x be the number of units of a product produced

Total cost function → C(x)

Total revenue function→ R(x)

Total profit function→ $P(x) = R(x) - C(x)$

Marginal cost function → $C'(x)$

- ✓ Marginal cost is the rate of change in total cost per unit change in production at an output level of x-units.
- ✓ Marginal revenue function→ $R'(x)$
- ✓ Marginal profit function→ $P'(x) = R'(x) - C'(x)$
- ✓ Average cost→ $\check{c}(x) = \frac{C(x)}{x}$
- ✓ Marginal average cost→ $\check{c}'(x)$
- ✓ Average revenue → $\check{R}(x) = \frac{R(x)}{x}$
- ✓ Marginal average revenue→ $\check{R}'(x$
- ✓ Average profit → $\bar{P}(x) = \frac{P(x)}{x}$
- ✓ Marginal average profit → $\bar{P}'(x)$

Example-1

A company manufactures and sells x transistor radios per week. Its weekly cost and demand equations are:

$C(x) = 5000 + 2x$

$P = 10 - \frac{x}{1000}$ find

a) Production level that maximizes revenue and the maximum revenue.

b) The production level that maximizes profit and the maximum profit.

c) The MR and MC at the profit maximizing output level.

d) The average cost per unit if 1000 radios are produced.

e) The marginal average cost at a production level of 1000 radios and interpret the result.

Solution:

a) $R(x) = P.X = (10-x)\frac{(x)}{1000}$

$R(x)\ 10x - \frac{x^2}{1000}$

$R(x) = 10 - \frac{x}{5000} = 0$

$10 = \frac{x}{500} \rightarrow x = 5000$ units

$R'(x) = -1/5000$; $R'(x) < 0$ x= 5000 units is the revenue maximizing output level.

Maximum revenue is at x = 5000 units

$R(x) = 10x - \frac{x^2}{1000}$

$R(5000) = 10(5000) - \frac{(5000)^2}{1000}$

$= \textit{Birr } 25000$

b) $P(x) = R(x) - C(x)$

$[10x - \frac{x^2}{1000}] - [5000+2x]$

$= 10x - \frac{x^2}{1000} - 5000 - 2x$

$P(x) = 8x - \frac{x^2}{1000} - 5000$

$P'(x)\ 8 - \frac{2x}{1000} = \frac{8-x}{500}$

$8 - \frac{x}{500} = 0$

$8 = \frac{x}{500}$

$x = \textit{4000 units}$

$P''(x) = \frac{-1}{500}$ $P''(x) < 0$ x = 4000 units is the profit maximizing output level

At x = 4000 units $P(x) = 8x - \frac{x^2}{1000} - 5000$

$= 8(4000) - \frac{(4000)^2}{1000} - 5000$

$= \textit{Birr 11,000}$

c) $C'(x) = 2$ production cost increases by birr 2 at each level of out put

$R'(x) = 10 - \frac{2x}{1000} = 10 - \frac{x}{500}$

At x = 4000 units

$R'(x) = 10 - \frac{4000}{500} = 10 - 8 = 2$ birr

- At each level of output TR increases by birr 2
- At the profit maximization; MR = MC, i.e. 2 Birr

d) x =1000 radios;

- ✓ Average cost $\check{c}(x) = \frac{C(x)}{x}$

$= \frac{5000 + 2x}{x}$

At x = 1000 radios; $\check{c}(x) = \frac{5000 + 2(1000)}{1000}$

$= 7$ Birr

e) Marginal average cost = $\check{c}'(x)$

$$MAC = -\frac{5000}{x^2}$$

$$\check{c}'(x) = -\frac{5000}{(1000)^2} = -0.005 \text{ Birr}$$

Interpretation: At a production level of 1000 units a unit increase in production will decrease average cost by approximately 0.5 cents or by 0.005 Birr.

Self-test 5.1.

A certain manufacturing company has the following information: Average total cost is given by the equation:

$\check{c}(q) = 0.5q - 500 + \frac{5000}{q}$ and,

The demand function is:

P = 2500-0.5q

A. Find the firm's: i. Total profit function ii. Marginal cost function iii. Marginal average cost function

B. Find the quantity level that: i. maximizes total revenue ii. Maximizes total profit iii. Minimizes total cost

C. Find the firm's: i. Maximum revenue ii. Maximum profit

D. Find the price level that leads to maximum: i. Revenue ii. Profit

5.3. Integral Calculus

Integral Calculus - which deals with the problem of finding a quantity, given that we know the rate at which it is changing.

Integral calculus is the reciprocal of the differential calculus. Given the rate of change f '(x), by integral calculus we can find the original function f(x).

Indefinite Integral: Given F(x) which is the anti-derivative of f(x), the indefinite integral of f(x) is defined to be:

$\int f(x)\, d_x = F(x) + C$,

Where;

$\int$ = the integral symbol

f(x) = the integrand (the function to be integrated)

F(x) = the integral (the outcome of integration)

C = the constant of integration

d_x = indicates the variable to be integrated

The rules of integration:

1. **A constant function rule**;

If f(x) = k

$$\int f(x)d_x = \int (k)d_x = \frac{kx^{0+1}}{0+1} + C = kx + C$$

E.g. If f(x) = 5

$$\int f(x)\, d_x = \int (5)d_x = \frac{5x^{0+1}}{0+1} + C = 5x + C$$

2. **The power rule**

If $f(x) = x^n$

$$\int f(x)\, d_x = \int (x^n)d_x = \frac{x^{n+1}}{n+1} + C$$

E.g. If $f(x) = x^5$;

$$\int f(x)d_x = \int (x^5)d_x = \frac{x^{5+1}}{5+1} + C = \frac{x^6}{6} + C$$

3. **A constant times a function rule;**

If $f(x) = ax^n$

$$\int (x)d_x = \int (ax^n)d_x = a\int (x^n)d_x = a\left(\frac{x^{n+1}}{n+1}\right) + C$$

E.g. If $f(x) = 3x^3$

$$\int f(x)d_x = \int (3x^3)d_x = 3\int (x^3)d_x = 3\left(\frac{x^{3+1}}{3+1}\right) + C$$

$$= 3/4x^4 + C$$

4. **The sum and difference rule**

If f(x) =g(x) ± h(x)

$$\int f(x)d_x = \int [g(x) \pm h(x)]$$

$$= \int g(x)d_x \pm \int (h(x)d_x$$

E.g. If $f(x) = 5x+9$

$\int f(x)d_x = \int(5x+9)d_x = \int(5x)d_x + \int(9)d_x$

$= \frac{5x^{1+1}}{1+1} + \frac{9x^{0+1}}{0+1}$

$= 5/2\ x^2 + 9x + C$

5. **The product rule**

If $f(x) = (ax+b)^n$

$\int f(x)d_x = \int(ax+b)^n = \frac{(ax+b)^{n+1}}{a(n+1)} + C$

E.g. If $f(x) = (x+2)^2 = \int f(x)d_x = \int(x+2)^2 d_x$

$= \frac{(x+2)^{2+1}}{(2+1)} + C = \frac{(x+2)^3}{3} + C$

6. **The quotient rule;**

If $f(x) = \frac{g(x) + h(x)}{k(x)}$

$\int f(x)d_x = \int(\frac{g(x)}{k(x)})d_x + \int(\frac{h(x)}{k(x)})d_x$

E.g. If $f(x) = \frac{8+x^3}{x^2}$

$\int f(x)dx = \int(\frac{8}{x^2})d_x + \int(\frac{x^3}{x^2})d_x$

$\int(8x^{-2})d_x + \int(x)d_x$

$= \frac{8x^{-2+1}}{-2+1} = \frac{x^{1+1}}{1+1} + C$

$= \frac{8x^{-1}}{-1} + \frac{x^2}{2} + C$

$= -8x^{-1} + 1/2x^2 + C$

$= -8/x + \frac{x^2}{2} + C$

Indefinite integral for finding total functions:

Example: The function describing the marginal cost of producing a product is given by $f(x) = x+100$, where x is the number of units produced, determine the total cost function if the total cost of producing 100 units is birr 40 000.

Solution:

$C(x) = \int f(x)d_x = f(x) = x+100$

$\int(x+100)d_x = \frac{x^{1+1}}{} + \frac{100x^{0+1}}{} + C$

$\frac{x^{2}}{2}$ (1+1) + 100x (0+1) + C

C(x) =1/2x^2 + 100 x +c → (fixed cost)
40,000 = ½$(100)^2$+100(100) + C
40,000 = 5000+ 10,000+ C
40,000 = 15000+ C
40,000 -15000 = C
C = 25000 Birr
Therefore; C(x) = ½x^2+100x+25,000

Self-test 5.2.

Dear student, attempt this question. The marginal revenue function for a company's product is given by f(x) = 50,000-x, where x is the number of units produced. Develop the total revenue function if revenue is zero when no units are produced and sold.

Definite integral:

Definition: If f(x) is a continuous function on the interval [a, b], the definite integral of f(x) is defined as $\int_a^b$ f (x)d_x = F(b) –F(a)
Where

F(x) = the anti-derivative for f(x)
F(b) = the upper limit
F(a) = the lower limit
F′(x) = f(x)

A definite integral has a single numerical value associated with it and can be obtained through the indefinite integral by using the following steps.

*Step-1: g*et the indefinite integral of the function

Step-2: substitute the value x = a in the indefinite integral

Step-3: substitute x = b in the indefinite integral

Step-4: subtract the numerical value obtained in *Step 2* from step 3 and the result gives the definite integral value of the function between the limits x = a x = b.

Example

If marginal revenue is given by:

F(q) = 200-6q , what extra total revenue is obtained by increasing sales (q) from 15 to 20?

Solution:

Extra revenue = $\int_{15}^{20} f(q)d_q = \int_{15}^{20} (200-6q)d_q$

$= 200q - \frac{6q^{1+1}}{1+1} + C$

$= [200(20) - 3(20)^2 + C] - [200(15) - 3(15)^2 + C]$

$= 2800 + C - 2325 - C$

= *Birr475*

5.4. Summary

The major formulas for application of concept of calculus in marginal analysis and optimization problems include the following.

- *Marginal cost* is the rate of change in total cost per unit change in production at an output level of x-units.
- Marginal revenue function→ $R'(x)$
- Marginal profit function→ $P'(x) = R'(x) - C'(x)$
- Average cost→ $\check{c}(x) = \frac{C(x)}{x}$
- Marginal average cost→ $\check{c}'(x)$
- Average revenue → $\check{R}(x) = \frac{R(x)}{x}$
- Marginal average revenue→ Ř′(x
- Average profit → $\bar{P}(x) = \frac{P(x)}{x}$
- Marginal average profit → $\bar{P}'(x)$

  - $\int f(x)\, d_x = F(x) + C$
  - $\int_a^b f(x) d_x = F(b) - F(a)$

5.5. Review Questions

1. One million Birr is deposited in to a savings account for 1 year at 12% compounded quarterly. If interest is added at the end of each quarter. Find the account's balance for each quarter. Where is the graph discontinuous?

2. The profit, P (in millions of Birr), gained from selling x (in thousands) units of a product is given by: $P(x) = -0.1x^2+4x-30$ $[10 \leq x \leq 30]$

a. Find the average rate of change of profit with respect to x.

b. Determine the average rate of change of profit with respect to x as x changes from x=12 to x=15.

c. Interpret your result.

3. A small machine shop manufactures drill bits that are used in petroleum industry. The shop manager estimates that the total daily cost (in Birr) of producing X bits is $C(x) = 1{,}000 + 25x – 0.1x^2$.

a. Find the daily average cost if x units are produced.

b. Find the average cost per unit if 10 drill bits are produced.

c. Find the marginal average cost function.

d. Find the marginal average cost if 10 drill bits are produced, and interpret the results.

4. From past experience, an apple grower knows that if the apples are harvested now, each tree will yield on average 130 pounds and the grower will sell the apples for Birr 0.64 per pound. However, for each additional week that the grower waits before harvesting, the yield per tree will increase by 5 pounds, while the price per pound will decrease by Birr 0.02. How many weeks should the grower wait before harvesting the apples in order to maximize the sales revenue per tree? What is the maximum sales revenue per tree?

5. The function describing the marginal profit from producing and selling a product is $f(x) = - 6x+750$ where x = the number of units produced and sold. Moreover, when 100 units are produced and sold, total profit equals Birr 25,000. Determine the total profit function.

6. A truck carrying natural gas gets stuck at a low underpass and leaks natural gas at the rate of $L'(t) = 10t +20$ cubic feet per minute, where t denotes time (in minutes) elapsed since the gas first began leaking.

a. Find the total amount of natural gas that has leaked during the first five minutes.

b. Find the total amount of natural gas that has leaked during the first 10 minutes.

c. Find the total amount of gas that has leaked during the fifth minute.

d. Find the total amount of gas that has leaked during the sixth minute

e. If the amount of natural gas that the truck was carrying was 5,100 cubic feet, how many minutes will be elapsed before the truck is empty?

7. The annual profit of a certain hotel is given by

$$P(x, y) = 100x^2 + 4y^2 + 2x + 5y + 100{,}000$$

Where x is the number of rooms available for rent and y is the monthly advertising expenditures. Presently, the hotel has 90 rooms available and is spending Birr 1000 per month on advertising.

a) If an additional room is constructed in an unfinished area, how will this affect annual profits?

b) If an additional Birr is spent on monthly advertising expenditures, how will this affect profit?

7. The revenue, z, derived from selling x units of calculators and y units of adding machines is given by the function

$$Z = f(x, y) = -x^2 + 8x - 2y^2 + 6y + 2xy + 50$$

a) How many calculators and adding machines should be sold in order to maximize sales revenue?

b) What is the maximum sales revenue?

Answer for Review Questions

1.a. Answer: Birr 1,030,000, 1,060,900, 1,092,720, and 1,125,508.81.

b. Answer: The graph is discontinuous at the end of the first, second, third and fourth quarters.

2. a. Answer: $-0.2x - 0.1\Delta x + 4$

b. **Answer**: Birr 1,300,000

c. Answer: As x changes from x=12, 000 to x= 15,000, an additional unit sold yields Birr 1,300, on the average

3.

a. $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{1{,}000 + 25x - 0.1x^2}{x} = -0.1x + 25 + \dfrac{1{,}000}{x}$

b. $\overline{C}(10) = -0.1(10) + 25 + \dfrac{1{,}000}{10} = Birr\,124$

c. $\overline{C}'(x) = \frac{dC(x)}{dx} = -0.1 - \frac{1{,}000}{x^2}$

d. $\overline{C}'(10) = -0.1 - \frac{1{,}000}{(10)^2} = -Birr\ 10.10$

A unit increase in production will decrease the average cost per unit by approximately Birr 10.10 at a production level of 10 units.

4, Let x be the number of weeks the grower waits. The sales revenue per tree is given by R = number of pounds per tree * price per pound

R(x) = (130+5x) (0.64 - 0.02x)

Critical values

Using the product rule we find

R'(x) = (130+5x) (-0.02) + (0.64 - 0.02x) (5)

= 0.6 - 0.2x

0 = 0.6 - 0.2x

0.6 = 0.2x

x = 3

Test for absolute extrema (Second derivative test)

R"(x) = -0.2 <0

Absolute maximum is at x = 3

R (3) = [130 +5(3)] [0.64 - .2(3)]

= 145 * 0.58

= Birr 84.10

5. MP = - 6x+750

TP = ∫MP dx

= ∫(-6x + 750)dx

= ∫-6xdx + ∫750dx + C

= -6x/2 + 750x + C

$= -3x^2+750x+c$

$25{,}000 = -3(100^2) + 750(100) + C$

25,000 = 45,000 + C

C = -20,000

$P(x) = -3x^2+750x-20{,}000$

1. **a. Answer: 225 cubic feet**

   b. Answer: 700 cubic feet

   c. Answer: 65 cubic feet

   d. Answer: 75 cubic feet

   e. Answer: 30 minutes

7. a. Answer: Birr 18,002

b. **Answer**: Birr 8,005

8. Critical points

a) we first calculate f_x and f_y, as follows:

$$f_x = -2x + 8 + 2y$$

$$f_y = -4y + 6 + 2x$$

Setting f_x and f_y equal to 0, we have

$$0 = -2x + 8 + 2y$$

$$0 = -4y + 6 + 2x$$

Solving for this linear system for x and y, we obtain x = 11 and y = 7. Thus, the only critical point is (11, 7).

Second derivative test

We calculate $f_{xx} = -2$, $f_{yy} = -4$, and $f_{xy} = 2$.

Since the partial critical point is (11, 7), then

$$A = f\ (11, 7) = -2$$

$$B = f\ (11, 7) = -4$$

$$C = f\ (11, 7) = 2.$$

$$AB - C^2 = -2(-4) - 2^2 = 4$$

Since $AB - C^2 > 0$ and $A < 0$, then, according to the second derivative test, a relative maximum occurs at (11, 7). Thus, in order to maximize revenue, x = 11 calculators and y = 7 adding machines must be sold.

b) The maximum sales revenue is

$$Z = f\ (11, 7) = -(11)^2 + 8(11) - 2(7)^2 + 6(7) + 2(11)(7) + 50 = \text{Birr } 115.$$

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