

CHAPTER 4

ANALYSIS OF INDETERMINATE STRUCTURES: KANI'S ROTATION CONTRIBUTION METHOD

INTRODUCTION

As we saw in Chapter 2, Hardy Cross moment distribution method consists essentially in solving the simultaneous equations in the slope deflection method by successive approximations. Since the introduction of moment distribution method, various special techniques have been developed. Some of these techniques are for the reduction of the number of iterations involved in the moment distribution process while others help to harmonize the analysis procedures for sway and for non-sway frames. One such technique was the one developed by Gasper Kani, a German, in mid 19th century. This technique, known as **rotation contribution method**, is a properly organized iterative procedure for relatively rapid solution of the slope deflection equations. It is especially convenient for use in the analysis of multi-storey and/or multi-bay frames, and for structures subject to sway (or lateral translation of joints).

Sign Convention

In this method, we shall follow the same sign convention adopted for both the slope deflection method of Chapter 1 and the moment distribution method of Chapter 2. Thus:

- Moments are positive when counterclockwise;
- Rotations are positive when counterclockwise.

DEVELOPMENT OF THE METHOD

STRUCTURES WITHOUT TRANSLATION OF JOINTS

Consider a member AB in a rigid-jointed structure of Fig.3.1(a).

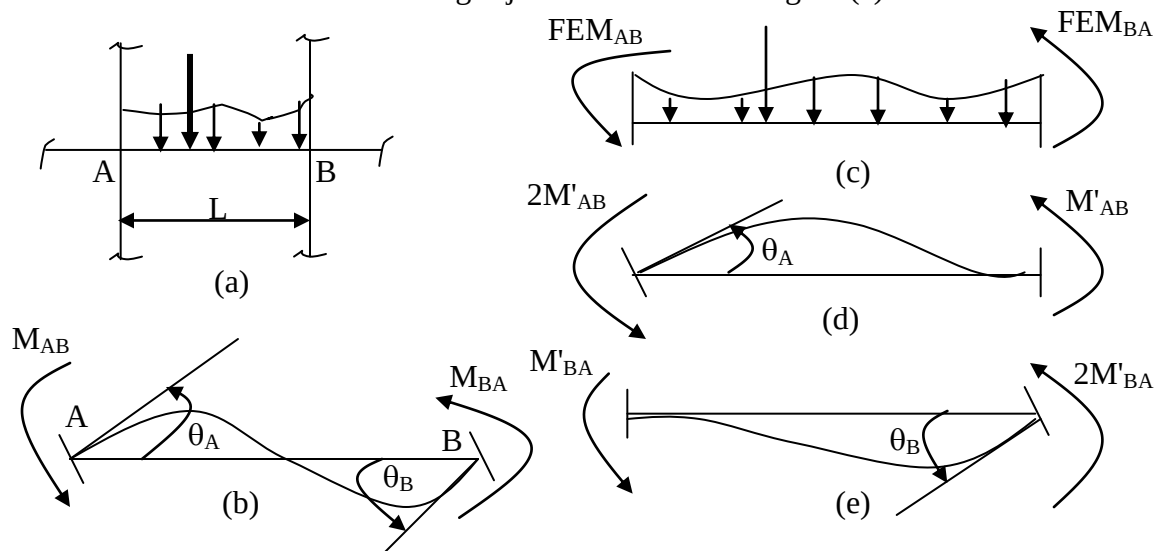


Fig.3.: (a) Member AB in a rigid-jointed structure;
 (b) Elastic curve as ends undergo only rotation; (c) Fixed-end moments (FEM);
 (d) Only end A rotates; (e) only end B rotates.

If we assume that the joints can only rotate without undergoing any translation, then the total end moments for member AB will be the algebraic sum of the following component end moments:

1. Fixed-end moments FEM_{AB} and FEM_{BA} (corresponding to the actual loading on span AB), respectively for member ends A and B, with the member assumed fully fixed (Fig.3.1(c));
2. End moments $2M'_{AB}$ and M'_{AB} respectively for ends A and B, with end A assumed rotated by θ_A while end B remained fixed (Fig.3.1(d)). M'_{AB} is known as rotation moment (or rotation contribution) at end A;
3. End moments M'_{BA} and $2M'_{BA}$ respectively for ends A and B with end B assumed rotated by θ_B while end A remained fixed (Fig.3.1 (e)). M'_{BA} is the rotation moment (or rotation contribution) at end B.

The total end moments are therefore given as follows:

$$M_{AB} = FEM_{AB} + 2M'_{AB} + M'_{BA} \quad (3.1)$$

$$\text{and } M_{BA} = FEM_{BA} + 2M'_{BA} + M'_{AB}$$

Thus, the moment at each end of a member is given by the algebraic sum of the following components:

- The fixed-end moment at the member end being considered, due to the actual loading on the member;
- Twice the rotation moment (or rotation contribution) at that end; and
- The rotation moment at the far end.

Consider now a multi-storey frame as shown in Fig.3.2. If there is no joint translation, the eqn (3.1) applies to all the members.

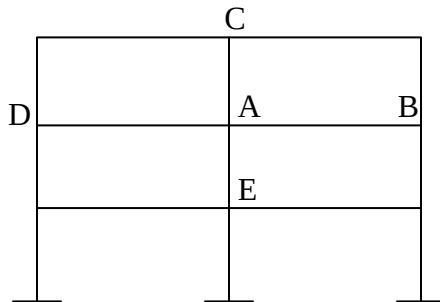


Fig.3.2 Multi-storey frame

Thus, the end moments for members meeting at A will be obtained as follows:

$$\begin{aligned} M_{AB} &= FEM_{AB} + 2M'_{AB} + M'_{BA} \\ M_{AC} &= FEM_{AC} + 2M'_{AC} + M'_{CA} \\ M_{AD} &= FEM_{AD} + 2M'_{AD} + M'_{DA} \\ M_{AE} &= FEM_{AE} + 2M'_{AE} + M'_{EA} \end{aligned} \quad (3.2)$$

For joint A to be in equilibrium, the sum of end moments at A ($\sum M_A$) of all the members meeting at A must be equal to zero. Thus:

$$\sum M_A = 0;$$

$$\sum FEM_A + 2\sum RM_A + \sum RM_{f.end} = 0 \quad (3.3)$$

where $\sum FEM_A = FEM_{AB} + FEM_{AC} + FEM_{AD} + FEM_{AE}$;

$$\sum RM_A = M'_{AB} + M'_{AC} + M'_{AD} + M'_{AE};$$

$$\sum RM_{f.end} = M'_{BA} + M'_{CA} + M'_{DA} + M'_{EA}.$$

Here, $\sum FEM_A$ = Algebraic sum of fixed-end moments at A of all the members meeting at A;

$\sum RM_A$ = Algebraic sum of rotation moments at A of all the members meeting at A;

$\sum RM_{f.end}$ = Algebraic sum of rotation moments at far ends of all the members meeting at A.

From eqn (3.3), we obtain:

$$\sum RM_A = -\frac{1}{2}(\sum FEM_A + \sum RM_{f.end}) \quad (3.4)$$

Consider now the beam shown in Fig.3.3.

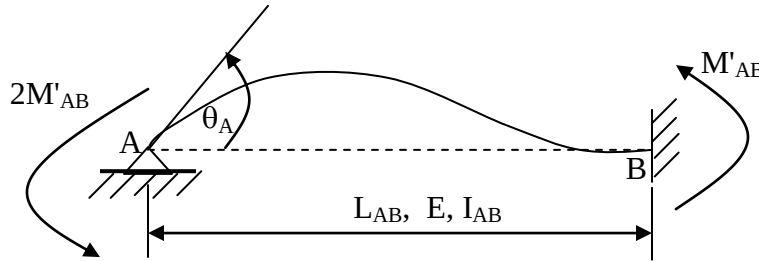


Fig.3.3 Beam with far end fixed.

From the force-displacement (or moment-rotation) relationship given in Table 1.1, for the beam of Fig.3.3 we can write the following:

$$2M'_{AB} = \frac{4EI_{AB}\theta_A}{L_{AB}} = 4EK_{AB}\theta_A$$

where K_{AB} = relative stiffness of member AB $\left(\because K_{AB} = \frac{I_{AB}}{L_{AB}} \right)$

$$\therefore M'_{AB} = 2EK_{AB}\theta_A \quad (3.5)$$

If at a joint A in a structure, more than one member meets (see Fig.3.2), all the members at the joint will undergo the same rotation θ_A . If E is the same for all the members, we can write:

$$\sum RM_A = 2E\theta_A \sum K_A \quad (3.6)$$

Where $\sum K_A$ = sum of relative stiffness of all members meeting at A.

Dividing eqn (3.5) by eqn (3.6), we obtain:

$$\frac{M'_{AB}}{\sum RM_A} = \frac{K_{AB}}{\sum K_A} \quad (3.7)$$

or $M'_{AB} = \frac{K_{AB}}{\sum K_A} \sum RM_A \quad (3.8)$

Substituting for $\sum RM_A$ from eqn (3.4), we obtain:

$$M'_{AB} = \left(-\frac{1}{2}\right) \frac{K_{AB}}{\sum K_A} (\sum FEM_A + \sum RM_{f.end}) \quad (3.9)$$

Where, the ratio $\left(-\frac{1}{2}\right) \frac{K_{AB}}{\sum K_A}$ is known as the rotation factor for member AB at joint A.

If we denote rotation factor by RF, then:

$$RF = \left(-\frac{1}{2}\right) \frac{K_{AB}}{\sum K_A} \quad (3.10)$$

and eqn (3.9) can be re-written as:

$$M'_{AB} = RF_{AB} (\sum FEM_A + \sum RM_{f.end}) \quad (3.11)$$

In eqn (3.11) the term $\sum FEM_A$ is a known quantity since fixed-end moments can be calculated directly from the loading. However, at the initial stage the far end rotation moments $\sum RM_{f.end}$ are not known but trial values are assumed for them. At the very start of the calculations, the trial values for the far end rotation moments are usually taken as zero. With the far end rotation moments and the sum of the fixed-end moments at a joint known, the rotation moment at members' ends attached to the joint can now be calculated using eqn (3.11). Similarly, the rotation moments at other joints are also approximately determined by repeatedly applying eqn (3.11) to the joints, with one joint at a time being considered the near joint. The rotation moment calculated for a member end when the adjacent joint is considered the "near" joint, is taken as a trial value when the joint becomes a "far" joint to another "near" joint to which eqn (3.11) will be applied.

Having determined approximate values for the rotation moments, in every subsequent cycle more accurate value of the rotation moment at A for member AB can be determined using eqn (3.11). The cycles of determination of rotation moment at every joint such as A continues until the values for the rotation moments for every member end in two successive cycles of calculations become the same or nearly the same. These accurate values of the rotation moments are then substituted in eqn (3.2) for the computation of the final end moments.

In using the rotation contribution method it is pertinent to note the following:

- The sum of the rotation factors at a joint is equal to $\left(-\frac{1}{2}\right)$

- At a fixed end of a member, the rotation moment is zero since a fixed end is not subject to rotation.
- A hinged or pinned member end can be more conveniently assumed to be fixed but with the relative stiffness taken as $\frac{3I}{4L}$

Example 3.1

Determine the member-end moments for the continuous beam shown in Fig.3.4.

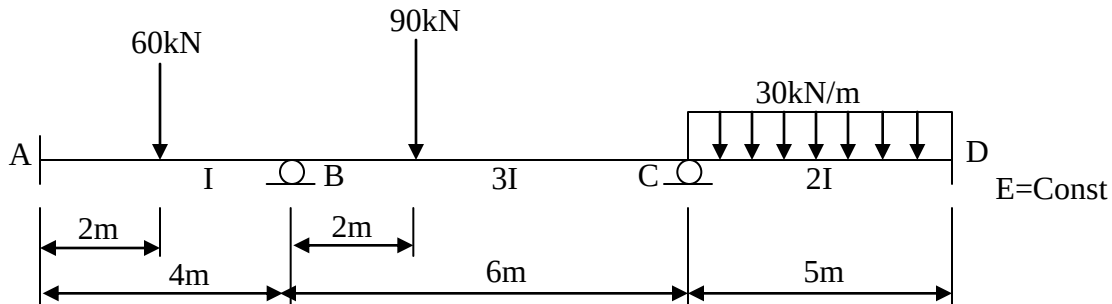


Fig.3.4 Given beam and loading.

SOLUTION

Fixed-end moments (FEM)

$$FEM_{AB} = \frac{60 \times 4}{8} = 30 \text{ kNm}$$

$$FEM_{BA} = -30 \text{ kNm}$$

$$FEM_{BC} = \frac{90 \times 2 \times 4^2}{6^2} = 80 \text{ kNm}$$

$$FEM_{CB} = -\frac{90 \times 4 \times 2^2}{6^2} = -40 \text{ kNm}$$

$$FEM_{CD} = \frac{30 \times 5^2}{12} = 62.5 \text{ kNm}$$

$$FEM_{DC} = -62.5 \text{ kNm}$$

Rotation Factors (RF)

Joint B

$$K_{BA} = \frac{I}{4}; \quad K_{BC} = \frac{3I}{6} = \frac{I}{2}; \quad \sum K_B = \frac{I}{4} + \frac{I}{2} = \frac{3I}{4}$$

$$RF_{BA} = \left(-\frac{1}{2} \right) \frac{K_{BA}}{\sum K_B} = -\frac{1}{6}$$

$$RF_{BC} = \left(-\frac{1}{2} \right) \frac{K_{BC}}{\sum K_B} = -\frac{1}{3}$$

Check: $-\frac{1}{6} - \frac{1}{3} = -\frac{1}{2}$ (Satisfied)

Joint C

$$K_{CB} = \frac{3I}{6} = \frac{I}{2}; \quad K_{CD} = \frac{2I}{5}; \quad \sum K_C = \frac{I}{2} + \frac{2I}{5} = \frac{9I}{10}$$

$$RF_{CB} = \left(-\frac{1}{2} \right) \frac{K_{CB}}{\sum K_C} = -\frac{5}{18}$$

$$RF_{CD} = \left(-\frac{1}{2} \right) \frac{K_{CD}}{\sum K_C} = -\frac{4}{18}$$

Check: $-\frac{5}{18} - \frac{4}{18} = -\frac{1}{2}$ (Satisfied)

Sum of fixed-end moments at joints ($\sum FEM$)

At Joint B

$$\sum FEM_B = -30 + 80 = 50 \text{ kNm}$$

At Joint C

$$\sum FEM_C = -40 + 62.5 = 22.5 \text{ kNm}$$

The fixed-end moments (FEM), rotation factors (RF) and the algebraic sum of the fixed-end moments ($\sum FEM$) at each joint are all recorded as shown in Fig.3.5. In the figure, a horizontal line is drawn to show the beam axis with each joint represented by two rectangles consisting of an outer and an inner one. The fixed-end moment is shown for each member end, on top of the member axis at the corresponding member end. The algebraical sum of the fixed-end moments at each joint is recorded inside the inner rectangle corresponding to the joint, while the rotation factors are recorded in the spaces between the inner and the outer rectangles at their respective member ends.

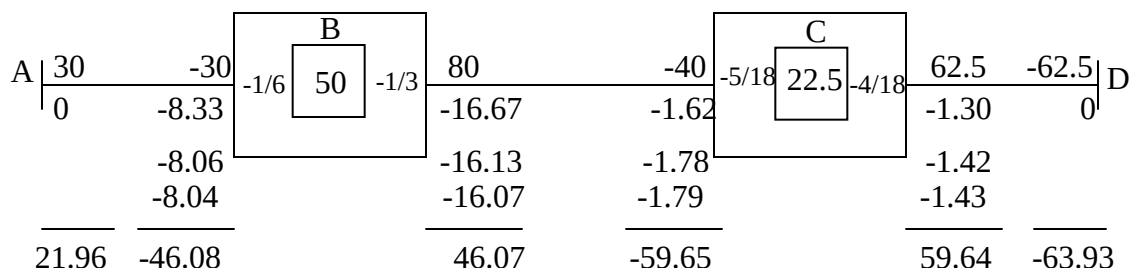


Fig.3.5 Scheme for the computation of end-moments (kNm) for the beam of Example 3.1

The Iteration Process

Eqn (3.11) will now be applied to joints B and C which are subject to rotation.

Cycle 1Joint B

For joint B, eqn (3.11) will take the following specific forms:

$$\begin{aligned} M'_{BA} &= RF_{BA} \left(\sum FEM_B + \sum RM_{Bf.end} \right) \\ \text{and } M'_{BC} &= RF_{BC} \left(\sum FEM_B + \sum RM_{Bf.end} \right) \end{aligned} \quad (a)$$

Here, $\sum RM_{Bf.end}$ = sum of the rotation moments of the far ends of members connected to joint B, i.e., $\sum RM_{Bf.end} = M'_{AB} + M'_{CB}$. In the first cycle although the rotation factors are known and the sum of the fixed-end moments are known, the rotation moments of member far ends are not known for the starting joint so they are taken as zero as a first approximation. We also note that the rotation moment of a fixed end is zero since a fixed end cannot rotate. Thus for joint B the following apply:

$$RF_{BA} = -\frac{1}{6}; \quad RF_{BC} = -\frac{1}{3}; \quad \sum FEM_B = 50kNm; \quad \sum RM_{Bf.end} = M'_{AB} + M'_{CB}$$

$$\therefore M'_{AB} = 0 \quad (\because \text{Joint A is fixed})$$

$$\text{and } M'_{CB} = 0 \quad (\text{First approximation})$$

Substituting these values in eqn (a) above, we obtain:

$$M'_{BA} = -\frac{1}{6}(50+0) = -8.33$$

$$\text{and } M'_{BC} = -\frac{1}{3}(50+0) = -16.67.$$

These rotation moments are recorded below the member axis at their respective member ends as shown in Fig.3.5.

Joint C

Applying the general equation (3.11) to this particular joint, we obtain the following:

$$M'_{CB} = -\frac{5}{18}(22.5 - 16.67 + 0)$$

or $M'_{CB} = -1.62kNm$. The "-16.67" within the parentheses above represents the rotation moment at far end B and "0" represents the rotation moment at far end D.

$$\text{Similarly, } M'_{CD} = -\frac{4}{18}(22.5 - 16.67 + 0)$$

$$\text{or } M'_{CD} = -1.30kNm$$

Again these rotation moments are recorded at their respective member end positions in Fig.3.5.

Cycle 2

The entire process carried out in cycle 1 will be repeated here, starting once again from joint B. The approximate values obtained for the rotation moments in cycle 1 will now be used in the cycle 2 computations, thereby enabling us to obtain more accurate

values of the rotation moments in cycle 2. as soon as new rotation moment values are obtained, the values obtained in the first cycle will be discountenanced.

Joint B

Sum of the rotation moments at far ends, i.e., $\sum RM_{Bf.end} = M'_{AB} + M'_{CB}$

Here, $M'_{AB} = 0$; $M'_{CB} = -1.62kNm$

$$\therefore \sum RM_{Bf.end} = 0 - 1.62 = -1.62kNm$$

$$M'_{BA} = -\frac{1}{6}(50 + 0 - 1.62) \\ = -8.06kNm$$

$$\text{and } M'_{BC} = -\frac{1}{3}(50 + 0 - 1.62) \\ = -16.13kNm$$

These rotation moment values, which now supersede the earlier ones of "-8.33" and "-16.67", are recorded accordingly as shown in Fig.3.5.

Joint C

$$M'_{BC} = -16.13kNm; \quad M'_{DC} = 0$$

$$\therefore \sum RM_{Cf.end} = -16.13 + 0 = -16.13kNm$$

$$M'_{CB} = -\frac{5}{18}(22.5 - 16.13) = -1.78kNm$$

$$M'_{CD} = -\frac{4}{18}(22.5 - 16.13) = -1.42kNm$$

These values accordingly replace the earlier ones of "-1.62" and "-1.30" and are recorded as shown in Fig.3.5.

Cycle 3

The values of the rotation moments to be obtained in this cycle will be more accurate than those obtained in cycle 2, and will accordingly replace the cycle 2 values.

Joint B

$$M'_{BA} = -\frac{1}{6}(50 + 0 - 1.78) = -8.04kNm$$

$$M'_{BC} = -\frac{1}{3}(50 + 0 - 1.78) = -16.07kNm$$

Joint C

$$M'_{CB} = -\frac{5}{18}(22.5 - 16.07 + 0) = -1.79kNm$$

$$M'_{CD} = -\frac{4}{18}(22.5 - 16.07 + 0) = -1.43kNm$$

The above rotation moment values once again replace the corresponding cycle 2 values. The earlier values are thus discarded.

At this point it is seen that at joint B the greatest difference between the immediate previous and present values is "0.06" while at joint C the difference is "0.01". Considering these differences as not sufficient to affect the desired accuracy of the final moment values, we decide to stop the iteration at this point, i.e., at cycle 3.

The final end moments are obtained using eqn (3.1) as follows:

$$M_{AB} = 30 + 2(0) - 8.04 = 21.96 \text{ kNm}$$

$$M_{BA} = -30 + 2(-8.04) + 0 = -46.08 \text{ kNm}$$

$$M_{BC} = 80 + 2(-16.07) - 1.79 = 46.07 \text{ kNm}$$

$$M_{CB} = -40 + 2(-1.79) - 16.07 = -59.65 \text{ kNm}$$

$$M_{CD} = 62.5 + 2(-1.43) + 0 = 59.64 \text{ kNm}$$

$$M_{DC} = -62.5 + 2(0) - 1.43 = -63.93 \text{ kNm}$$

The final end moments are recorded in the last row of Fig.3.5.

We now show the procedure for the analysis of beams with hinged end supports.

Example 3.2

Determine the support moments for the beam shown in Fig.3.6.

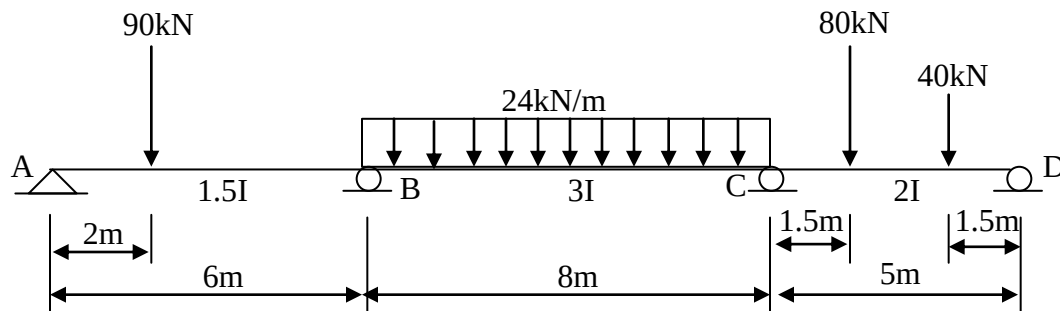


Fig.3.6 Given beam and loading.

SOLUTION

Notice that this beam was analysed in Example 2.5 using the moment distribution method. We now show how the same beam can be analysed using the rotation contribution method.

Since supports A and D are respectively a hinge and a roller, it is convenient to consider those ends as fixed and take the relative stiffnesses of spans AB and CD respectively as $\frac{3}{4}K_{AB}$ and $\frac{3}{4}K_{CD}$. In addition, the modified fixed-end moments will be used for those spans in accordance with eqn (2.11) of Chapter 2.

Fixed-end Moments, FEM

$$FEM_{AB} = \frac{90 \times 4^2 \times 2}{6^2} = 80 \text{ kNm}; \quad FEM_{BA} = -\frac{90 \times 4 \times 2^2}{6^2} = -40 \text{ kNm};$$

$$FEM_{BC} = \frac{24 \times 8^2}{12} = 128 \text{ kNm}; \quad FEM_{CB} = -128 \text{ kNm}$$

$$FEM_{CD} = \frac{80 \times 3.5^2 \times 1.5}{5^2} + \frac{40 \times 1.5^2 \times 3.5}{5^2} = 71.4 \text{ kNm};$$

$$FEM_{DC} = -\left(\frac{40 \times 3.5^2 \times 1.5}{5^2} + \frac{80 \times 1.5^2 \times 3.5}{5^2} \right) = -54.6 \text{ kNm}.$$

Modified Fixed-end Moments, FEM*

The fixed-end moments for members AB and CD will be modified to account for the hinged and roller supports at ends A and D respectively. Thus:

$$\begin{aligned} FEM^*_{BA} &= FEM_{BA} - \frac{1}{2} FEM_{AB} \\ &= -40 - \frac{1}{2}(80) = -80 \text{ kNm} \end{aligned}$$

$$\begin{aligned} FEM^*_{CD} &= FEM_{CD} - \frac{1}{2} FEM_{DC} \\ &= 71.4 - \frac{1}{2}(-54.6) = 98.7 \text{ kNm} \end{aligned}$$

$$FEM^*_{AB} = FEM^*_{DC} = 0 \quad (\because \text{the hinge and roller supports are assumed to be fixed but having zero fixed-end moments})$$

Rotation Factors, RF

At joint B

$$K_{BA} = \frac{3}{4} \times \frac{1.5I}{6} = \frac{3I}{16}; \quad K_{BC} = \frac{3I}{8}; \quad \sum K = K_{BA} + K_{BC} = \frac{9I}{16}$$

$$RF_{BA} = \left(-\frac{1}{2} \right) \frac{K_{BA}}{\sum K} = -\frac{1}{6}; \quad RF_{BC} = \left(-\frac{1}{2} \right) \frac{K_{BC}}{\sum K} = -\frac{1}{3}.$$

At joint C

$$K_{CD} = \frac{3}{4} \times \frac{2I}{5} = \frac{3I}{10}; \quad K_{CB} = \frac{3I}{8}; \quad \sum K = K_{CB} + K_{CD} = \frac{27I}{40}$$

$$RF_{CD} = \left(-\frac{1}{2} \right) \frac{K_{CD}}{\sum K} = -\frac{2}{9}; \quad RF_{CB} = \left(-\frac{1}{2} \right) \frac{K_{CB}}{\sum K} = -\frac{5}{18}.$$

The fixed-end moments and the rotation factors are recorded as shown in Fig.3.7.

The computations are carried out as follows:

Cycle 1

Joint B

$$M'_{BA} = -\frac{1}{6}(48 + 0 + 0) = -8$$

$$M'_{BC} = -\frac{1}{3}(48 + 0 + 0) = -16$$

Joint C

$$M'_{CB} = -\frac{5}{18}(-29.3 - 16 + 0) = 12.58$$

$$M'_{CD} = -\frac{2}{9}(-29.3 - 16 + 0) = 10.07$$

Cycle 2

Joint B

$$M'_{BA} = -\frac{1}{6}(48 + 0 + 12.58) = -10.10$$

$$M'_{BC} = -\frac{1}{3}(48 + 0 + 12.58) = -20.19$$

Joint C

$$M'_{CB} = -\frac{5}{18}(-29.3 - 20.19 + 0) = 13.75$$

$$M'_{CD} = -\frac{2}{9}(-29.3 - 20.19 + 0) = 11.0$$

Cycle 3

Joint B

$$M'_{BA} = -\frac{1}{6}(48 + 0 + 13.75) = -10.29$$

$$M'_{BC} = -\frac{1}{3}(48 + 0 + 13.75) = -20.58$$

Joint C

$$M'_{CB} = -\frac{5}{18}(-29.3 - 20.58) = 13.86$$

$$M'_{CD} = -\frac{2}{9}(-29.3 - 20.58) = 11.08$$

Cycle 4Joint B

$$M'_{BA} = -\frac{1}{6}(48 + 13.86) = -10.31$$

$$M'_{BC} = -\frac{1}{3}(48 + 13.86) = -20.62$$

Joint C

$$M'_{CB} = -\frac{5}{18}(-29.3 - 20.62) = 13.87$$

$$M'_{CD} = -\frac{2}{9}(-29.3 - 20.62) = 11.09$$

The iteration will end at this 4th cycle since the rotation moment values for the last two cycles are very close to one another. The final end moments are calculated using eqn (3.1) as before and recorded in Fig.3.7.

		B		C			
A	0	-80	-1/6 48 -1/3	128	-128	-5/18 -29.3 -2/9	98.7 0 D
		-8		-16	12.58		10.07
		-10.1		-20.19	13.75		11.00
		-10.29		-20.58	13.86		11.08
		-10.31		-20.62	13.87		11.09
	0	-100.62	100.63	-120.88	120.88	0	

Fig.3.7 End-moments for the beam of Example 3.2.

Compare the final end moments with those obtained in Example 2.5 using moment distribution method.

BEAMS WITH OVERHANGS

In the analysis of continuous beams with overhangs, the overhanging ends are considered to have zero stiffness. Besides this, the rest of the analysis is as before. The procedure will now be illustrated by means of the following example.

Example 3.3

Determine the support moments for the beam shown in Fig.3.8.

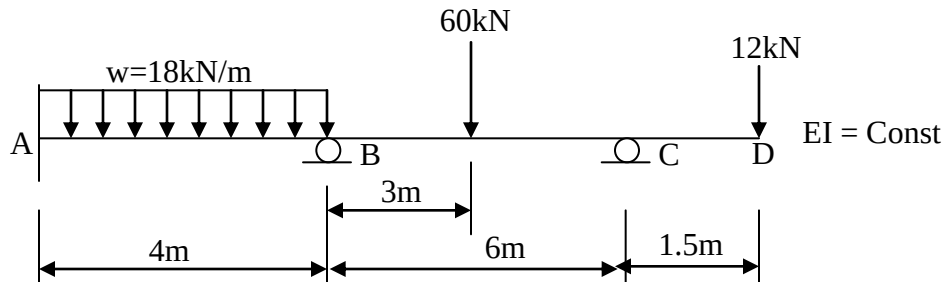


Fig.3.8 Given beam and loading

SOLUTION**Fixed End Moments**

This example is the same beam and loading analyzed in Example 2.3 using moment distribution method. We therefore pick the fixed-end moments from there.

Thus:

$$FEM_{AB} = -FEM_{BA} = 24 \text{ kNm}$$

$$FEM_{BC} = -FEM_{CB} = 45 \text{ kNm}$$

$$FEM_{CD} = 18 \text{ kNm}$$

Rotation Factors, RF

$$RF_{BA} = -\frac{1}{2} \left[\frac{\frac{I}{4}}{\frac{I}{4} + \frac{I}{6}} \right] = -\frac{1}{2} \left[\frac{\frac{I}{4} \times \frac{12}{5I}}{\frac{12}{4} + \frac{12}{6}} \right] = -\frac{3}{10}$$

$$RF_{BC} = -\frac{1}{2} \left[\frac{\frac{I}{6}}{\frac{I}{4} + \frac{I}{6}} \right] = -\frac{1}{2} \left[\frac{\frac{I}{6} \times \frac{12}{5I}}{\frac{12}{4} + \frac{12}{6}} \right] = -\frac{1}{5}$$

$$RF_{CB} = -\frac{1}{2} \left[\frac{\frac{I}{6}}{\frac{I}{6} + 0} \right] = -\frac{1}{2} \quad \text{(Remembering that the stiffness of the overhanging span is zero)}$$

The final end moments were obtained after 5 cycles, as shown in Fig.3.9. Compare these moments with those of Example 2.3.

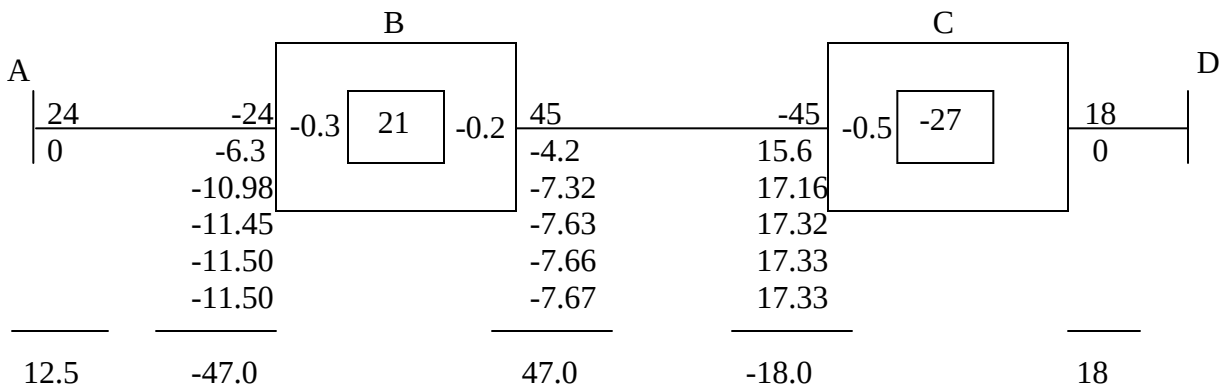


Fig.3.9 Final end moments for the beam of Example 3.3

MEMBERS WITH RELATIVE LATERAL TRANSLATION OF ENDS

Consider a frame member AB (Fig.3.10) with displacements at its two ends A and B but with the end rotation restrained.

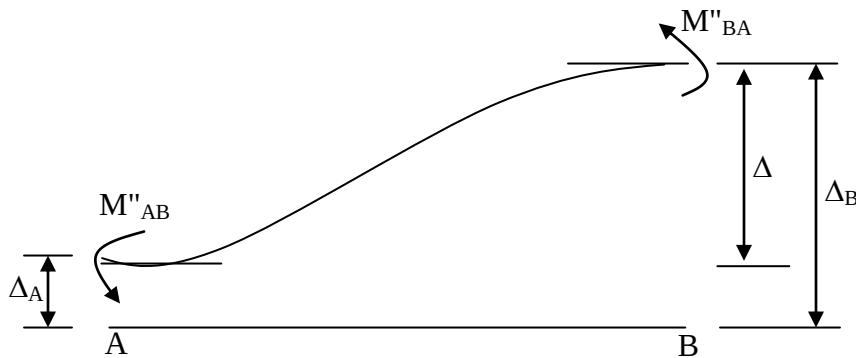


Fig.3.10 Frame member with end translations but no rotations.

The fixed-end moments for the member due to the translations are as follows:

$$M''_{AB} = M''_{BA} = \frac{6EI\Delta}{L^2} \quad (3.12)$$

If the joints rotate as well as translate, the end moments will be:

$$\begin{aligned} M_{AB} &= FEM_{AB} + 2M'_{AB} + M'_{BA} + M''_{AB} \\ \text{and } M_{BA} &= FEM_{BA} + 2M'_{BA} + M'_{AB} + M''_{BA} \end{aligned} \quad (3.13)$$

Here $M''_{AB} = M''_{BA}$ is referred to as the displacement moment of member AB.

If more than two members meet at a joint A, such as in Fig.3.2, the equilibrium equation for joint A will be as follows:

$$\sum M_A = 0;$$

$$\sum FEM_A + 2\sum RM_A + \sum RM_{f.end} + \sum TM_A = 0$$

or
$$\sum RM_A = \left(-\frac{1}{2}\right)(\sum FEM_A + \sum RM_{f.end} + \sum TM_A) \quad (3.14)$$

Here $\sum FEM_A$, $\sum RM_A$, and $\sum RM_{f.end}$ have the same meaning as before, and $\sum TM_A$ is the algebraic sum of the translation moments of all the members meeting at joint A.

Let us represent eqn(3.8) here as eqn (3.15). Thus:

$$M'_{AB} = \frac{K_{AB}}{\sum K_A} \sum RM_A \quad (3.15)$$

Substituting for $\sum RM_A$ from eqn (3.14) into eqn (3.15), we have:

$$M'_{AB} = \frac{K_{AB}}{\sum K_A} \left(-\frac{1}{2}\right)(\sum FEM_A + \sum RM_{f.end} + \sum TM_A)$$

or
$$M'_{AB} = RF_{AB}(\sum FEM_A + \sum RM_{f.end} + \sum TM_A) \quad (3.16)$$

Similarly,
$$M'_{BA} = RF_{BA}(\sum FEM_B + \sum RM_{f.end} + \sum TM_B)$$

The rotation moments for members with relative end translation are thus obtained using eqn (3.16). When the value of the translation Δ is known, eqn (3.12) is used to calculate the translation moment. If however the value of the translation is not known, additional equations (or translation contribution) equations are required. The nature of this equation depends on the type of structure and loading. this aspect will be considered later in this chapter.

Example 3.4

Obtain the end moments for the beam shown in Fig.3.11 if under the given loading support B sinks by 5mm. Take $E=210 \times 10^6 \text{ kN/m}^2$, $I=360 \times 10^{-6} \text{ m}^4$.

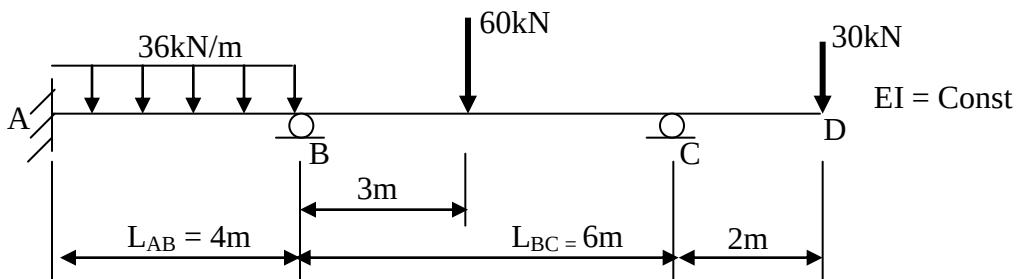


Fig.3.11 Given beam and loading

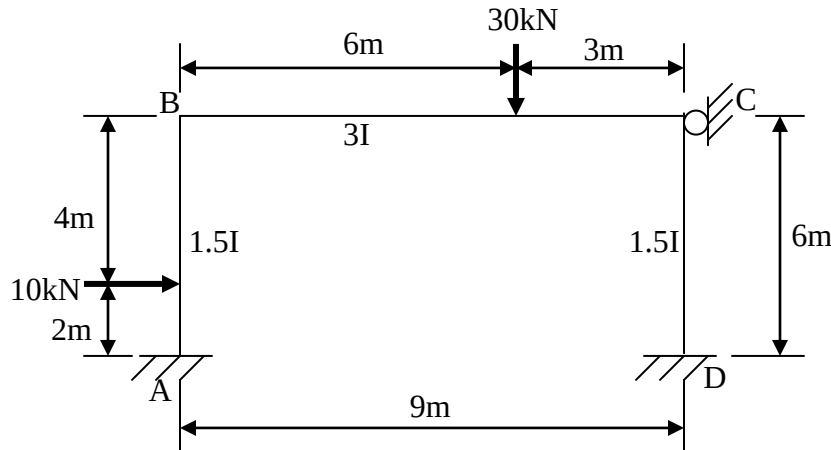


Fig.3.13 Given frame and loading

SOLUTION

Observe that this frame had previously been analyzed by the slope-deflection method (Example 1.5) and by the Hardy Cross moment distribution method (Example 2.6). The fixed-end moments are:

$$FEM_{AB} = 8.89 \text{ kNm}; \quad FEM_{BA} = -4.44 \text{ kNm};$$

$$FEM_{BC} = 20 \text{ kNm}; \quad FEM_{CB} = -40 \text{ kNm}.$$

The above fixed-end moments, the rotation factors, and the rest of the computations are shown in Fig.3.14.

B		C	
15.56	-4/14	-40	-4/14
-3/14		-3/14	
-4.44	20	-40	
	-4.45	12.70	
	-8.07	13.73	
	-8.37	13.82	
	-8.39	13.83	
	-8.40	13.83	
	17.03	-20.74	
-3.33		9.53	
-6.06		10.30	
-6.28		10.37	
-6.30		10.37	
-6.30		10.37	
-17.04		20.74	
2.59		10.37	
8.89	A	0.0	D

Fig.3.14 Final end moments of the frame of Example 3.5.

Compare now the above final end moments with those obtained in Examples 1.5 and 2.6.

SYMMETRICAL FRAMES SUBJECTED TO SYMMETRICAL LOADING

Much computational effort can be saved if use is made of symmetry of frame and loading in the analysis of such frames. Two cases of symmetry are possible. They include the case when the axis of symmetry passes through the centreline of the beams, and the case when the axis of symmetry passes through the column line. We now consider each of the two cases in turn.

1. Axis of Symmetry passes through the beam centreline

Let the horizontal member AB (Fig.3.15a) represent part of a frame and let the axis of symmetry of the frame pass through the centreline of this horizontal member.

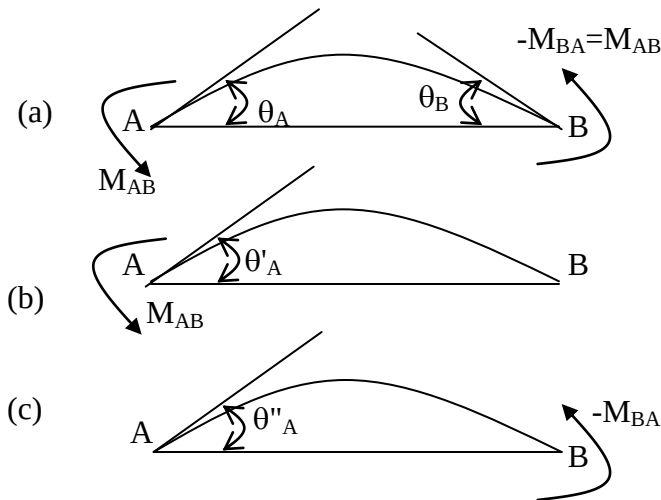


Fig.3.15: (a) End moments and rotations; (b) Rotation at end A due to moment M_{AB} ; (c) Rotation at end A due to moment M_{BA} .

Let M_{AB} and M_{BA} be the end moments. Owing to symmetry of deformation, M_{AB} and M_{BA} are equal in magnitude but are opposite in sense. The slope θ_A is the algebraic sum of the rotations due to M_{AB} (i.e. θ'_A) and $-M_{BA}$ (i.e. θ''_A) as shown in Fig.3.15. Thus:

$$\theta_A = \theta'_A + \theta''_A$$

At this point we recall that from moment rotation relationship, the rotation at the end of a member when moment is applied to that end (near end) is twice the magnitude of the rotation at that same end of the member if the same magnitude of moment is applied at the 'far end' of the member. Consequently, with reference to Fig.3.15 we can write the following:

$$\theta'_A = \frac{M_{AB}L}{3EI} \quad \text{and} \quad \theta''_A = -\frac{M_{BA}L}{6EI} = \frac{M_{AB}L}{6EI}$$

$$\text{Therefore } \theta_A = \theta'_A + \theta''_A = \frac{M_{AB}L}{2EI} \quad (a)$$

We now replace member AB by member AB' whose end A will undergo rotation θ_A due to moment M_{AB} applied at end A while end B' is being restrained (Fig.3.16). The substitute member (or member AB') will have the same value of I as for the original member.

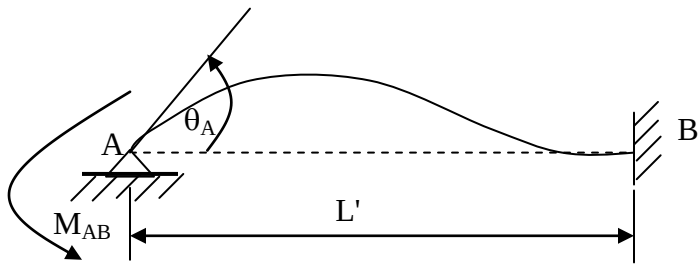


Fig.3.16 Substitute member AB'

For such a beam the force-displacement relationship is as follows:

$$\theta_A = \frac{M_{AB}L'}{4EI} \quad (b)$$

where L' is the length of the substitute member.

For the equality of rotation between the original member AB and the substitute member AB', we have:

$$\theta_A = \frac{M_{AB}L}{2EI} = \frac{M_{AB}L'}{4EI}$$

From the above relation we have: $\frac{I}{L} = \frac{2I}{L'}$

$$\text{or } K = 2K'$$

$$\text{or } K' = \frac{K}{2} \quad (3.17)$$

Thus, if K is the relative stiffness of the original member AB, this member can be replaced by substitute member AB' having relative stiffness $K/2$. With this substitute member, the analysis then needs to be carried out for only one half of the frame considering the line of symmetry as the fixed end.

Example 3.6

Obtain the end moments for the frame and loading shown in Fig.3.17, taking advantage of symmetry.

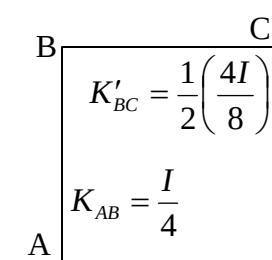
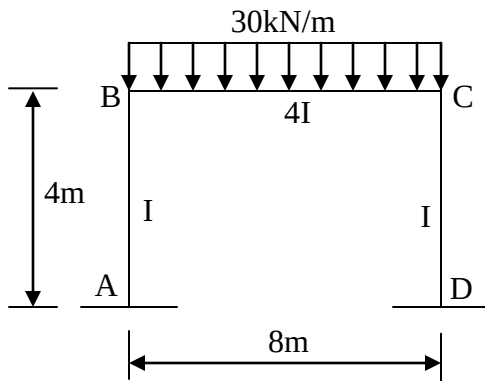


Fig.3.17: (a) Frame and loading; (b) Substitute frame.

SOLUTION

Here there is symmetry of frame and loading and so lateral displacement of the frame does not occur. Since the frame is symmetrical about the beam centreline, only one half of the frame needs be analyzed. The substitute frame is shown in Fig.3.17(b).

Fixed-end moments

$$FEM_{BC} = \frac{30 \times 8^2}{12} = 160 \text{ kNm}$$

The actual rotation moments are obtained in the first cycle of distribution as shown in Fig.3.18(a). The computation of the final end moments using eqn (3.1) is shown in Fig.3.18(b) and the final end moments are extracted and shown on the body of the frame in Fig.3.18(c).

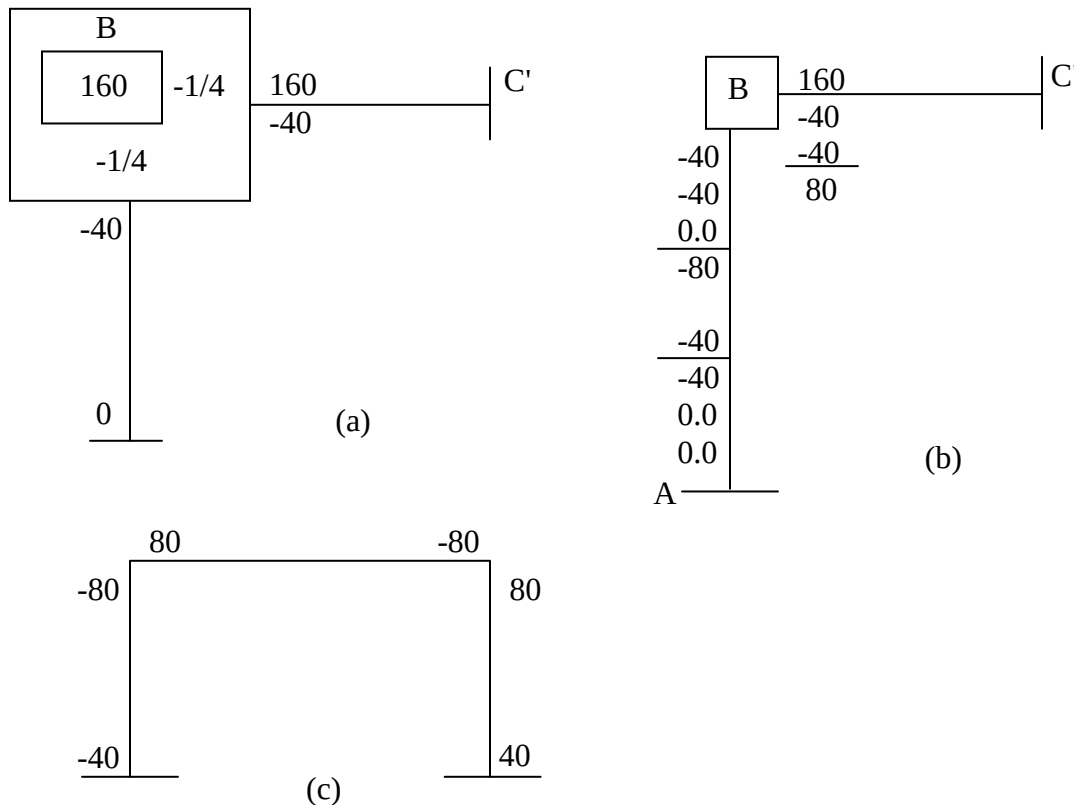


Fig.3.18 End moments for the frame of Example 3.16:

- (a) Scheme for the computation of rotation moments;
- (b) Calculation of final end moments;
- (c) Final end moments shown on frame member ends.

Example 3.7

Determine the end moments of the frame shown in Fig.3.19(a) taking advantage of symmetry.

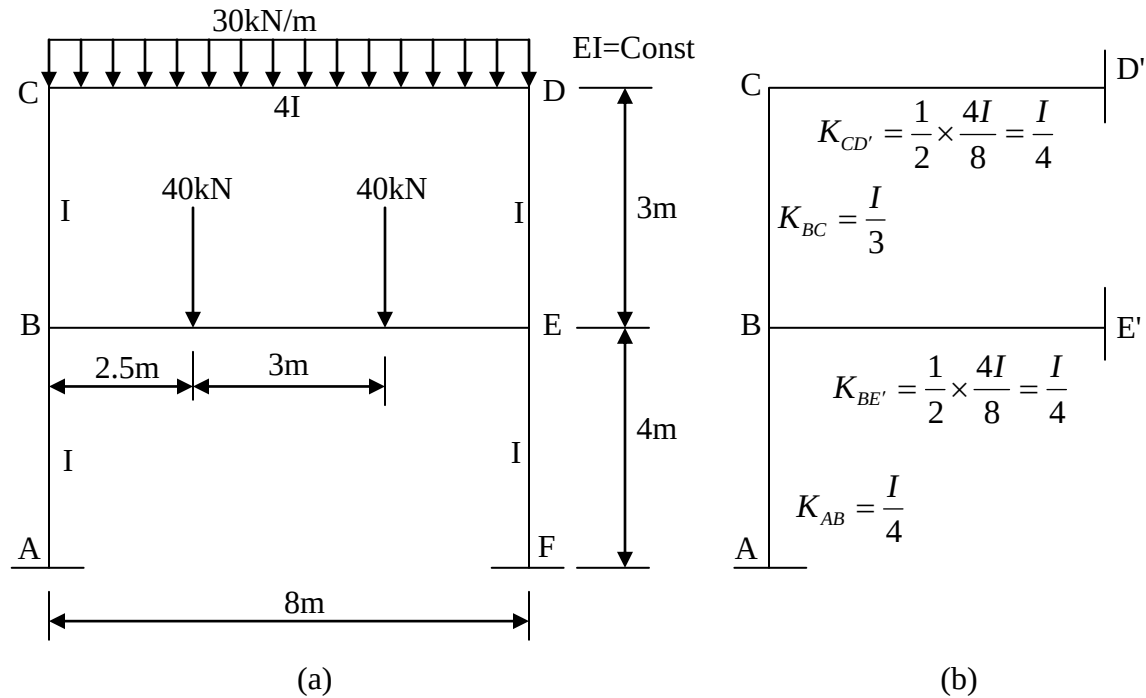


Fig.3.19: (a) Given frame and loading; (b) Substitute frame.

SOLUTION

The substitute frame is shown in Fig.3.19(b).

Fixed-end Moments

$$FEM_{BE} = \frac{40 \times 2.5 \times 5.5^2}{8^2} + \frac{40 \times 5.5 \times 2.5^2}{8^2} = 68.75 \text{ kNm};$$

$$FEM_{CD} = \frac{30 \times 8^2}{12} = 160 \text{ kNm}.$$

Relative Stiffness

These are shown in Fig.3.19(b) on the substitute frame.

Rotation Factors

These are shown in the computation scheme of Fig.3.20(a). The rotation moments are determined for joints B and C and the final end moments were obtained after 4 cycles, as shown in Fig.3.20(a). The final member end moments for the entire frame are shown on the body of the frame in Fig.3.20(b). Observe that the end moments for the left and the right halves of the frame are equal in magnitude but opposite in sign.

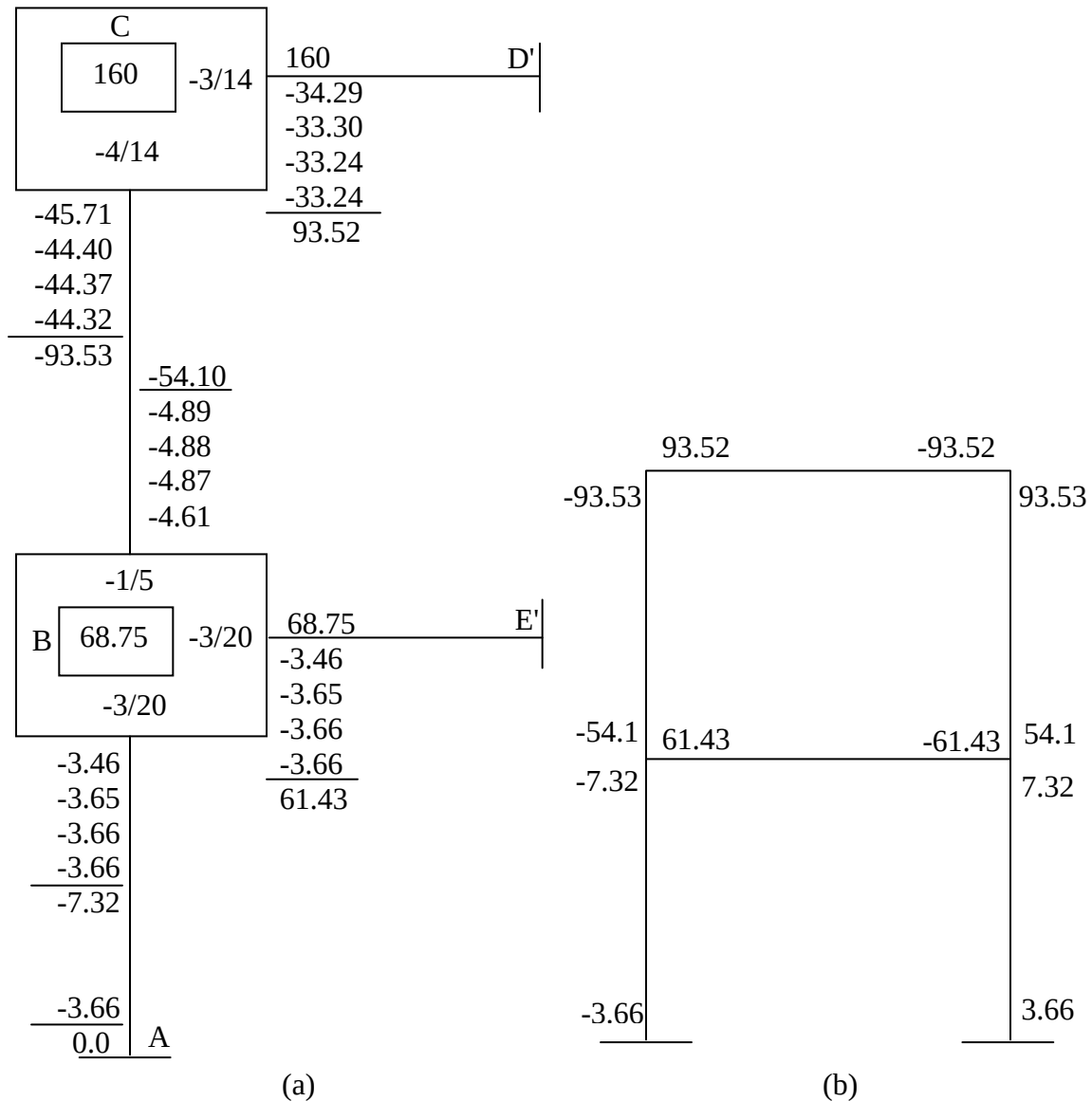


Fig.3.20 End moments for the frame of Example 3.7:

- (a) Computation scheme for rotation moments;
 (b) Final end moments for the entire frame.

2. Axis of Symmetry Passes through the Column

This situation presents when a frame has an even number of bays. Due to symmetry of frame and loading, the joints on the axis of symmetry do not rotate and so it is sufficient to analyze only one half of the frame, considering the joints at the line of symmetry as fixed.

Example 3.8

Determine the end moments for the frame of Fig.3.21(a), taking advantage of symmetry.

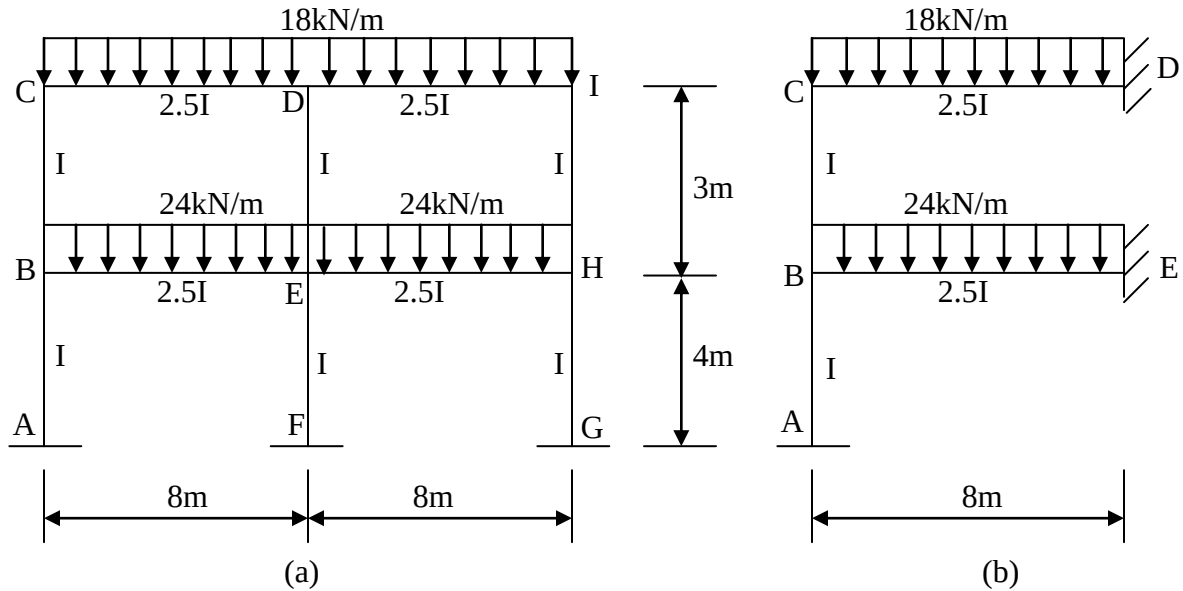


Fig.3.21: (a) Given frame and loading; (b) Frame for analysis.

SOLUTION

Only half of the frame is considered for analysis as shown in Fig.3.21(b). Joints D, E, and F are considered fixed since under the loading they are not subject to rotation.

Fixed-end Moments

$$FEM_{CD} = -FEM_{DC} = \frac{18 \times 8^2}{12} = 96 \text{ kNm}; \quad FEM_{BE} = -FEM_{EB} = \frac{24 \times 8^2}{12} = 128 \text{ kNm}.$$

The rest of the procedure is as before and the results are shown in Fig.3.22.

C 96 -0.242 -0.258		96	-96	D	59.3	-114.35	114.35	-59.3	59.3
-24.77	-19.81	-19.58	-23.23						
-19.56	-19.56	-19.56	-18.59	-114.35	-59.3	-114.35	114.35	-59.3	59.3
-59.29	-59.29	-59.29	-18.36	-18.35	-59.9	-146.87	146.87	-90.26	90.26
			-18.35	59.30	-30.36				
			-20.17						
			-20.17						
			-20.12						
			-19.20						
B 128 -0.174 -0.14		128	-128	E	-15.18	15.18			
-14.45	-15.15	-15.18	-17.96	0.0					
-15.18	-15.18	-15.18	-18.83	-114.35					
			-18.87						
			-18.87						
			-15.18	90.26					
			0.0						

(a) (b)

A Fig.3.22: (a) Calculation of rotation moments; (b) Final end moments.

FRAMES WITH LATERAL TRANSLATION OF JOINTS

Two types of loading cases can cause lateral translation of frame joints, namely, vertical and horizontal loadings. We now consider each of these cases in turn.

VERTICAL LOADING

Consider a multistorey frame such as shown in Fig.3.23(a). Let AB represent a vertical member in any storey of the frame. M_{AB} and M_{BA} are the column moments at A and B ends of the member. Let the horizontal force (or shear) exerted by the frame on column AB be H.

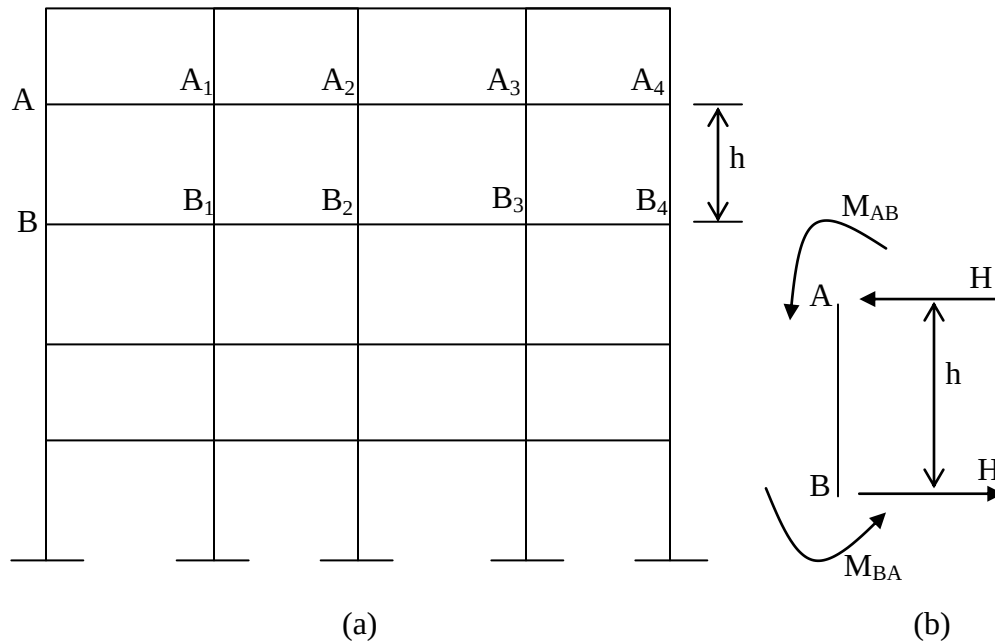


Fig.3.23: (a) Multistorey frame; (b) End moments of a typical column.

If the column height is h, from the equilibrium consideration of the free-body diagram of member AB we can obtain an expression for the shear H in terms of M_{AB} , M_{BA} , and h. Thus:

$$M_{BA} + M_{AB} + Hh = 0 \quad (3.18)$$

$$\text{or } H = -\frac{(M_{AB} + M_{BA})}{h} \quad (3.19)$$

Referring now to the multistorey frame of Fig.3.23(a), let AB, A₁B₁, A₂B₂, A₃B₃, and A₄B₄ denote all the columns in a particular storey. Application of eqn (3.19) to all the columns of the storey yields:

$$\sum H = -\frac{(\sum M_{AB} + \sum M_{BA})}{h} \quad (3.20)$$

where $\sum H$ = shear in all the columns in the storey under consideration. If we denote the shear in the r^{th} storey as Q_r , then:

$$Q_r = -\frac{(\sum M_{AB} + \sum M_{BA})}{h_r} \quad (3.21)$$

Here, h_r = height of columns of the r^{th} storey;

$\sum M_{AB}$ = sum of the end moments at the top ends of all the columns in the r^{th} storey;

$\sum M_{BA}$ = sum of the end moments at the bottom ends of all the columns in the r^{th} storey.

For the case when the external loading is vertical, $Q_r = 0$. Furthermore, each of the columns of the r^{th} storey is of height h_r . Therefore, from eqn (3.21), we obtain:

$$\sum M_{AB} + \sum M_{BA} = 0 \quad (3.22)$$

for the r^{th} storey.

We now recall that the general expression for end moments for member AB is given by eqn (3.13) as:

$$\begin{aligned} M_{AB} &= FEM_{AB} + 2M'_{AB} + M'_{BA} + M''_{AB} \\ \text{and } M_{BA} &= FEM_{BA} + 2M'_{BA} + M'_{AB} + M''_{BA} \end{aligned} \quad (3.23)$$

Referring to the above equation, we note that for a vertical column, $FEM_{AB} = FEM_{BA} = 0$ since the loading on the frame is vertical. We also note that for any prismatic member, the displacement moments M''_{AB} and M''_{BA} are equal. Therefore from eqn (3.23), we have:

$$\sum M_{AB} + \sum M_{BA} = 3\sum RM_{AB} + 3\sum RM_{BA} + 2\sum TM_{AB} \quad (3.24)$$

where $\sum RM_{AB} = \sum M'_{AB}$ is the sum of the rotation moments at the upper ends of all the columns in the storey;

$\sum RM_{BA} = \sum M'_{BA}$ is the sum of the rotation moments at the lower ends of all the columns in the storey;

$\sum TM_{AB} = \sum M''_{AB}$ is the sum of the translation moments of all the columns in the storey.

Taking cognisance of eqn (3.22), we can write:

$$3\sum RM_{AB} + 3\sum RM_{BA} + 2\sum TM_{AB} = 0$$

$$\text{or } \sum TM_{AB} = -\frac{3}{2}(\sum RM_{AB} + \sum RM_{BA}) \quad (3.25)$$

Equation (3.25) expresses the relationship between the rotation and displacement moments.

From the works in Chapters 1 and 2, we know that the relative lateral displacement Δ is the same for all the columns in any one storey. For a given column, the translation moment is given by:

$$M''_{AB} = \frac{6EI\Delta}{h^2} = \frac{6EI\psi}{h}$$

where $\psi = \frac{\Delta}{h}$.

Thus, the translation moment of a column in a storey is proportional to its relative stiffness $K = I/h$. Therefore:

$$\frac{M''_{AB}}{\sum TM_{AB}} = \frac{K_{AB}}{\sum K_{AB}}$$

or $M''_{AB} = \frac{K_{AB}}{\sum K_{AB}} \sum TM_{AB}$ (3.26)

Substituting for $\sum TM_{AB}$ from eqn (3.25) into eqn (3.26), we have:

$$M''_{AB} = \frac{K_{AB}}{\sum K_C} \left(-\frac{3}{2} \right) (\sum RM_{AB} + \sum RM_{BA})$$

or $M''_{AB} = TF_{AB} (\sum RM_{AB} + \sum RM_{BA})$ (3.27)

where $TF_{AB} = \left(-\frac{3}{2} \right) \frac{K_{AB}}{\sum K_C}$ is known as the translation factor of member AB;

$(\sum RM_{AB} + \sum RM_{BA})$ is the sum of the rotation moments at the upper and the lower ends of all the columns of the storey being considered;

$\sum K_C$ is the sum of the relative stiffnesses of all the columns in the storey being considered.

Thus, eqn (3.25) is used to calculate the sum of the displacement moments of the columns of a storey while the individual displacement contributions of the columns are determined by distributing this sum among the columns of the storey in proportion to their K values, in accordance with eqn (3.27). From eqn (3.27) it is understandable that the sum of the translation factors of all the columns of a storey is equal to (-3/2).

A **summary** of the relationships obtained so far in this chapter is as follows:

$$\begin{aligned} M'_{AB} &= RF_{AB} (\sum FEM_A + \sum RM_{f.end} + \sum TM_A) \\ M'_{BA} &= RF_{BA} (\sum FEM_B + \sum RM_{f.end} + \sum TM_B) \end{aligned} \quad (3.28)$$

$$M''_{AB} = M''_{BA} = TF (\sum RM_{AB} + \sum RM_{BA}) \quad (3.29)$$

$$\begin{aligned} M_{AB} &= FEM_{AB} + 2M'_{AB} + M'_{BA} + M''_{AB} \\ M_{BA} &= FEM_{BA} + 2M'_{BA} + M'_{AB} + M''_{BA} \end{aligned} \quad (3.30)$$

$\sum TM_A$; $\sum TM_B$ = sum of translation moments of all the columns attached to joints A and B respectively;

$\sum RM_{AB}$; $\sum RM_{BA}$ = sum of the rotation moments at the upper and the lower ends respectively, of the columns in a storey.

Equations (3.28) and (3.29) are used to determine the rotation and the translation moments for all the storeys in turn, by iteration. With the acceptable values of rotation and translation moments, the final end moments are then determined with the aid of eqn (3.30).

Example 3.9

Determine the member end moments for the sway frame shown in Fig.3.24.

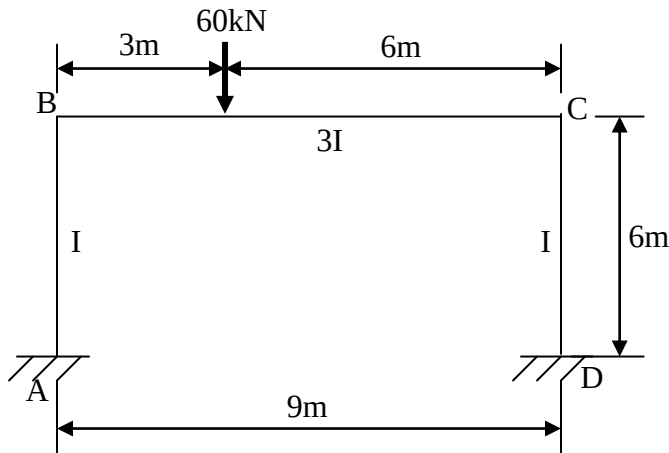


Fig.3.24. Given frame and loading

SOLUTION

The solution will be in the following order. First the fixed end moments, the rotation factors and the translation factors will be computed. Next, the cycles of calculations for the rotation and the translation moments will be carried out in the following sequence: joint B, joint C, and the Storey. Note that only joints B and C can rotate in the given frame and therefore the rotation moment for each of joints A and D is equal to zero.

Fixed-end Moments

$$FEM_{BC} = \frac{60 \times 6^2 \times 3}{9^2} = 80 \text{ kNm}; \quad FEM_{CB} = \frac{60 \times 3^2 \times 6}{9^2} = -40 \text{ kNm}.$$

Rotation Factors, RF

At Joint B

$$K_{BA} = \frac{I}{6}; \quad K_{BC} = \frac{3I}{9} = \frac{I}{3}; \quad \sum K_B = \frac{I}{6} + \frac{I}{3} = \frac{I}{2}$$

$$RF_{BA} = -\frac{1}{2} \times \frac{K_{BA}}{\sum K_B} = -\frac{1}{2} \times \frac{I}{6} \times \frac{2}{I} = -\frac{1}{6};$$

$$RF_{BC} = -\frac{1}{2} \times \frac{K_{BC}}{\sum K_B} = -\frac{1}{2} \times \frac{I}{3} \times \frac{2}{I} = -\frac{1}{3}.$$

At Joint C

$$RF_{CB} = RF_{BC} = -\frac{1}{3}; \quad RF_{CD} = RF_{BA} = -\frac{1}{6}.$$

Translation Factor, TF

$$\sum K_{col} = \frac{I}{6} + \frac{I}{6} = \frac{I}{3}; \quad TF_{BA} = TF_{CD} = -\frac{3}{2} \times \frac{K_{BA}}{\sum K_{col}} = -\frac{3}{2} \times \frac{K_{CD}}{\sum K_{col}} = -\frac{3}{2} \times \frac{I}{6} \times \frac{3}{I} = -\frac{3}{4}$$

The above fixed-end moments, rotation and translation factors are recorded as shown in Fig.3.25(a).

Cycle 1Joint B

Sum of FEM = 80.

At the start, all far end rotation moments and the translation moment are assumed to be zero. Therefore:

$$M'_{BA} = -\frac{1}{6}(80) = -13.33;$$

$$M'_{BC} = -\frac{1}{3}(80) = -26.67$$

Joint C

Sum of FEM = -40

Rotation moment at B = -26.67

Rotation moment at D = 0 ($\because$ fixed end)

Translation moment of column CD = 0 (assumed)

Total = -66.67

Therefore: $M'_{CB} = -\frac{1}{3}(-66.67) = 22.22;$

$$M'_{CD} = -\frac{1}{6}(-66.67) = 11.11.$$

Storey

Rotation moment at upper end of column AB = -13.33

Rotation moment at upper end of column CD = 11.11

Rotation moment at lower ends of columns = 0

Total = -2.22

Therefore: $M''_{AB} = M''_{CD} = -\frac{3}{4}(-2.22) = 1.67$

Cycle 2Joint B

Sum of FEM = 80

Rotation moment at A = 0 (fixed end)

Rotation moment at C = 22.22

Translation moment of column AB = 1.67

Total = 103.89

Therefore: $M'_{BA} = -\frac{1}{6}(103.89) = -17.32;$

$$M'_{BC} = -\frac{1}{3}(103.89) = -34.63.$$

Joint C

Sum of FEM	= -40
Rotation moment at B	= -34.63
Rotation moment at D	= 0
Translation moment of column CD	= 1.67
Total	= -72.96

Therefore: $M'_{CB} = -\frac{1}{3}(-72.96) = 24.32;$
 $M'_{CD} = -\frac{1}{6}(-72.96) = 12.16.$

Storey

Rotation moment at upper end of column AB	= -17.32
Rotation moment at upper end of column CD	= 12.16
Rotation moment at lower ends of columns	= 0
Total	= -5.16

Therefore: $M''_{AB} = M''_{CD} = -\frac{3}{4}(-5.16) = 3.87.$

Cycle 3

Joint B

Sum of FEM	= 80
Rotation moment at A	= 0
Rotation moment at C	= 24.32
Translation moment of column AB	= 3.87
Total	= 108.19

Therefore: $M'_{BA} = -\frac{1}{6}(108.19) = -18.03;$
 $M'_{BC} = -\frac{1}{3}(108.19) = -36.06.$

Joint C

Sum of FEM	= -40
Rotation moment at B	= -36.06
Rotation moment at D	= 0
Translation moment of column CD	= 3.87
Total	= -72.19

Therefore: $M'_{CB} = -\frac{1}{3}(-72.19) = 24.06;$
 $M'_{CD} = -\frac{1}{6}(-72.19) = 12.03.$

Storey

Rotation moment at upper end of column AB	= -18.06
---	----------

$$\begin{aligned}
 \text{Rotation moment at upper end of column CD} &= 12.03 \\
 \text{Rotation moment at lower ends of columns} &= 0 \\
 \text{Total} &= -6.03
 \end{aligned}$$

Therefore: $M''_{AB} = M''_{CD} = -\frac{3}{4}(-6.03) = 4.52.$

Cycle 4

Joint B

$$\begin{aligned}
 \text{Sum of FEM} &= 80 \\
 \text{Rotation moment at A} &= 0 \\
 \text{Rotation moment at C} &= 24.06 \\
 \text{Translation moment of column AB} &= 4.52 \\
 \text{Total} &= 108.58
 \end{aligned}$$

Therefore: $M'_{BA} = -\frac{1}{6}(108.58) = -18.10;$

$$M'_{BC} = -\frac{1}{3}(108.58) = -36.19.$$

Joint C

$$\begin{aligned}
 \text{Sum of FEM} &= -40 \\
 \text{Rotation moment at B} &= -36.19 \\
 \text{Rotation moment at D} &= 0 \\
 \text{Translation moment of column CD} &= 4.52 \\
 \text{Total} &= -71.67
 \end{aligned}$$

Therefore: $M'_{CB} = -\frac{1}{3}(-71.67) = 23.89;$

$$M'_{CD} = -\frac{1}{6}(-71.67) = 11.95.$$

Storey

$$\begin{aligned}
 \text{Rotation moment at upper end of column AB} &= -18.10 \\
 \text{Rotation moment at upper end of column CD} &= 11.95 \\
 \text{Rotation moment at lower ends of columns} &= 0 \\
 \text{Total} &= -6.15
 \end{aligned}$$

Therefore: $M''_{AB} = M''_{CD} = -\frac{3}{4}(-6.15) = 4.61.$

Cycle 5

Joint B

$$\begin{aligned}
 \text{Sum of FEM} &= 80 \\
 \text{Rotation moment at A} &= 0 \\
 \text{Rotation moment at C} &= 23.89 \\
 \text{Translation moment of column AB} &= 4.61 \\
 \text{Total} &= 108.5
 \end{aligned}$$

Therefore: $M'_{BA} = -\frac{1}{6}(108.5) = -18.08;$
 $M'_{BC} = -\frac{1}{3}(108.5) = -36.17.$

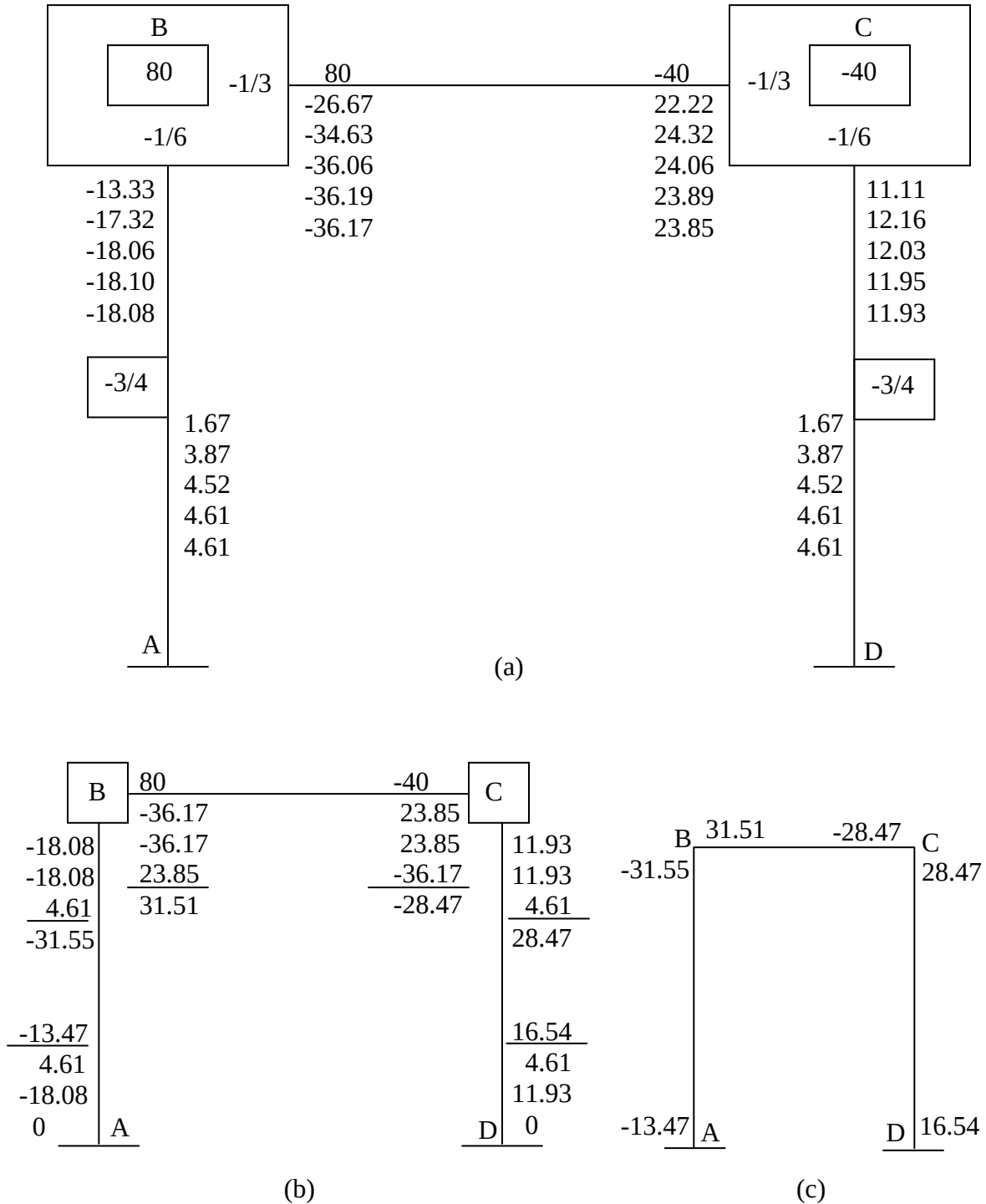


Fig.3.25 Results of the analysis of the frame of Example 3.9:

- (a) Scheme for the computation of rotation and translation moments;
- (b) Computation of final moments;
- (c) Final end moments.

Joint C

Sum of FEM	= -40
Rotation moment at B	= -36.17
Rotation moment at D	= 0
Translation moment of column CD	= 4.61
Total	= -71.56

Therefore: $M'_{CB} = -\frac{1}{3}(-71.56) = 23.85;$

$M'_{CD} = -\frac{1}{6}(-71.56) = 11.93.$

Storey

Rotation moment at upper end of column AB	= -18.08
Rotation moment at upper end of column CD	= 11.93
Rotation moment at lower ends of columns	= 0
Total	= -6.15

Therefore: $M''_{AB} = M''_{CD} = -\frac{3}{4}(-6.15) = 4.61.$

Thus the rotation and the translation moment values for the fourth and fifth cycles are seen to be either the same or very close and consequently further cycles are unnecessary. The final moments are computed using eqn (3.30). The computations for the final moments are shown in Fig.3.25(b) while the final moments are shown in Fig.3.25(c).

HORIZONTAL LOADING

When a frame is subjected to horizontal loading, the storey shear $\sum H = Q_r$ (Fig.3.26).

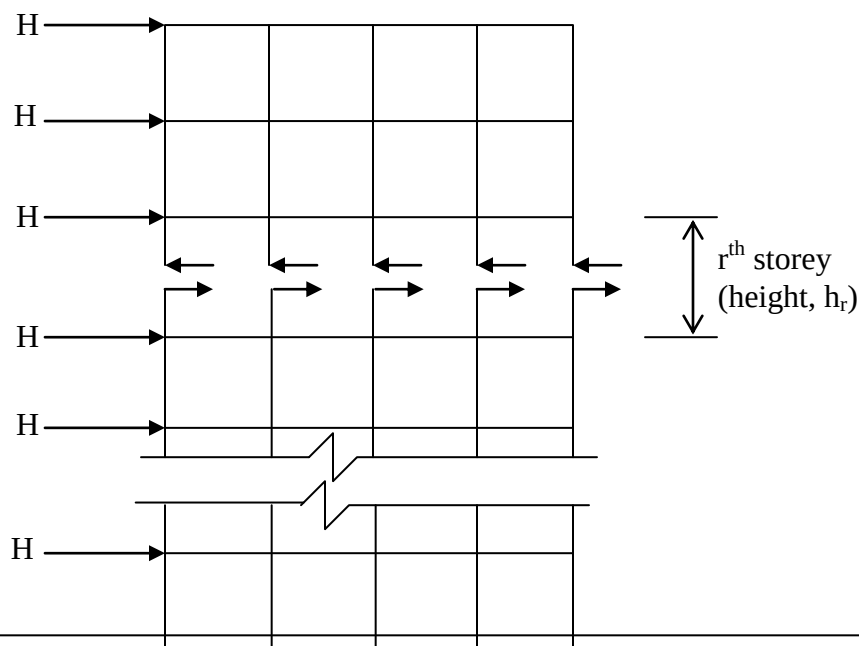


Fig.3.26 Shear in columns.

If all the columns of the storey are of height h_r , we can write the equilibrium equation:

$$\sum M = 0;$$

$$\sum M_{AB} + \sum M_{BA} + Q_r(h_r) = 0 \quad (3.31)$$

$$\text{or} \quad Q_r h_r = -(\sum M_{AB} + \sum M_{BA}) \quad (3.32)$$

Using eqn. (3.23), and noting that $FEM_{AB} = FEM_{BA} = 0$ for columns, the column moments can be expressed as follows:

$$Q_r h_r = -\sum_r \{3(M'_{AB} + M'_{BA}) + 2M''_{AB}\} \quad (3.33)$$

Summation $\sum_r$ is for all the columns in the r^{th} storey.

$$\text{or} \quad \frac{Q_r h_r}{3} = -\sum_r \left\{ (M'_{AB} + M'_{BA}) + \frac{2}{3} M''_{AB} \right\} \quad (3.34)$$

which gives

$$\sum M''_{AB} = -\frac{3}{2} \left\{ \frac{Q_r h_r}{3} + \sum_r (M'_{AB} + M'_{BA}) \right\} \quad (3.35)$$

$$\text{or} \quad \sum M''_{AB} = -\frac{3}{2} \left\{ M_r + \sum_r (M'_{AB} + M'_{BA}) \right\} \quad (3.36)$$

Here $M_r = \frac{Q_r h_r}{3}$ is known as the **storey moment**. The storey moment is positive when Q acts from right to left. From eqn. (3.36) we can write:

$$M''_{AB} = \frac{K_{AB}}{\sum_r K_{AB}} \left(-\frac{3}{2} \right) \left\{ M_r + \sum_r (M'_{AB} + M'_{BA}) \right\} \quad (3.37)$$

$$\text{or} \quad M''_{AB} = TF_{AB} \left\{ M_r + \sum_r (M'_{AB} + M'_{BA}) \right\} \quad (3.38)$$

The difference in the analysis of a multistorey building frame with horizontal loading compared to that of a frame with vertical loading consists only in the fact that for the former, in the determination of the translation moments, the sum of the rotation moments of all member ends of the storey must also contain storey moment M_r . The following example illustrates the procedure for the analysis of multistorey frame with horizontal loading.

Example 3.10

Determine the member end moments for the two-storey frame shown in Fig.3.27.

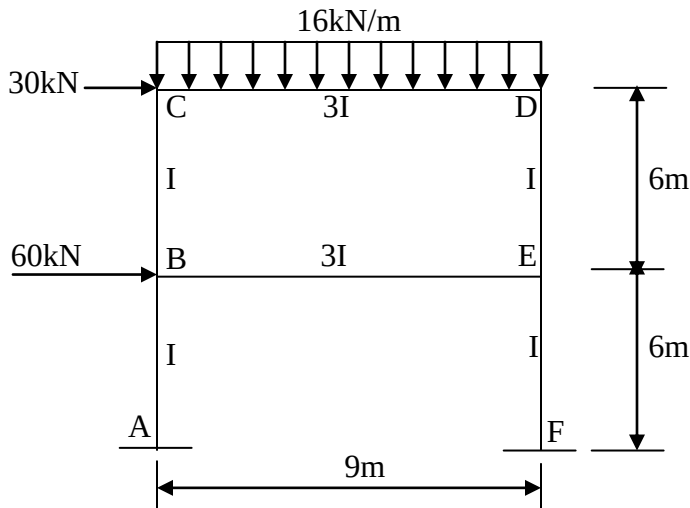


Fig.3.27 Given frame and loading.

SOLUTION

The solution will be in the following order. First the fixed end moments, the storey moments, the K-values, the rotation factors, and the translation factors will be computed. Next, the cycles of calculations for the rotation and the translation moments will be carried out in the following sequence: joint B, joint C, joint D, joint E, storey 2, and storey 1. Note that joints A and F cannot rotate in the given frame and consequently the rotation moment for each of them is equal to zero.

Fixed-end Moments

$$FEM_{CD} = -FEM_{DC} = \frac{16 \times 9^2}{12} = 108 \text{ kNm.}$$

Storey MomentsStorey 2

$$Q_2 = -30 \text{ kN}; \quad M_r = -\frac{Q_r h_r}{3} = -\frac{30 \times 6}{3} = -60 \text{ kNm.}$$

Storey 1

$$Q_1 = -(30 + 60) = -90 \text{ kN}; \quad M_r = -\frac{90 \times 6}{3} = -180 \text{ kNm.}$$

K valuesAt Joints B and E

$$K_{BA} = K_{BC} = K_{ED} = K_{EF} = \frac{I}{6}; \quad K_{BE} = K_{EB} = \frac{3I}{9} = \frac{I}{3};$$
$$\sum K_B = \sum K_E = \frac{I}{6} + \frac{I}{6} + \frac{I}{3} = \frac{2I}{3}$$

At Joints C and D

$$K_{CB} = K_{DE} = \frac{I}{6}; \quad K_{CD} = K_{DC} = \frac{I}{3}$$
$$\sum K_C = \sum K_D = \frac{I}{6} + \frac{I}{3} = \frac{I}{2}$$

Columns

$$\sum K_{col} = \frac{I}{6} + \frac{I}{6} = \frac{I}{3}$$

Rotation Factors, RF

At Joints B and E

$$RF_{BA} = RF_{BC} = RF_{EF} = RF_{ED} = -\frac{1}{2} \times \frac{K_{BA}}{\sum K_B} = -\frac{1}{2} \times \frac{I}{6} \times \frac{3}{2I} = -\frac{1}{8}$$
$$RF_{BE} = RF_{EB} = -\frac{1}{2} \times \frac{K_{BE}}{\sum K_B} = -\frac{1}{2} \times \frac{I}{3} \times \frac{3}{2I} = -\frac{1}{4}.$$

At Joints C and D

$$RF_{CB} = RF_{DE} = -\frac{1}{2} \times \frac{K_{CB}}{\sum K_C} = -\frac{1}{2} \times \frac{I}{6} \times \frac{2}{I} = -\frac{1}{6}.$$
$$RF_{CD} = RF_{DC} = -\frac{1}{2} \times \frac{K_{CD}}{\sum K_C} = -\frac{1}{2} \times \frac{I}{3} \times \frac{2}{I} = -\frac{1}{3}.$$

Translation Factor, TF

There are two storeys and only two columns in each storey. Each of the two columns have the same K value of I/6. Therefore, the translation factor for each column is the same. Thus:

$$TF_{BA} = TF_{BC} = TF_{EF} = TF_{ED} = -\frac{3}{2} \times \frac{K_{BA}}{\sum K_{col}} = -\frac{3}{2} \times \frac{I}{6} \times \frac{3}{I} = -\frac{3}{4}$$

The fixed end moments, storey moments, rotation and translation factors are all indicated on the computation scheme shown in Fig.3.28.

Cycle 1

Joint B

The rotation moments and translation moments are initially assumed to be zero.

Sum of FEM = 0

Sum of rotation moments at far ends:

A = 0 (fixed end)

$$C = 0 \text{ (assumed)}$$

$$E = 0 \text{ (assumed)}$$

$$\text{Translation moment of upper column BC} = 0 \text{ (assumed)}$$

$$\text{Translation moment of lower column BA} = 0 \text{ (assumed)}$$

$$\textbf{Total} = 0$$

Application of eqn (3.28) gives:

$$M'_{BA} = M'_{BC} = M'_{BE} = 0$$

Joint C

$$\text{Sum of FEM} = 108$$

$$\text{Rotation moment at B} = 0$$

$$\text{Rotation moment at D} = 0$$

$$\text{Translation moment of column CB} = 0$$

$$\textbf{Total} = 108$$

$$\text{Therefore: } M'_{CB} = -\frac{1}{6}(108) = -18;$$

$$M'_{CD} = -\frac{1}{3}(108) = -36.$$

Joint D

$$\text{Sum of FEM} = -108$$

$$\text{Rotation moment at C} = -36$$

$$\text{Rotation moment at E} = 0 \text{ (assumed)}$$

$$\text{Translation moment of column DE} = 0$$

$$\textbf{Total} = -144$$

$$\text{Therefore: } M'_{DC} = -\frac{1}{3}(-144) = 48;$$

$$M'_{DE} = -\frac{1}{6}(-144) = 24.$$

Joint E

$$\text{Sum of FEM} = 0$$

$$\text{Rotation moment at D} = 24$$

$$\text{Rotation moment at B} = 0$$

$$\text{Rotation moment at F} = 0$$

$$\text{Translation moment of upper column DE} = 0$$

$$\text{Translation moment of lower column EF} = 0$$

$$\textbf{Total} = 24$$

$$\text{Therefore: } M'_{ED} = M'_{EF} = -\frac{1}{8}(24) = -3;$$

$$M'_{EB} = -\frac{1}{4}(24) = -6.$$

We now continue the first cycle by calculating the translation moments, first for storey 2, and then for storey 1.

Storey 2

Storey moment	= - 60
Rotation moment at upper end of column BC	= -18
Rotation moment at lower end of column BC	= 0
Rotation moment at upper end of column DE	= 24
Rotation moment at lower end of column DE	= -3
Total	= -57

Application of eqn (3.29) yields:

$$M''_{BC} = M''_{DE} = -\frac{3}{4}(-57) = 42.75.$$

Storey 1

Storey moment	= - 180
Rotation moment at upper end of column AB	= 0
Rotation moment at lower end of column AB	= 0
Rotation moment at upper end of column EF	= -3
Rotation moment at lower end of column EF	= 0
Total	= -183

Application of eqn (3.29) yields:

$$M''_{AB} = M''_{EF} = -\frac{3}{4}(-183) = 137.25.$$

Cycle 2

Joint B

Sum of FEM = 0

Sum of rotation moments at far ends:

A = 0 (fixed end)

C = -18

E = -6

Translation moment of upper column BC = 42.75

Translation moment of lower column BA = 137.25

Total = 156

Therefore: $M'_{BC} = M'_{BA} = -\frac{1}{8}(156) = -19.5;$

$$M'_{BE} = -\frac{1}{4}(156) = -39.$$

Joint C

Sum of FEM	= 108
Rotation moment at B	= -19.5
Rotation moment at D	= 48
Translation moment of column CB	= 42.75
Total	= 179.25

Therefore: $M'_{CB} = -\frac{1}{6}(179.25) = -29.88;$

$M'_{CD} = -\frac{1}{3}(179.25) = -59.75.$

Joint D

Sum of FEM	= - 108
Rotation moment at C	= -59.75
Rotation moment at E	= -3
Translation moment of column DE	= 42.75
Total	= -128

Therefore: $M'_{DC} = -\frac{1}{3}(-128) = 42.67;$

$M'_{DE} = -\frac{1}{6}(-128) = 21.33.$

Joint E

Sum of FEM	= 0
Rotation moment at D	= 21.33
Rotation moment at B	= -39
Rotation moment at F	= 0
Translation moment of upper column ED	= 42.75
Translation moment of lower column EF	= 137.25
Total	= 162.33

Therefore: $M'_{ED} = M'_{EF} = -\frac{1}{8}(162.33) = -20.29;$

$M'_{EB} = -\frac{1}{4}(162.33) = -40.58.$

Storey 2

Storey moment	= - 60
Rotation moment at upper end of column BC	= -29.88
Rotation moment at lower end of column BC	= -19.5
Rotation moment at upper end of column DE	= 21.33
Rotation moment at lower end of column DE	= -20.29
Total	= -108.34

Application of eqn (3.29) yields:

$$M''_{BC} = M''_{ED} = -\frac{3}{4}(-108.34) = 81.26.$$

Storey 1

Storey moment	= - 180
Rotation moment at upper end of column AB	= -19.5
Rotation moment at lower end of column AB	= 0
Rotation moment at upper end of column EF	= -20.29
Rotation moment at lower end of column EF	= 0
Total	= -219.79

Application of eqn (3.29) yields:

$$M''_{BA} = M''_{EF} = -\frac{3}{4}(-219.79) = 164.84.$$

Cycle 3

Joint B

Sum of FEM	= 0
Sum of rotation moments at far ends:	
	A = 0 (fixed end)
	C = -29.88
	E = -40.58
Translation moment of upper column BC	= 81.26
Translation moment of lower column BA	= 164.84
Total	= 175.64

Therefore: $M'_{BC} = M'_{BA} = -\frac{1}{8}(175.64) = -21.96;$

$$M'_{BE} = -\frac{1}{4}(175.64) = -43.91.$$

Joint C

Sum of FEM	= 108
Rotation moment at B	= -21.96
Rotation moment at D	= 42.67
Translation moment of column CB	= 81.26
Total	= 209.97

Therefore: $M'_{CB} = -\frac{1}{6}(209.97) = -35.0;$

$$M'_{CD} = -\frac{1}{3}(209.97) = -69.99.$$

Joint D

Sum of FEM	= - 108
Rotation moment at C	= -69.99
Rotation moment at E	= -20.29
Translation moment of column DE	= 81.26
Total	= -117.02

Therefore: $M'_{DC} = -\frac{1}{3}(-117.02) = 39.01;$
 $M'_{DE} = -\frac{1}{6}(-117.02) = 19.50.$

Joint E

Sum of FEM	= 0
Rotation moment at D	= 19.50
Rotation moment at B	= -43.91
Rotation moment at F	= 0
Translation moment of upper column ED	= 81.26
Translation moment of lower column EF	= 164.84
Total	= 221.69

Therefore: $M'_{ED} = M'_{EF} = -\frac{1}{8}(221.69) = -27.71;$
 $M'_{EB} = -\frac{1}{4}(221.69) = -55.42.$

Storey 2

Storey moment	= - 60
Rotation moment at upper end of column BC	= -35.0
Rotation moment at lower end of column BC	= -21.96
Rotation moment at upper end of column DE	= 19.50
Rotation moment at lower end of column DE	= -27.71
Total	= -125.17

Application of eqn (3.29) yields:

$$M''_{BC} = M''_{ED} = -\frac{3}{4}(-125.17) = 93.88.$$

Storey 1

Storey moment	= - 180
Rotation moment at upper end of column AB	= -21.96
Rotation moment at lower end of column AB	= 0
Rotation moment at upper end of column EF	= -27.71
Rotation moment at lower end of column EF	= 0
Total	= -229.67

Application of eqn (3.29) yields:

$$M''_{BA} = M''_{EF} = -\frac{3}{4}(-229.67) = 172.25.$$

The values of the rotation and translation moments for 7 cycles are shown in the computation scheme of Fig.3.28. The values for the 7th cycle are taken as the final values with which the final moments are computed using eqn (3.30). The computations for the final moments are shown in Fig.3.29 while the final end moments are shown on the body of the frame in Fig.3.30.

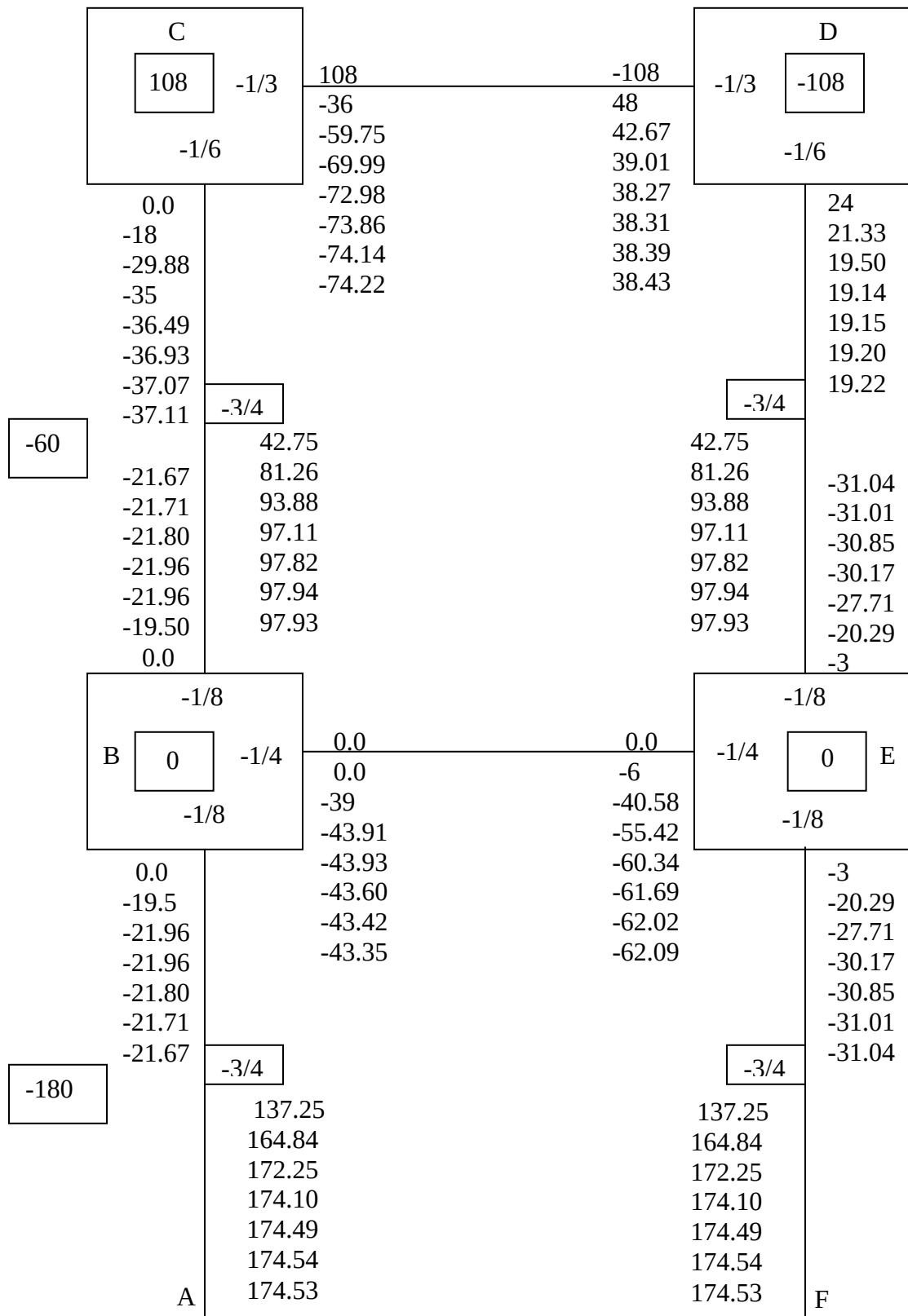


Fig.3.28 Computation scheme for the frame of Example 3.10

	C	108	-108	D	
		-74.22	38.43		
-37.11		-74.22	38.43		19.22
-37.11		38.43	-74.22		19.22
-21.67		-2.01	-105.36		-31.04
97.93					97.93
2.04					105.33
	97.93		97.93		
17.48					55.07
97.93					97.93
-37.11					19.22
-21.67					-31.04
-21.67					-31.04
	B	-43.35	-62.09	E	
		-43.35	-62.09		-31.04
-21.67		-62.09	-43.35		-31.04
-21.67		-148.79	-167.53		0.0
0.0					174.53
174.53					112.45
131.19					
	174.53		174.53		
152.86					143.49
174.53					174.53
-21.67					-31.04
0.0	A			F	0.0

Fig.3.29 Computations for the final moments.

	-2.01	-105.36	
2.04			105.33
17.48	-148.79	-167.53	55.07
131.19			112.45
152.86			143.49

Fig.3.30 Final moments for the frame of Example 3.10.