

CHAPTER 3

ANALYSIS OF INDETERMINATE STRUCTURES: HARDY CROSS MOMENT DISTRIBUTION METHOD

2.1. INTRODUCTION

As we saw in chapter 1, the slope deflection method consists of equations that relate joint displacements to member-end moments. However, since every member has two ends and accordingly two corresponding slope deflection equations, the direct use of this method for the analysis of rigid structures with many members (such as multi-storey and/or multi-bay) frames involves the simultaneous solution of many equations. The amount of computation involved often makes the use of slope-deflection method for the analysis of such structures tedious and time-consuming. The moment distribution method is a procedure for solving the simultaneous equations in the slope-deflection method by successive approximations. This iterative procedure was one of the most outstanding contributions to the development of structural analysis in the twentieth century. The method was originally presented in 1932 by Prof. Hardy Cross of Illinois University for the analysis of rigid frames. Before the advent of computers and the development of matrix methods of analysis, the moment distribution method was since its introduction, the most popular method among users because the calculations involved are minimal and does not require the solution of any simultaneous equations for beams, and for frames not subject to lateral translation. Although its popularity has diminished owing to the "universal" nature of the computer-oriented matrix methods, moment distribution method remains popular among users, especially for relatively simple structures, which are easily analyzed without the aid of computers. Besides, even when a computer is used to analyse a frame with a large number of redundant, it is still useful to analyse parts of the frame manually in order to check (or validate) computer results. Moment distribution method remains very useful in such situations.

The procedure consists in first identifying all the joints of the structure that are free to rotate, and then in the imagination, clamping (or fixing) them against rotation. Next, the fixed-end moments due to the external loading are computed for each beam or frame member. At the clamped joints, the fixed-end moments are not in equilibrium and constitute unbalanced joint moments since in reality the joints are not clamped but free to rotate. For this reason, the clamped joints are released, one at a time, i.e., as one joint is released, the rest remain clamped. The release of a joint is achieved by evaluating the unbalanced moment (which is equal to the sum of the fixed-end moments at member ends meeting at the joint) at the joint and balancing the joint by introducing there a moment equal in magnitude but opposite in sign to the unbalanced joint moment. This new moment is known as *balancing moment*. As the joint has now been released by the balancing moment, it begins to rotate under the effect of this balancing moment. This rotation induces moments on the members at their ends connected to the joint. The induced end moments, known as *distributed moments*, cause bending in the appropriate members. The bending in turn causes moments, known as *carryover moments*, to be developed at the far ends of the various members connected to the joint. In this temporarily balanced (or equilibrium) state the joint is once again clamped in its rotated position and another of the previously clamped joints is released and balanced as done for

the first joint. This process of releasing, balancing, and re-clamping is performed for each of the joints that were originally free to rotate, until all the joints have been considered. The first *cycle* is then said to be completed.

In the second cycle a joint is identified which is unbalanced due to carryover moments from the far ends of the members connected to the joint. The joint is released, balanced and re-clamped, as done in the first cycle. This process is repeated for every joint which remains unbalanced, until all such joints have been considered. The second cycle is thus completed. The cycle is terminated when the unbalanced moments at the joints become sufficiently small as to be considered negligible for the desired accuracy. The *final end moments* are then obtained by algebraically *summing* up the *fixed-end moments*, the *distributed moments*, and the *carryover moments* for each member end.

SIGN CONVENTION

The sign convention adopted for this method is the same as the one used in the slope-deflection method (refer to Chapter 1).

2.2. BASIC TERMINOLOGY AND DEFINITIONS

FIXED-END MOMENTS

These are moments due to external loads and they develop at the ends of a member whose ends are fixed against rotation. For more detail, refer to Chapter 1.

MEMBER STIFFNESS

General

In Chapter 1, we employed the moment-area method to show that the moment applied at a hinged end of a beam with the far end fixed, M_{AB} (see Fig.1.3) is given by

$$M_{AB} = \frac{4EI\theta_A}{L} \quad \text{and the moment induced at the far fixed end, } M_{BA} \text{ is given by}$$
$$M_{BA} = \frac{2EI\theta_A}{L}. \quad \text{These moments are given by eqn (1.2). These components of end}$$

moments were added to the components given by eqns (1.3), (1.5) and the fixed-end moments to obtain the slope deflection equations (1.6) or as expressed by the single equation (1.7). It should be noted that since the slope-deflection equation has been derived, it can be used to easily obtain the various components of end-moments as given by eqns (1.2), (1.3) and (1.5) without further reference to the moment-area method or indeed to any other method of analysis.

To illustrate the use of the slope-deflection equation to obtain the end moments of eqn. (1.2), consider again Fig.1.3 (redrawn below as Fig.2.1). Using the slope-deflection equation (eqn.1.7), we substitute:

$$M_{nf} = M_{AB}, \theta_n = \theta_A = \theta, \theta_f = \psi = FEM_{nf} = 0 \quad \text{to obtain:}$$

$$M_{AB} = \frac{4EI\theta}{L} \quad (2.1)$$

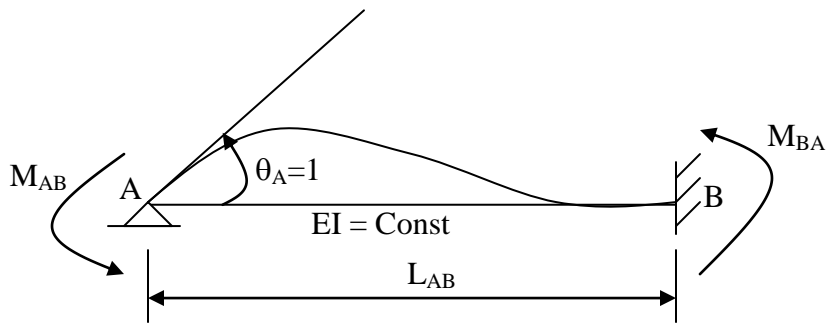


Fig.2.1 Member with far end fixed

Bending Stiffness, S

The bending stiffness S, of a member is the moment M, required to be applied at a member end to cause a unit rotation of that end. Thus, if in eqn.(2.1) we set $\theta = 1\text{rad}$, we obtain the bending stiffness of the beam of Fig.2.1 to be:

$$S = \frac{4EI}{L} \quad (2.2)$$

In cases where the modulus of elasticity E for all members of a structure is the same, it is more convenient to work with the relative bending stiffness rather than the bending stiffness.

Relative Bending Stiffness, K

The relative bending stiffness K, of a member is the bending stiffness, divided by 4E. Thus, the relative bending stiffness of the beam of Fig.2.1 is given by:

$$K = \frac{S}{4E} = \frac{I}{L} \quad (2.3)$$

In general, for a prismatic member A-B, the relative stiffness is given by:

$$K_{AB} = K_{BA} = \frac{I_{AB}}{L_{AB}} \quad (2.3a)$$

MODIFIED STIFFNESS

Let us consider a beam with the far end hinged, as shown in Fig.2.2 (the same beam of Fig.1.4b(i)).

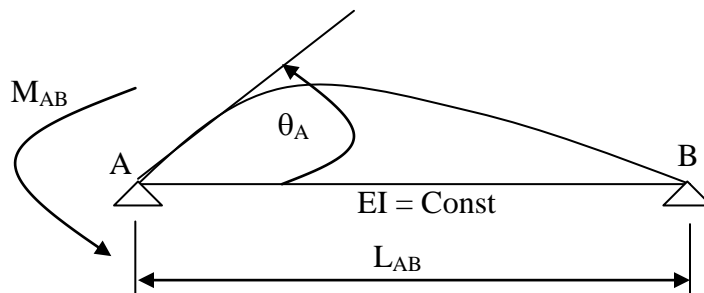


Fig.2.2 Beam with far end hinged

The moment at A is given by eqn.(1.3b) as:

$$M = M_{AB} = \frac{3EI\theta_A}{L} \quad (2.4)$$

By definition if we set $\theta_A = 1$ rad, we obtain the bending stiffness of the beam of Fig.2.2 as:

$$S = \frac{3EI}{L} \quad (2.5)$$

This bending stiffness for a member with far end hinged (given by eqn.(2.5)) is known as the modified stiffness of the member. The relative bending stiffness for this case will then be:

$$K = \frac{S}{4E} = \frac{3I}{4L} \quad (2.6)$$

The moment given by eqn.(2.4) may alternatively be obtained by using the slope-deflection equation, modified for the case of member with far end hinged (Eqn.1.7a). Equation (1.7a) is re-written below as eqn.(2.7):

$$M_{rh} = \frac{3EI}{L}(\theta_r - \psi) + \left(FEM_{rh} - \frac{FEM_{hr}}{2} \right) \text{ and} \quad (2.7)$$

$$M_{hr} = 0$$

In eqn.(2.7) the subscript **r** refers to the rigidly connected end of the member where the moment M_{rh} is applied, and the subscript **h** refers to the hinged end of the member. As explained in Chapter 1, moment at the far hinged end $M_{hr} = 0$. To obtain the moment at the rigidly connected end using eqn.(2.7), we set:

$$M_{rh} = M_{AB}, \quad \theta_r = \theta_A, \quad \psi = FEM_{rh} = FEM_{hr} = 0.$$

Thus,

$$M_{AB} = \frac{3EI\theta_A}{L}, \text{ which is the same as eqn.(2.4).}$$

A little later in this section (under "Modified fixed-end moment"), it will be shown that the term in the second pair of parentheses in eqn.(2.7) is also valid. This will be considered after discussing the carryover moment and carryover factor.

CARRYOVER MOMENT AND CARRYOVER FACTOR

It was shown in Chapter 1 (see Fig.1.3 and eqn.(1.2)) that when a moment M_{AB} is applied at the hinged end A of a beam member, a moment M_{BA} is induced at the far (fixed) end of the member. The moment M_{BA} is called **carryover moment**.

Recall that according to eqn.(1.2), $M_{AB} = \frac{4EI\theta_A}{L}$ and $M_{BA} = \frac{2EI\theta_A}{L}$.

$$\text{From here, } \frac{M_{BA}}{M_{AB}} = \frac{1}{2}, \text{ or } M_{BA} = \frac{M_{AB}}{2} \quad (2.8)$$

Thus, equation (2.8) shows that when a moment M_{AB} is applied at the hinged end of a member, it induces a moment $M_{BA} = \frac{1}{2}M_{AB}$ at the far (fixed) end of the member. The carryover moment M_{BA} is therefore a fraction of the applied moment M_{AB} , which is carried over from the hinged to the fixed end. The fraction itself, which is $+1/2$ in the case of a prismatic beam member with the far end fixed, is known as the **carryover**

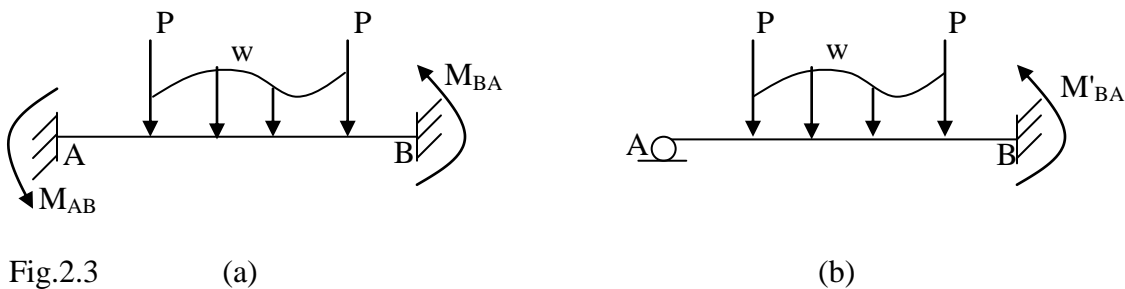
factor. The positive sign indicates that the carryover moment acts in the same direction as the applied moment. In the case of a member with the far end hinged (see Fig.2.2), the carryover moment M_{BA} will be zero since the hinge at the far end B does not support a moment. From the foregoing discussion, carryover factor can therefore be defined as the factor by which the moment at a simply supported end A, i.e., M_{AB} , is multiplied to obtain the moment carried over to the other end, i.e., M_{BA} . Thus:

$$M_{BA} = C_{AB}M_{AB} \quad (2.9)$$

where C_{AB} is the carryover factor.

MODIFIED (OR ADJUSTED) FIXED-END MOMENT

Consider again a beam similar to the beam of Fig.1.6 (now redrawn in Fig.2.3(a)), subjected to the same transverse loading but with the support conditions of Fig.2.3(b).



In Fig.2.3 (b), end A is hinged and consequently can rotate hence only one end moment M'_{BA} needs be evaluated. To abridge the moment distribution process in some cases it is convenient to use the modified fixed-end moment M'_{BA} together with the modified stiffness K (see Kani's Rotation Contribution Method in Chapter 3).

First, let us consider the beam of Fig.2.3 (a) with ends fixed against displacement. The end moments are M_{AB} and M_{BA} . Let us now allow end A to rotate by an amount such that a moment $-M_{AB}$ is induced at end A. The corresponding moment induced at B will be $-C_{AB}M_{AB}$. If the end moment induced by the rotation is algebraically added to the end moment due to fixity at the end, the moment at A will be zero, which is similar to the condition of the beam of Fig.2.3. (b). The modified FEM at B when end A is hinged can therefore be written as follows:

$$M'_{BA} = M_{BA} - C_{AB}M_{AB} \quad (2.10)$$

Where M_{AB} and M_{BA} are the fixed-end moments of the same beam with the two ends fixed, and C_{AB} is the carryover factor from A to B. For a prismatic member (for which the carryover factor is $1/2$), eqn.(2.10) becomes:

$$M'_{BA} = M_{BA} - \frac{1}{2}M_{AB} \quad (2.11)$$

DISTRIBUTION FACTOR, DF

If a moment M is applied at a joint where two or more members of a structure meet (Fig.2.4a), the joint undergoes a rotation θ , which is the same for all the members meeting at the joint. Nothing that the bending stiffness S of a member is given by

$S = \frac{4EI}{L}$ (see eqn. 2.2) and that this stiffness is the moment required to cause a unit rotation at a member end, we can write:

$$M = S = \frac{4EI\theta}{L}, \text{ when } \theta = 1 \quad (2.11) \text{ or}$$

$$M = 4EK \quad (2.12)$$

Where $K = \frac{I}{L}$ (see eqn.2.3)

From eqns.(2.11) and (2.12), it is clear that the difference in the fractions of the moment M that will be induced at the joint A ends of members of the frame of Fig.2.4(a) will be due only to the difference in their various I/L values, i.e., their relative stiffness values. These fractions represent the factors by which the applied joint moment needs to be multiplied to obtain the induced moment at the joint A end of various members meeting at the joint. The fraction for each member end is known as the **distribution factor** for the member end.

To formally derive an expression for the distribution factor, let us consider the free-body diagram of joint A as shown in Fig.2.4. (b).

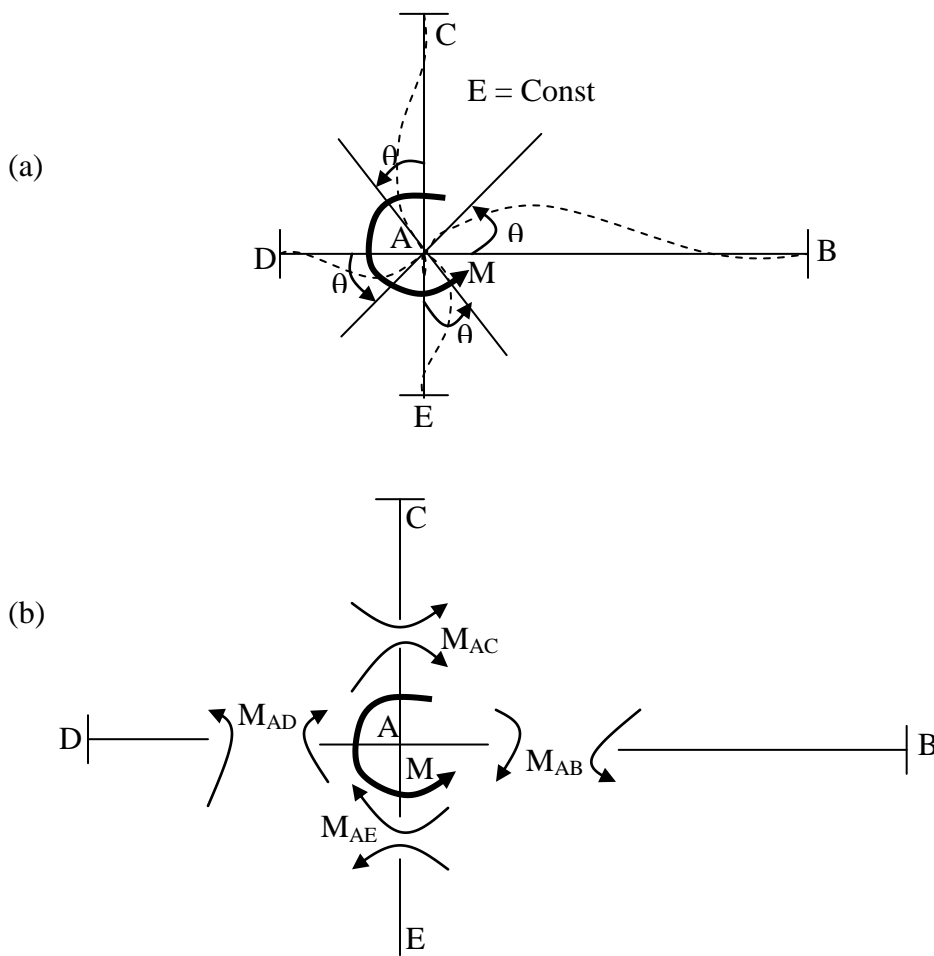


Fig.2.4. (a) Frame with applied moment M at joint A; (b) Free-body diagram of joint A

For the equilibrium of joint A,

$$\sum M_A = 0; \quad M_{AB} + M_{AC} + M_{AD} + M_{AE} - M = 0 \quad (2.13)$$

Taking cognizance of eqns.(2.11) and (2.12), the member end moments can be written in terms of the angle of rotation θ as:

$$\begin{aligned} M_{AB} &= 4EK_{AB}\theta ; \\ M_{AC} &= 4EK_{AC}\theta ; \\ M_{AD} &= 4EK_{AD}\theta ; \\ M_{AE} &= 4EK_{AE}\theta . \end{aligned} \quad (2.14)$$

Substituting these expressions into eqn.(2.13), we obtain:

$$\begin{aligned} M &= 4E\theta(K_{AB} + K_{AC} + K_{AD} + K_{AE}) \text{ or} \\ M &= 4E\theta \sum K \end{aligned} \quad (2.15) \text{ or}$$

$$\theta = \frac{M}{4E \sum K} \quad (2.16)$$

Where $\sum K$ = sum of the relative stiffness of all the members meeting at joint A.

Substituting the expression for θ from eqn. (2.16) into the first of equations (2.14), we obtain:

$$\begin{aligned} M_{AB} &= 4EK_{AB} \frac{M}{4E \sum K} \text{ or} \\ M_{AB} &= \frac{K_{AB}}{\sum K} M \end{aligned} \quad (2.17)$$

The ratio $\frac{K_{AB}}{\sum K}$ represents the fraction by which the applied moment M is to be multiplied in order to obtain the moment resisted by member AB. This ratio, as defined earlier, is the distribution factor.

In general therefore, the distribution factor (DF) for any member AB is given by the formula:

$$DF_{AB} = \frac{K_{AB}}{\sum K} \quad (2.18)$$

Where:

K_{AB} is the relative stiffness of the member. Therefore the distribution factor (DF) for an end of a member that is rigidly connected to the adjacent joint is equal to the ratio of the relative bending stiffness of the member to the sum of the relative bending stiffness of all the members connected to that joint.

We now illustrate the basic procedure involved in the moment distribution method using the following example.

Example 2.1

Determine the member end moments of the continuous beam ABC, fixed at ends A and C and continuous over support B as shown in Fig.2.5. (a).

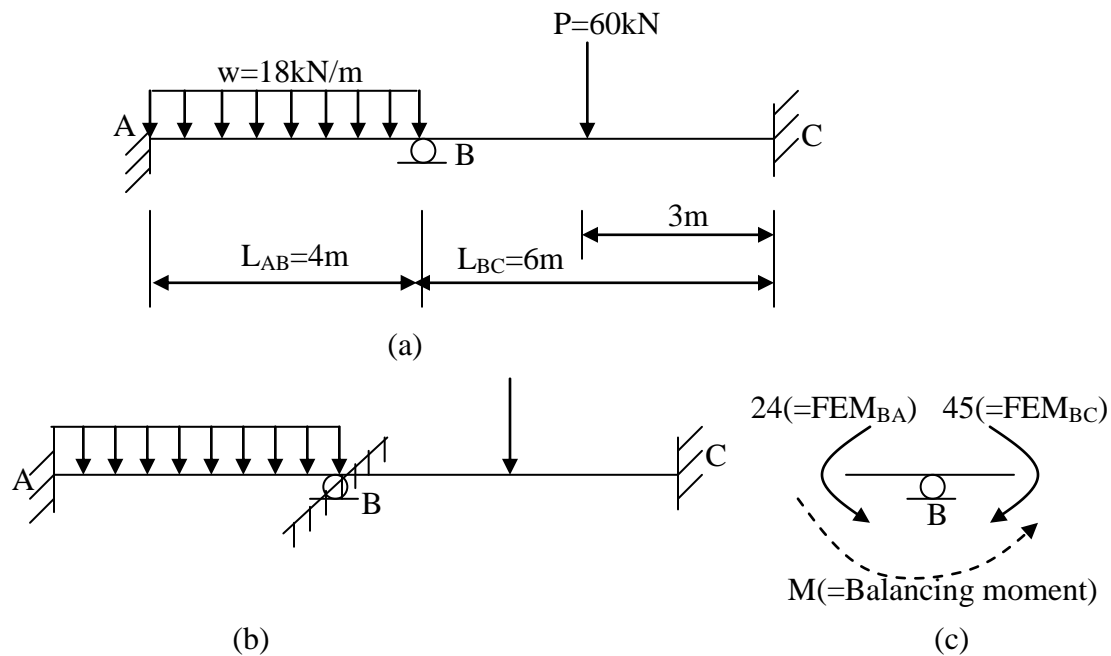


Fig.2.5. (a) Given beam and loading; (b) Joint B restrained against rotation; (c) Balancing of joint B

SOLUTION

The beam is fixed at support points A and C but is free to rotate at point B. Therefore we start by restraining joint B against rotation, as shown in Fig.2.5(b). Next, we compute the fixed-end moments due to the external loading (refer to Table 1.2). Taking cognizance of the adopted sign convention, we have:

$$FEM_{AB} = \frac{wL_{AB}^2}{12} = \frac{18 \times 4^2}{12} = 24 \text{ kNm}$$

$$FEM_{BA} = -24 \text{ kNm}$$

$$FEM_{BC} = \frac{PL_{BC}}{8} = \frac{60 \times 6}{8} = 45 \text{ kNm}$$

$$FEM_{CB} = -45 \text{ kNm}$$

The free-body diagram of joint B shown in Fig.2.5(c) shows how the above fixed-end moments act at the joint. The restraining moment required to prevent the rotation of joint B is equal to the algebraic sum of the fixed-end moments of all member ends meeting at joint B. In this example that sum is equal to 21kNm ($=45-24$). Knowing that in reality the joint is not restrained against rotation, we now release it to rotate by removing the restraining moment (also known as out-of-balance moment) of 21kNm. The effect achieved by removing this moment and allowing the joint to rotate is the same effect achieved by keeping joint B restrained and applying to it a moment equal but opposite to the 21kNm out-of-balance moment. In other words, we apply a moment equal to -21kNm. This is known as the **balancing moment**. From Fig.2.5(c) we can also obtain the required balancing moment through equilibrium consideration of joint B. Thus, noting

that according to our sign convention, clockwise joint moments are positive (since counterclockwise member end moments are positive), the following joint equilibrium equation can be written for joint B:

$$\begin{aligned}\sum M_B &= 0; \\ 45 - 24 - M &= 0, \text{ or} \\ M &= -21 \text{ kNm}\end{aligned}$$

This balancing moment causes the ends of the members at joint B to rotate through the same angle, thereby inducing end moments M_{BA} and M_{BC} (also known as *distributed moments*) at joint B ends of the members. The magnitude of each of these distributed moments is proportional to the relative stiffness of each of the members meeting at the joint.

The distribution factors (DF) are obtained from relative stiffnesses of members in accordance with eqn. (2.18), as follows:

$$\begin{aligned}DF_{BA} &= \frac{I/4}{I/4 + I/6} = 0.6 \text{ and} \\ DF_{BC} &= \frac{I/6}{I/4 + I/6} = 0.4\end{aligned}$$

From the above values of the distribution factors it can be seen that their sum is equal to 1. In fact, the sum of the distribution factors at every joint is always equal to 1. This condition may be used to check the correctness of the computation of the distribution factors at a joint.

The moment developed at the joint B end of each member is obtained by using eqn (2.17). Thus:

$$M_{BA} = 0.6(-21) = -12.6 \text{ kNm}$$

$$M_{BC} = 0.4(-21) = -8.4 \text{ kNm}$$

As explained earlier, the rotation of the member ends at joint B induces carryover moments at their far ends. The carryover moments can be computed using eqn (2.8), or more generally, eqn (2.9). Thus:

$$M_{AB} = \frac{1}{2}(-12.6) = -6.3 \text{ kNm}$$

$$M_{CB} = \frac{1}{2}(-8.4) = -4.2 \text{ kNm}$$

Since no moment is carried over to joint B member ends from their far ends (because the far ends are fixed and therefore have zero distribution factors, zero distributed moments and consequently zero moments to carryover), the joint remains balanced and the beam is now in equilibrium under the given external loading. Therefore the moment distribution process is ended here. In structures for which rotation can occur at more than one joint (for example, in beams that are continuous over more than one support), further cycles of distribution and carryover will need to be carried out.

The final end moments are obtained by algebraically adding the end moments in the restrained condition (FEM) to the moments caused by the rotation of joint B. Thus:

$$M_{AB} = 24 - 6.3 = 17.7 \text{ kNm}$$

$$M_{BA} = -24 - 12.6 = -36.6 \text{ kNm}$$

$$M_{BC} = 45 - 8.4 = 36.6 \text{ kNm}$$

$$M_{CB} = -45 - 4.2 = -49.2 \text{ kNm}$$

The results of analysis using moment distribution method are usually presented in a tabular form. Such a table is shown in Fig.2.6 for Example2.1.

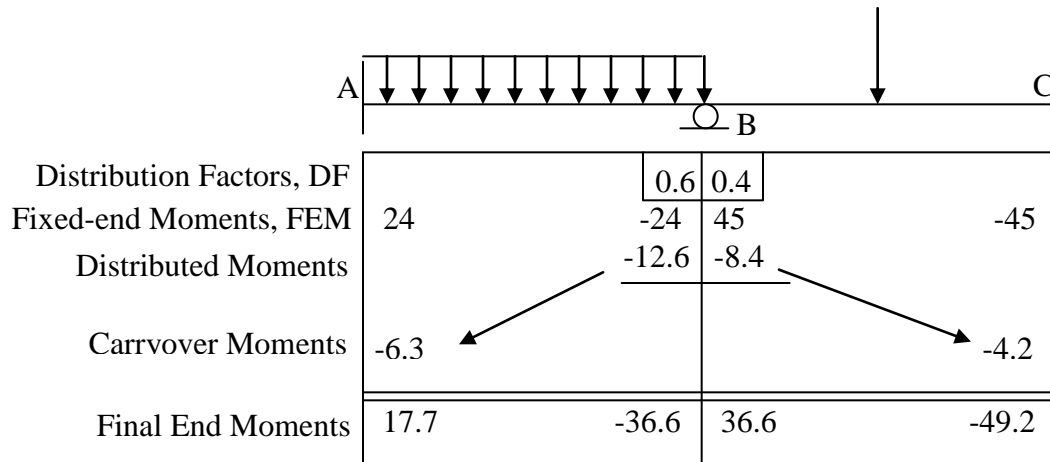


Fig.2.6 Given beam and Moment distribution table

Note that the distribution factors are recorded before the fixed-end moments and so it is convenient to calculate them first. In this example however the fixed-end moments were obtained first to illustrate the method since logically speaking we need to obtain the moments before distributing them. Also note that in the above table, to obtain the distributed moments, the fixed-end moments were algebraically added together to obtain +21kNm. The negative of this moment (-21kNm) was then shared among the joint B ends of the two members by multiplying the moment with the distribution factors. After distributing the moment, a line is drawn below the distributed values to indicate that the joint is balanced. Half of the distributed moments are carried over to the member far ends as carryover moments, as shown in fig.2.6. At a fixed joint, balancing is not necessary. This is because the member ends at fixed supports are imagined to be attached to a body of infinite rigidity, for which reason the distribution factor for a fixed end is zero.

Since the member end moments have been determined, if required, from the geometry of the simple beam bending moment diagram and that of the reactive moment diagram, for each span, the span moments can be calculated. Alternatively, equilibrium consideration can be used to determine the support reactions, span moments, member end shears, and member axial forces.

Let us now consider the case of a continuous beam for which rotation is possible at two joints.

Example 2.2

Determine the member end moments of the beam shown in Fig.2.7.

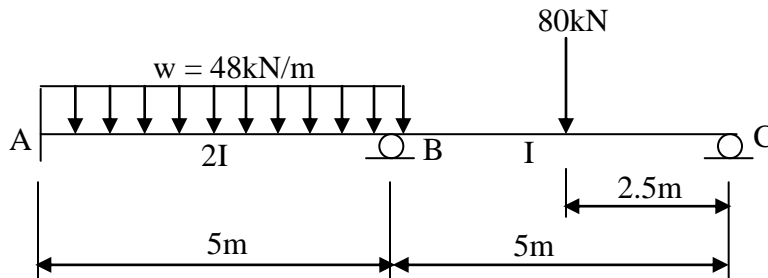


Fig.2.7 Given beam and loading.

SOLUTION

First, in the imagination we fix (or restrain) joints B and C which are free to rotate. The fixed-end moments are then computed as follows:

$$FEM_{AB} = \frac{48 \times 5^2}{12} = 100 \text{ kNm}$$

$$FEM_{BA} = -100 \text{ kNm}$$

$$FEM_{BC} = \frac{80 \times 5}{8} = 50 \text{ kNm}$$

$$FEM_{CB} = -50 \text{ kNm}$$

We now begin to release the artificially fixed joints, one after the other. Let us start with joint C (we can start with any of the joints). We now observe that joint C is actually a simple support and the final moment must be equal to zero since it cannot support moment. The unbalanced moment at the joint is balanced as shown in Fig.2.8. The magnitude of the balancing moment is $+25 \text{ kNm}$. Half of this balancing moment is carried over to joint B while joint C is once again temporarily fixed in its rotated position. At joint B this carryover moment is algebraically added to the joint's fixed-end moments to obtain an unbalanced moment of -25 kNm ($-100 + 50 + 25 = -25$). A balancing moment of opposite sign ($+25 \text{ kNm}$) is now distributed to the two members meeting at joint B by multiplying the moment with the appropriate distribution factors for the members. Joint B is now temporarily balanced and appropriate carryover moments (half of the distributed moments) are carried over to ends A and C. Balancing is not necessary at joint A because it is fixed but joint C once again needs to be released and equilibrium restored by balancing it once again. Half of the balancing moment (equal to -2.09 kNm) is carried over to joint B. Again joint B is released and it rotates due to the carryover moment of -2.09 kNm which now constitutes the new unbalanced moment at the joint. This unbalanced moment has however become much smaller at -2.09 kNm than the original unbalanced moment of 25 kNm . Joint C is once again released and a balancing moment distributed. The distributed moments are once again carried over to member far ends. This cycle of balancing, distributing and carrying over is repeated until the desired accuracy is achieved. The results are recorded in Fig.2.8.

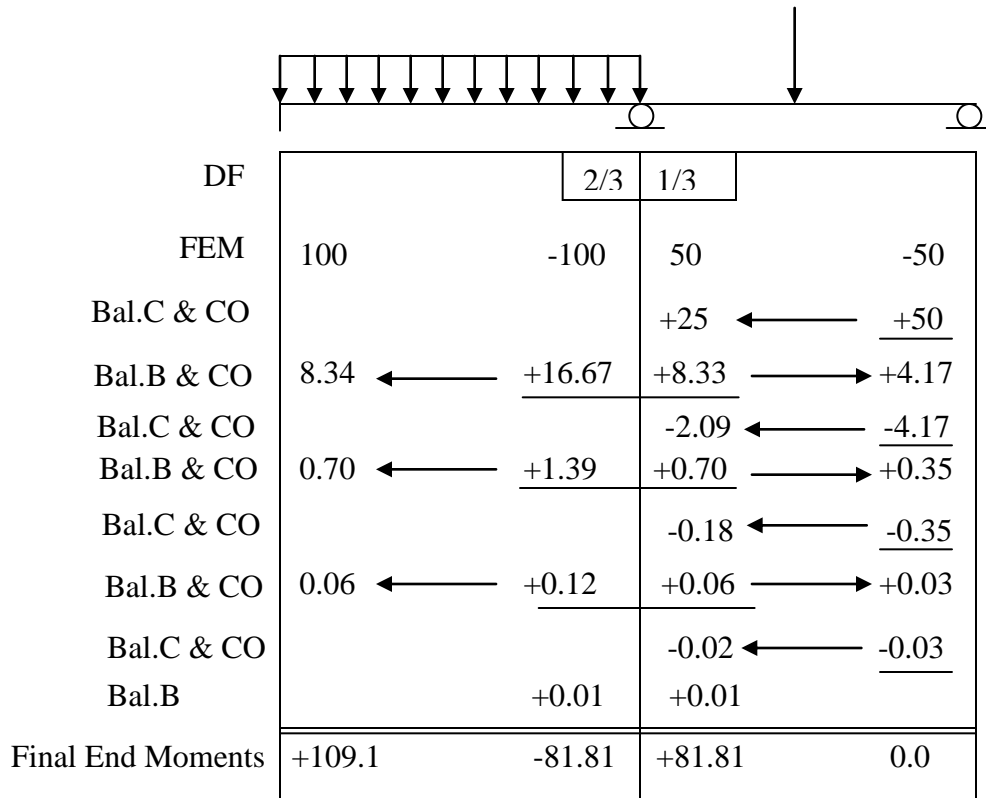


Fig.2.8 Given beam and Moment distribution table

In the Example 2 just considered the number of cycles was rather much because joint C is hinged and continuously sends some carryover moment to joint B. In such a structure where one end is simply supported, convergence can be achieved faster by using the modified (or reduced) stiffness factor, in this case for span BC.

Let us now re-work Example 2.2 by using the modified stiffness as given by eqn (2.6), i.e., $K = \frac{3}{4} \times \frac{I}{L}$, for span BC. The procedure is shown in Fig.2.9. Starting with joint C

once again, we release the joint, balance the -50kNm unbalanced moment with a moment equal to 50kNm, and carry over half of it to joint B. At this point we no longer fix joint C but allow it to freely rotate so that it develops no moment. At joint B, we release it, balance it and distribute moments to the members meeting at the joint. Appropriate moment is then carried over to joint A but none is carried over to joint C because the modified stiffness is used for span BC. Fig.2.9 shows that the final moments are sensibly the same as obtained in Fig.2.8 but considerable computation effort has been saved.

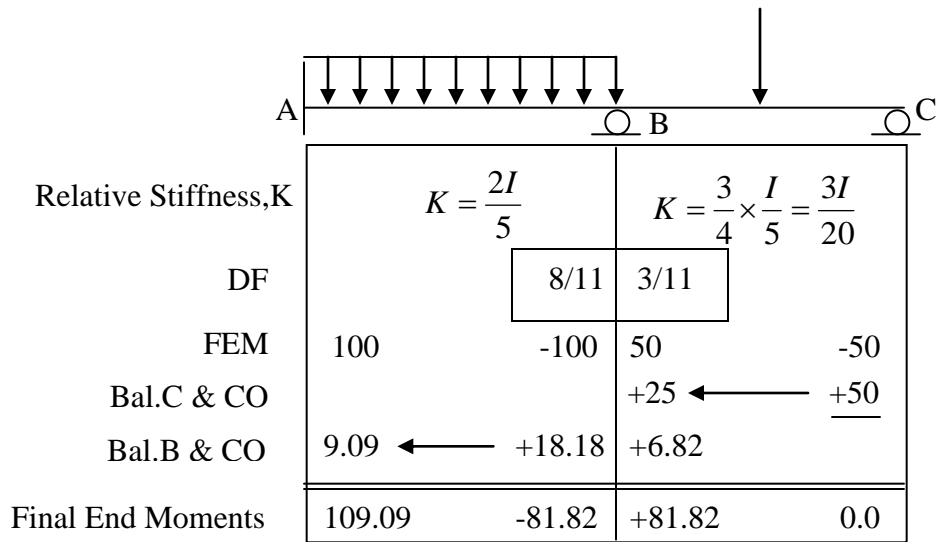


Fig.2.9 Given beam and table for alternative Moment distribution procedure using Modified Stiffness for span BC.

STRUCTURES WITH OVERHANGS

We now consider the analysis of structures with overhanging members. The important difference between the procedure used for such a structure and the procedure used for a structure with a simply supported end is that in the former case the overhang is replaced with an equivalent moment applied at the adjacent support point. The procedure then becomes similar to that used for a structure with a simply supported end, but with a known moment at that end. We now consider an example to show the procedure.

Example 2.3

Determine the support moments for the beam shown in Fig.2.10.

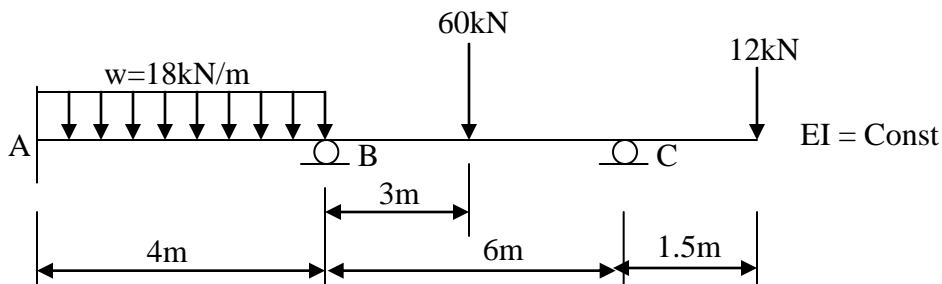


Fig.2.10 Given beam and loading

SOLUTION

First the overhanging span is replaced with its equivalent moment equal to 18kNm (12 x 1.5 = 18). This moment is positive in accordance with our adopted sign convention because the 12kN load tends to rotate joint C in a clockwise direction. The moment, i.e., is recorded on the same row as the fixed-end moments Fig.2.11).

The fixed-end moments are the same values used for the beam of Example 2.1. The modified stiffness factor of $\frac{3}{4}$ is used for span BC.

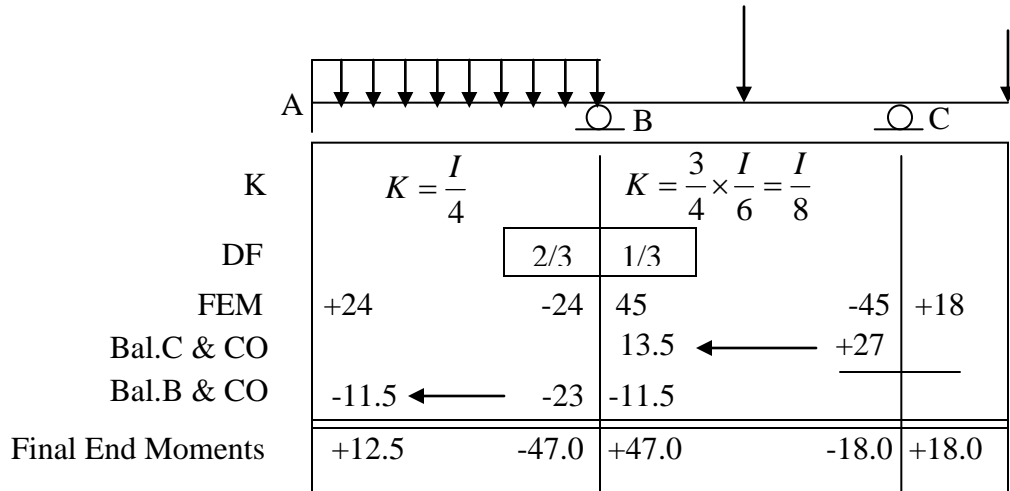


Fig.2.11. Moment distribution table of the given beam

STRUCTURES SUBJECT TO SUPPORT SETTLEMENT

We now consider the procedure involved in the analysis of a structure by moment distribution method when there is settlement of support. We illustrate this by means of the following example.

Example 2.4

Determine the support moments for the beam of Fig.2.12 if under the given loading support B sinks by 5mm. Take $E=210 \times 10^6 \text{ kN/m}^2$ and $I=360 \times 10^6 \text{ m}^4$.

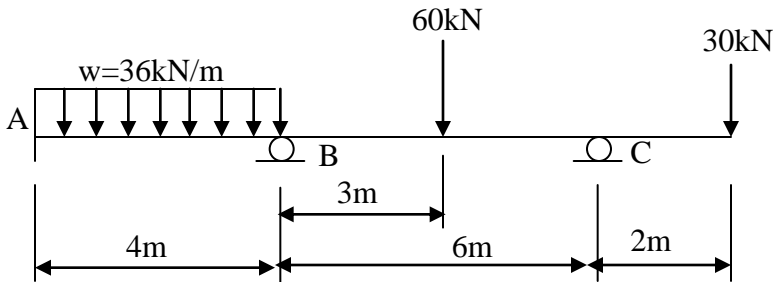


Fig.2.12 Given beam and loading.

SOLUTION

The relative stiffnesses and the distribution factors are shown respectively on the first and second rows of Fig.2.13.

The fixed-end moments are computed using table 1.2 as follows:

$$FEM_{AB} = \frac{36 \times 4^2}{12} = 48 \text{ kNm}$$

$$FEM_{BA} = -48 \text{ kNm}$$

$$FEM_{BC} = \frac{60 \times 6}{8} = 45 \text{ kNm}$$

$$FEM_{CB} = -45 \text{ kNm}$$

The overhang is replaced by a moment equal to 60kNm applied at joint C.

The fixed-end moments developed in the members due to the sinking of support B can be obtained by reference to Table 1.1 or by directly using the slope-deflection equation (see eqn.(1.7)), noting that the chord rotation $\psi = \Delta/L$, where Δ = relative translation between the two ends of the member in the direction perpendicular to the undeformed axis of the member and here it is equal to the sinking of the support. Note also that in directly applying eqn (1.7) the chord rotation ψ is positive if counterclockwise, in accordance with our adopted sign convention. Thus:

$$FEM_{AB} = \frac{6EI\Delta}{L^2} = \frac{6 \times 210 \times 10^6 \times 360 \times 10^{-6}}{4^2} \times \frac{5}{1000} = 141.75 \text{ kNm}$$

$$FEM_{BA} = 141.75 \text{ kNm}$$

$$FEM_{BC} = FEM_{CB} = \frac{-6EI\Delta}{L^2} = -\frac{6 \times 210 \times 10^6 \times 360 \times 10^{-6}}{6^2} \times \frac{5}{1000} = -63 \text{ kNm}$$

Note that here FEM_{AB} and FEM_{BA} are the same sign because with respect to span AB, as support B sinks the chord joining support A to the depressed position of support B is clockwise (negative) and when inserted into eqn (1.7) it multiplies with the negative sign already existing in the equation, to become positive. Similar reasoning also applies to FEM_{BC} and FEM_{CB} , both of which are negative.

Next, the fixed-end moments due to the sinking of support are algebraically added to the fixed-end moments due to external loading, after which the procedure becomes as for the examples considered previously. For clarity however, the two sets of FEM are recorded in separate rows while their sums are recorded in the third row, as shown in Fig.2.13, which also shows the final end (or support) moments.

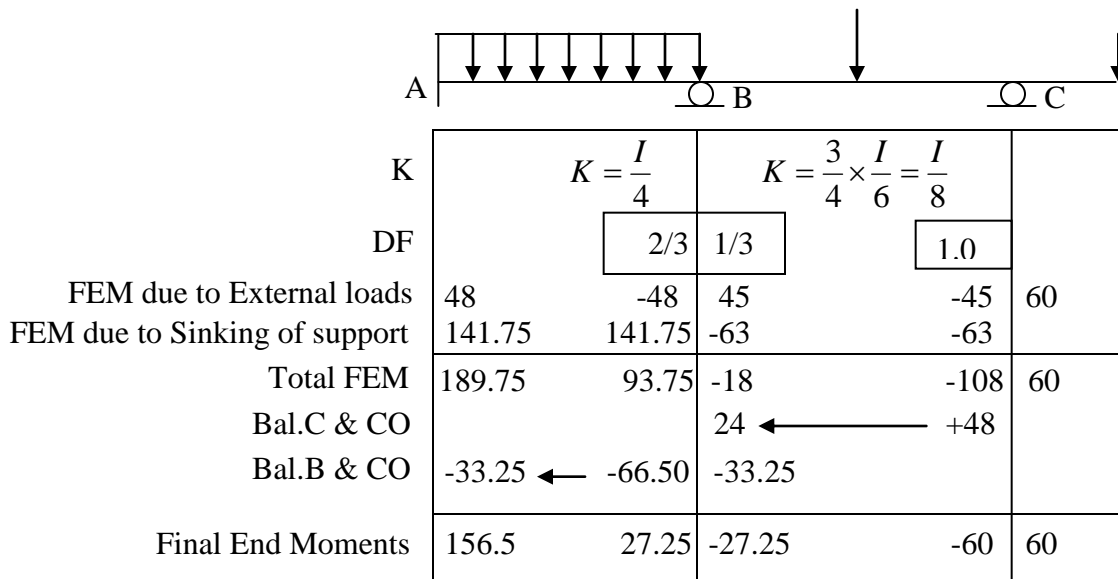


Fig.2.13 Given beam and Moment distribution table

More Practical Approach to the Moment Distribution Procedure

Up till now we have obtained member end moments by successively releasing and balancing one joint of the structure at a time. This approach was convenient to clearly explain the process involved. Although this approach can always be used, it is usually more convenient to use an alternative approach in which all the joints which were initially free to rotate are released and balanced simultaneously in the same step. all the carryover moments that are induced at the far ends of the members are then simultaneously determined in the following step. As in the earlier approach, the process of balancing the joints and carrying over moments is then repeated until the unbalanced moments at the joints become negligibly small. This alternative procedure is especially convenient for structures with many spans for which many joints are involved.

To illustrate the procedure involved, we consider an example. In this example we shall also obtain the support reactions and the span moments, as well as draw the bending moment diagram.

Example 2.5

For the continuous beam shown in Fig.2.14 obtain the member end moments. Calculate also the span moments and draw the final bending moment diagram (BMD). Determine also the final support reactions. In each cycle of moment distribution, balance all joints in one step and carry over moments from member ends in one step.

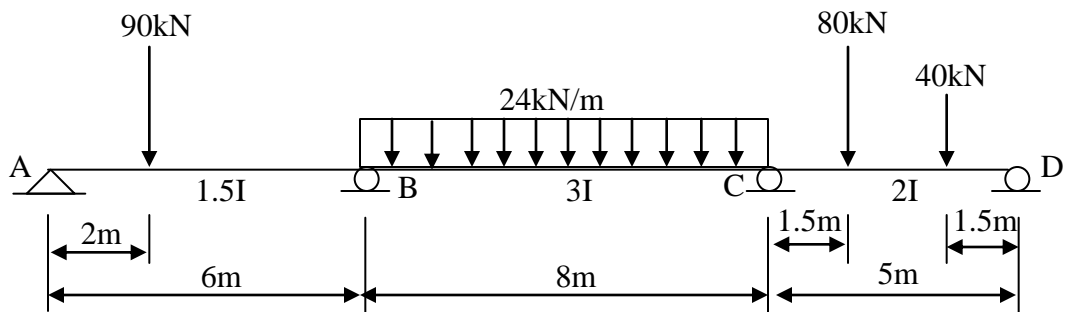


Fig.2.14 Given beam and loading.

SOLUTION

Distribution Factors, DF

At joint B

$$K_{BA} = \frac{3}{4} \times \frac{1.5I}{6} = \frac{3I}{16}; \quad K_{BC} = \frac{3I}{8}; \quad \sum K = K_{BA} + K_{BC} = \frac{9I}{16}$$

$$DF_{BA} = \frac{K_{BA}}{\sum K} = \frac{1}{3}; \quad DF_{BC} = \frac{K_{BC}}{\sum K} = \frac{2}{3}.$$

At joint C

$$K_{CD} = \frac{3}{4} \times \frac{2I}{5} = \frac{3I}{10}; \quad K_{CB} = \frac{3I}{8}; \quad \sum K = K_{CB} + K_{CD} = \frac{27I}{40}$$

$$DF_{CD} = \frac{K_{CD}}{\sum K} = \frac{4}{9}; \quad DF_{CB} = \frac{K_{CB}}{\sum K} = \frac{5}{9}.$$

Fixed-end Moments, FEM

$$FEM_{AB} = \frac{90 \times 4^2 \times 2}{6^2} = 80kNm; \quad FEM_{BA} = -\frac{90 \times 4 \times 2^2}{6^2} = -40kNm;$$

$$FEM_{BC} = \frac{24 \times 8^2}{12} = 128kNm; \quad FEM_{CB} = -128kNm.$$

$$FEM_{CD} = \frac{80 \times 3.5^2 \times 1.5}{5^2} + \frac{40 \times 1.5^2 \times 3.5}{5^2} = 71.4kNm;$$

$$FEM_{DC} = -\left(\frac{40 \times 3.5^2 \times 1.5}{5^2} + \frac{80 \times 1.5^2 \times 3.5}{5^2} \right) = -54.6kNm.$$

The distribution factors, fixed-end moments, the computations involved in the moment distribution process, as well as the final end moments are shown in Fig.2.15 while the simple beam reactions, reactions due to support moments, final support reactions, and the bending moment diagram are shown respectively in Figs.2.16(a), (b), (c) and (d). Note that the bending moment diagram was obtained by superimposing the BMD of simply supported beams on the BMD of the support moments.

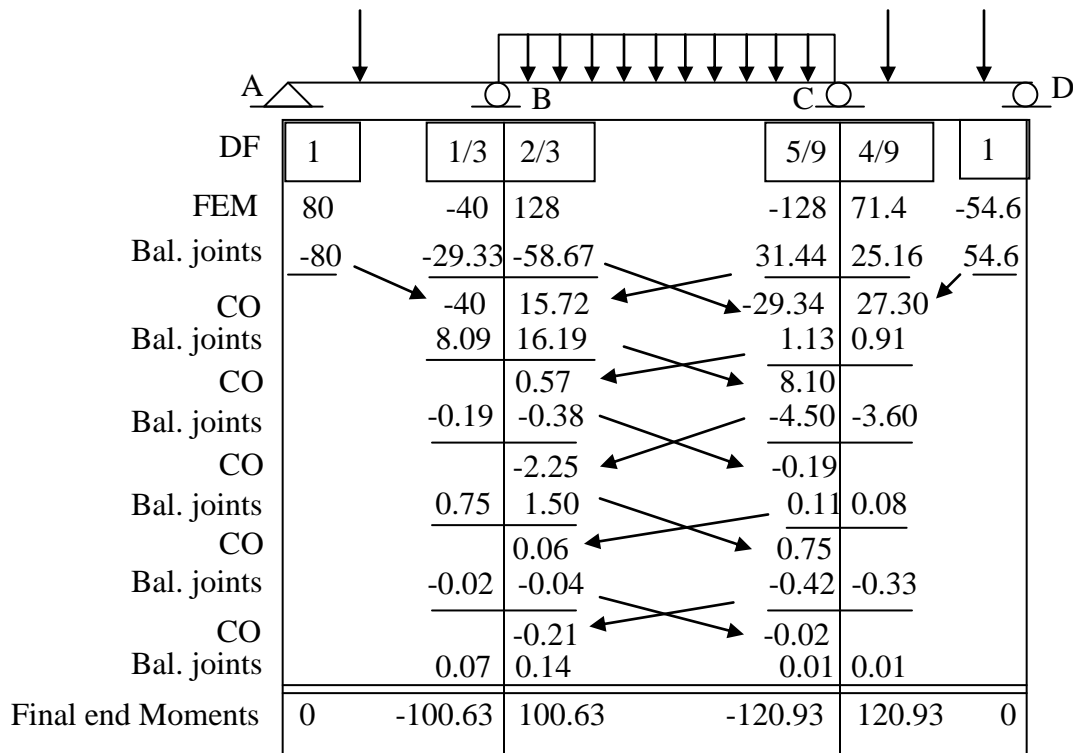


Fig.2.15 Given beam and Moment distribution table.

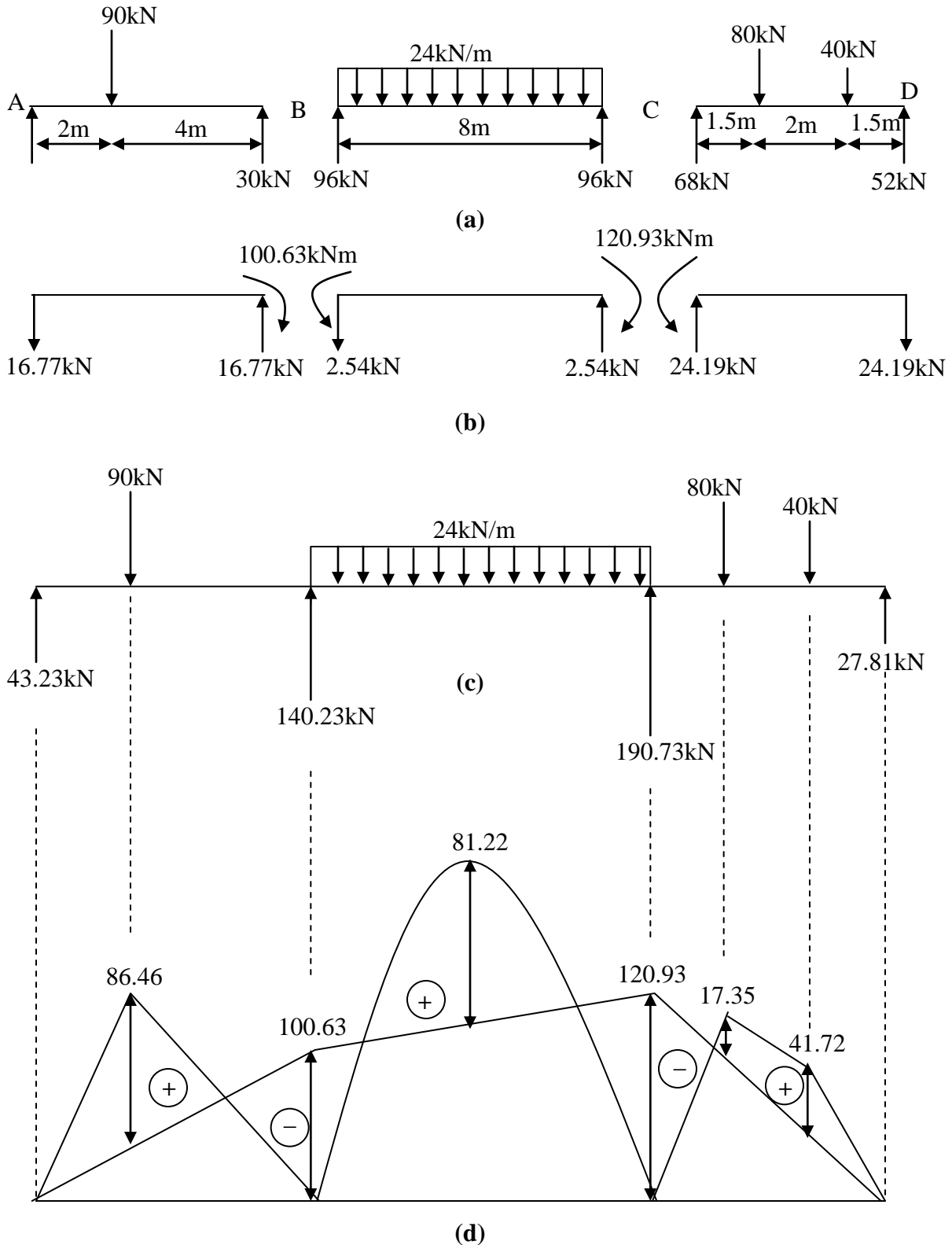


Fig.2.16: (a) Simple beam reactions due to external loads
 (b) Reactions due to support moments
 (c) Final (or total) beam reactions
 (d) Final bending moment diagram (KNm)

2.3. ANALYSIS OF FRAMES WITHOUT LATERAL TRANSLATION OF JOINTS

The procedure for the analysis of frames whose joint translations are prevented is similar to that of continuous beams. The entire process of calculating relative stiffness, distribution factor, fixed-end moment, and that of balancing joint moments and distributing them, as well as carrying over moments to members' far ends, are the same with the procedure for beams. For the purpose of recording computed values in the moment distribution table, in the case of a single bay single storey frame, it is convenient to spread out the frame legs to be horizontal with the girders, in which case the recordings are made exactly the same way as in the case of continuous beams. In frames with joints at which more than two members meet, it may be more convenient to adopt a recording pattern such as that shown in Fig.2.20 for Example 2.7. The procedure involved in the analysis of frames without lateral translation of joints is shown with the aid of the following example.

Example 2.6

Determine the support moments of the frame shown in Fig.2.17.

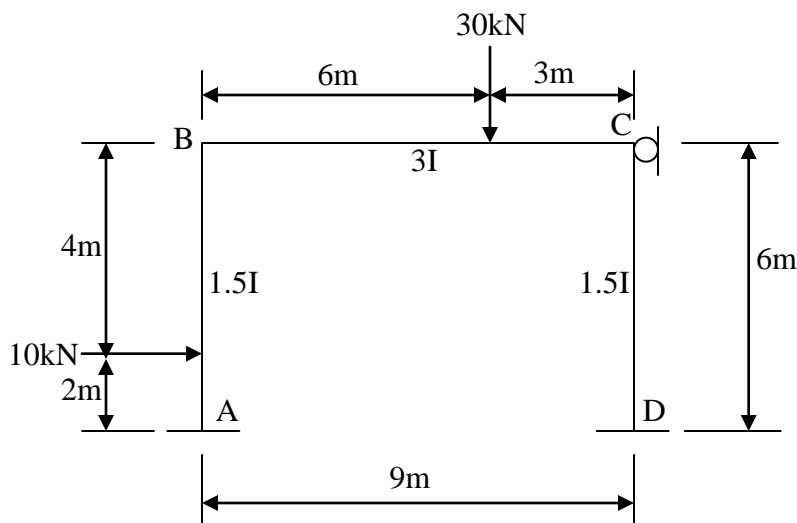


Fig 2.17. Given frame and loading.

SOLUTION

The support at C prevents the frame from undergoing lateral translation. The relative stiffness and distribution factors are calculated and shown in Fig.2.18. The fixed-end moments are calculated as follows:

$$FEM_{AB} = \frac{10 \times 2 \times 4^2}{6^2} = 8.89 \text{ kNm}$$

$$FEM_{BA} = -\frac{10 \times 4 \times 2^2}{6^2} = -4.44 \text{ kNm}$$

$$FEM_{BC} = \frac{30 \times 6 \times 3^2}{9^2} = 20 \text{ kNm}$$

$$FEM_{CB} = -\frac{30 \times 3 \times 6^2}{9^2} = -40 \text{ kNm}$$

$$FEM_{CD} = FEM_{DC} = 0 \text{ Since there is no transverse loading on member CD.}$$

The moment distribution table is shown in Fig.2.18. Observe that in this example the two joints (B and C) were balanced in the same step at each cycle and not one at a time.

K	$K = \frac{1.5I}{6} = \frac{I}{4}$		$K = \frac{3I}{9} = \frac{I}{3}$		$K = \frac{I}{4}$
DF	3/7 4/7		4/7 3/7		
FEM	8.89	-4.44	20	-40	0
Bal. joints		-6.67	-8.89	+22.86	+17.14
CO	-3.34		+11.43	-4.45	+8.57
Bal. joints		-4.9	-6.53	+2.54	+1.9
CO	-2.45		1.27	-3.27	0.95
Bal. joints		-0.54	-0.73	+1.87	+1.40
CO	-0.27		+0.94	-0.37	0.70
Bal. joints		-0.40	-0.54	+0.21	+0.16
CO	-0.20		0.11	-0.27	0.08
Bal. joints		-0.05	-0.06	+0.15	+0.12
Final End Moments	2.63	-17	17	-20.72	20.72

Fig.2.18. Moment distribution table for the frame of Fig.2.17

Let us now consider another example, this time, that of a frame having a joint where three members meet.

Example 2.7

Determine the member-end moments for the frame shown in Fig.2.19

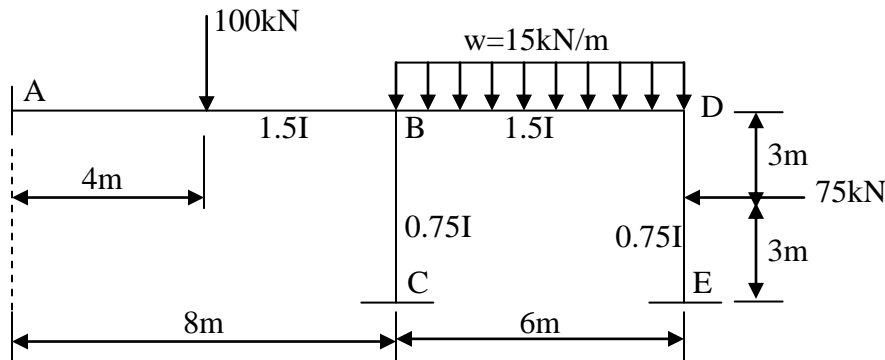


Fig.2.19 Given frame and loading

SOLUTION

Distribution Factors, DF

At Joint B

$$\sum K = K_{BA} = K_{BC} = K_{BD} = \frac{1.5I}{8} + \frac{0.75I}{6} + \frac{0.75I}{6} = 0.5625I$$

$$DF_{BA} = \frac{K_{BA}}{\sum K} = 0.33; \quad DF_{BC} = \frac{K_{BC}}{\sum K} = 0.22; \quad DF_{BD} = \frac{K_{BD}}{\sum K} = 0.45$$

Check: $DF_{BA} + DF_{BC} + DF_{BD} = 0.33 + 0.22 + 0.45 = 1$

At Joint D

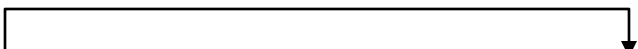
$$\sum K = K_{DB} + K_{DE} = \frac{1.5I}{6} + \frac{0.75I}{6} = 0.375I$$

$$DF_{DB} = \frac{K_{DB}}{\sum K} = 0.67; \quad DF_{DE} = \frac{K_{DE}}{\sum K} = 0.33;$$

Check: $DF_{DB} + DF_{DE} = 0.67 + 0.33 = 1$

The distribution factors are shown on the first row of the moment distribution table (Fig.2.20).

The fixed-end moments are calculated as done previously, and they are shown in Fig.2.19 together with the rest of the calculations, as well as the final end moments.



	AB	BA	BC	BD	DB	DE	ED	CB
DF		0.33	0.22	0.45	0.67	0.33		
FEM	100	-100	0	45	-45	56.25	-56.25	0
Bal. Joints		18.15	12.1	24.75	-7.54	-3.71		
CO	9.08			-3.77	12.38		-1.86	6.05
Bal. Joints		1.24	0.83	1.70	-8.29	-4.09		
CO	0.62			-4.15	0.85		-2.05	0.42
Bal. Joints		1.37	0.91	1.87	-0.57	-0.28		
CO	0.69			-0.29	0.94		-0.14	0.46
Bal. Joints		0.10	0.06	0.13	-0.63	-0.31		
CO	0.05			-0.32	0.07		-0.16	0.03
Bal. Joints		0.11	0.07	0.14	-0.05	-0.02		
Final Moments	110.44	-79.03	13.97	65.06	-47.84	47.84	-60.3	6.96

Fig.2.20. Moment distribution table for the frame of Fig.2.19.

2.4. ANALYSIS OF FRAMES WITH LATERAL TRANSLATION OF JOINTS (OR SWAY FRAMES)

In using the moment distribution method for the analysis of structures, up till now we have considered rigid frames (either continuous beams or frames) for which lateral translation is not allowed to occur. In practice however lateral translation (also known as sway) occurs at some joints in a frame. This sway may be as a result of lateral loading (such as wind load), lack of symmetry of loading, or lack of frame symmetry. For such frames the analysis consists of two parts.

To explain the procedure involved consider the sway frame shown in Fig.2.21(a). The first part of the analysis involves the application of an artificial restraint to prevent lateral translation of the frame (Fig.2.21. (b)). Under this condition the frame is analysed exactly as for frames without lateral translation and the final end moments are obtained for the condition of no-sway frame. However it is known that the actual frame is subject to sway. Therefore in the second part of the analysis, the restraint is removed and the force "R" (which is a support reaction in the direction of the restraining support) preventing sway, is applied to the support (Fig.2.21(c)).

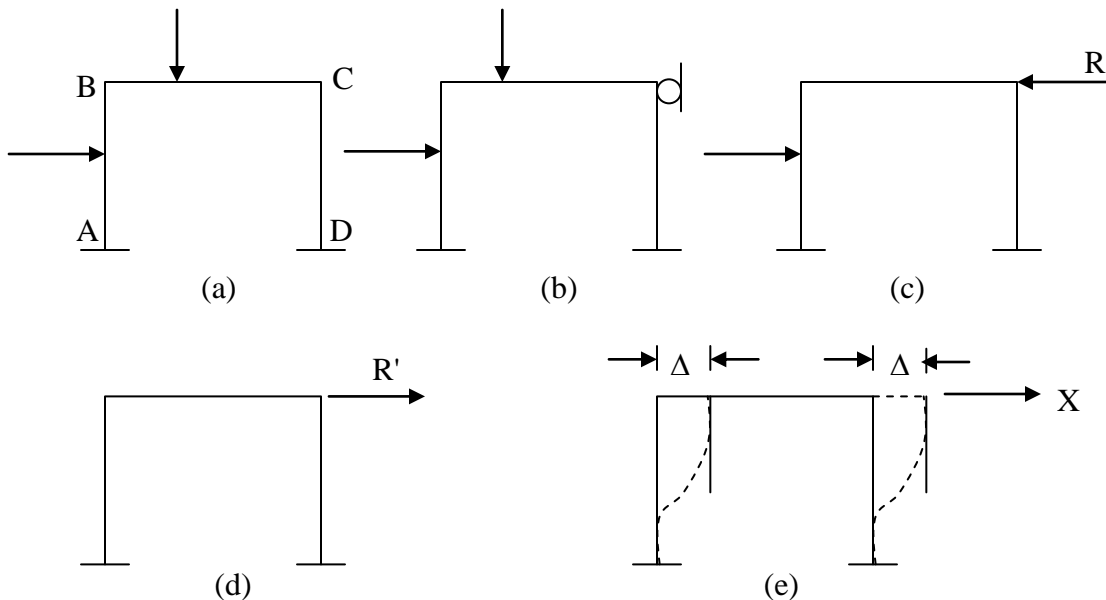


Fig.2.21: (a) Given sway frame; (b) Sway prevented by attaching a restraint at C.
 (c) Restraining support at C removed and replaced with support reaction R;
 (d) Force R' equal to R but opposite in direction is applied at joint C.
 (e) Unknown force X applied at C in the direction of R'.

The value of this reaction force is determined by considering the horizontal equilibrium of the free bodies of the frame legs under the action of all lateral forces, i.e., support shears and lateral loads. Since the actual frame is not restrained but subject to sway, a force equal but opposite to the restraining force is applied at the position and in the direction of R. This force is shown in Fig.2.21. (d) as R'. Next, the member end moments resulting from this condition are obtained. The algebraic sum of the frame member end moments obtained without joint sway and the end moments due to the lateral force R'

(which is responsible for the sway) gives the actual end moments for the given sway frame.

However, the moment distribution method cannot be used to obtain fixed-end moments due to such forces as R' since it is not a transverse force. For this reason an indirect approach is used as follows. An unknown force X is applied to the frame at the joint and in the direction of R' (Fig.2.21. (e)). The fixed-end moments in the columns can be obtained by restraining joints B and C from rotation and permitting only translation (Fig.2.21. (e)). If the axial deformation in the beam is neglected, the translation Δ is the same at joints B and C. The fixed-end moments under that condition can be obtained in terms of $EI\Delta$. Some arbitrary values can now be assigned to $EI\Delta$ and the fixed-end moments are evaluated for those arbitrary values. Moment distribution is carried out for these fixed-end moments to obtain the end moments due to X . Equilibrium consideration of the free bodies of the frame legs under the lateral force X and the support shears, developed due to the end moments resulting from the moment distribution of the fixed-end moments, is used to compute the force X . Since we now have the end moments due to the load X , the end moments due to the load R' , i.e., $M_{(R')}$ can be computed by proportion as follows:

$$M_{(R')} = \frac{R'}{X} \times \text{end moments due to } X \quad (2.19)$$

Algebraically summation of the end moments developed by the frame with sway prevented and the end moments due to sway caused by R' then gives the final or true end moments for the given sway frame. The procedure involved in the analysis of sway frames is further illustrated by the following example.

Example 2.8

Determine the support moments for the sway frame shown in Fig.2.22. (a).

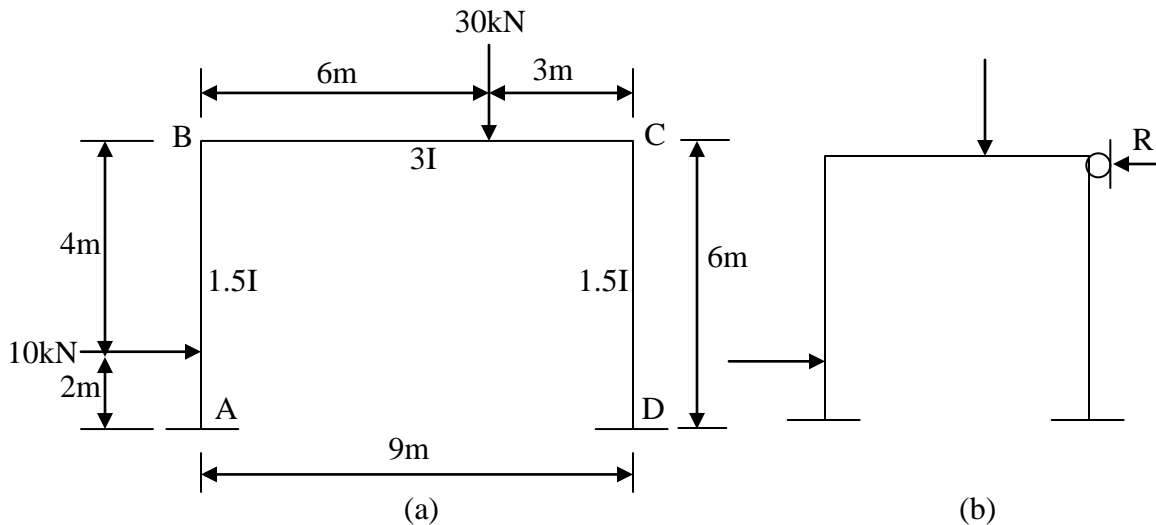


Fig.2.22: (a) Given frame and loading; (b) Restraint at joint C.

SOLUTION

Observe that this frame was analysed in Example 2.6 but as a frame without lateral translation of joints because a restraint was provided at C. In its present form, it represents the case of a symmetrical frame with non-symmetrical loading, hence there will be translation of joints. As explained earlier, the analysis consists of two parts. We now consider each of the parts in turn.

Part I: Joint translation Prevented

The analysis with joint translation prevented has already been carried out in Example 2.6 and its moment distribution table of Fig.2.18 will be used here. Fig.2.22(b) shows the joint restraint at C.

We now determine the restraining reaction force R from the equilibrium consideration of the columns under horizontal forces. To do that we first obtain the shears at A and D having applied the external horizontal loading as well as the end moments obtained from the analysis of the frame with sway prevented (Fig.2.23).

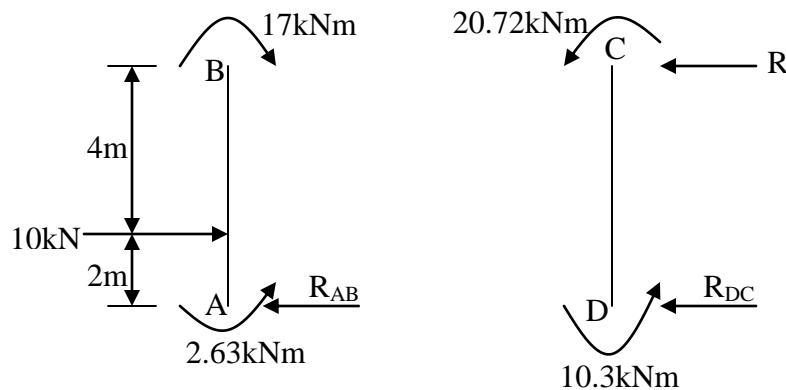


Fig.2.23 Free-body diagrams of the columns.

Considering the left column, we take moment about joint B to obtain R_{AB} .

Thus: $\sum M_B = 0;$

$$6R_{AB} - 2.63 - 10 \times 4 + 17 = 0$$

or $R_{AB} = 4.27kN.$

Similarly, for the right column we take moment about joint C.

Thus: $\sum M_C = 0;$

$$6R_{DC} - 10.3 - 20.72 = 0$$

or $R_{DC} = 5.17kN.$

To determine the restraining force R, we consider the horizontal equilibrium of the entire frame. Thus: $\sum F_x = 0;$

$$R + R_{DC} + R_{AB} - 10 = 0$$

or $R = 0.56kN$

Part II: Joint translation Permitted

We now remove the artificial restraint at C by applying a force equal in magnitude but opposite in direction to force R. So we apply the said force denoted by R' at C (Fig.2.24a). As explained earlier, the fixed-end moments due to force R' have to be obtained indirectly, for which purpose we apply an unknown force X at the position and in the direction of R' (Fig.2.24b).

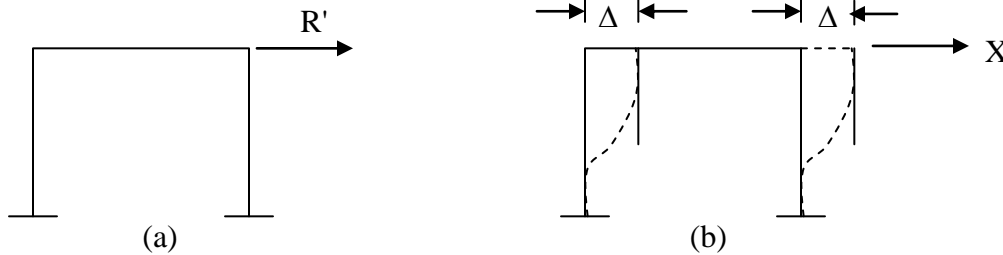


Fig.2.24: (a) Application of R' at joint C;
 (b) Translation of joints B and C permitted but with rotations restrained.

We now obtain the fixed-end moments under the condition that joints B and C only translate but do not rotate. Thus:

$$FEM_{AB} = FEM_{BA} = \frac{6EI\Delta}{L^2} = \frac{6(1.5)EI\Delta}{6^2} = 0.25EI\Delta$$

Also, $FEM_{DC} = FEM_{CD} = 0.25EI\Delta$

Letting $EI\Delta = 60$, $FEM_{AB} = FEM_{BA} = FEM_{DC} = FEM_{CD} = 0.25 \times 60 = 15kNm$

Note. Although the value of $EI\Delta$ is fixed arbitrarily, it is more convenient to fix the value such that the FEM obtained is within the range of values we are working with in the given problem.

Having obtained the FEM, we now carry out moment distribution using the same values of distribution factors as for the part I aspect of the analysis. The moment distribution for part II is shown in Fig.2.25.

Consideration of the horizontal equilibrium of the entire frame gives:

$$\sum F_x = 0; \quad X - R_{AB} - R_{DC} = 0$$

or $X = 7.5kN$

Recalling that the true value of the restraining force at C is 0.56kN we now obtain the end moments which it causes by multiplying the end moments caused by X with the factor $\frac{0.56}{7.5} = 0.0747$. The end moments obtained are then added to the end moments of

Fig.2.18 to get the actual total end moments of the given sway frame. The final end moments are therefore calculated as follows:

$$M_{AB} = 2.63 + 12.51(0.0747) = 3.56kNm$$

$$M_{BA} = -17 + 9.99(0.0747) = -16.25kNm$$

$$M_{BC} = 17 - 9.99(0.0747) = 16.25kNm$$

$$M_{CB} = -20.72 - 9.99(0.0747) = -21.47kNm$$

$$M_{CD} = 20.72 + 9.99(0.0747) = 21.47kNm$$

$$M_{DC} = 10.3 + 12.51(0.0747) = 11.23kNm$$

Checks.

As a check, the equilibrium of the entire frame under the action of the active (external) and the reactive (support shears) can be considered as follows:

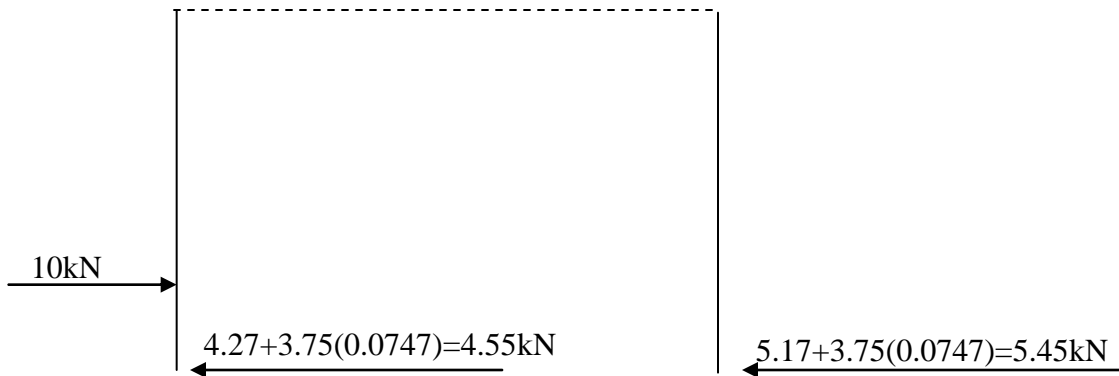


Fig.2.27 Check for frame horizontal equilibrium.

$$\sum F_x = 0; \quad 10 - 4.55 - 5.45 = 0 \text{ (Satisfied).}$$