Chapter Two: Yield Line Theory for Slabs

2.1 Introduction:

Rectangular one way or two way slabs under normal uniform loading can be analyzed and then designed using coefficients obtained from Tables published for this purpose.

In a situation where irregular shapes, varied support conditions, presence of openings, varied loading and more complex conditions are encountered, the established theory of elasticity or plasticity cannot be employed straight. For these circumstances, the yield line theory is found useful.

The yield line theory is an ultimate load method of analysis of slab, i.e. the BM at the verge of collapse is used as the basis for design. At collapse loads, an under reinforced slab begins to crack with the reinforcement yielding at points of high moment. The crack lines or the yield lines propagate with the increase in deflection until the slab is broken in to a number of segments.

A yield line is a line in the plane of the slab across which reinforcing bars have yielded and about which excessive deformation (*plastic rotation*) under constant limit moment (*ultimate moment*) continues to occur leading to failure.

2.2 Upper and lower bound theorem:

Plastic analysis methods such as the yield line theory derived from the general theory of structural plasticity, which states that the ultimate collapse load of a structure lies between two limits, an upper bound and a lower bound of the true collapse load.

The lower bound and upper bound theorem, when applied to slabs, can be stated as follows:

Lower bound theorem: If, for a given external load, it is possible to find a distribution of moments that satisfies equilibrium requirements, with the moment not exceeding the yield moment at any location, and if the boundary conditions are satisfied, then the given load is a lower bound of the true carrying capacity.

Upper bound theorem: If, for a small increment of displacement, the internal work done by the slab, assuming that the moment at every plastic hinge is equal to the yield moment and that boundary conditions are satisfied, is equal to the external work done by the given load for that same small increment of displacement, then that load is an upper bound of true carrying capacity.

If the lower bound conditions are satisfied, the slab can certainly carry the given load, although a higher load may be carried if internal distributions of moment occur. If the upper bound conditions are satisfied, a load greater than the given load will certainly cause failure, although a lower load may produce collapse if the selected failure mechanism is incorrect in any sense. Accordingly the yield line method of analysis for slabs is an upper bound method, and consequently the failure load calculated for a slab with known flexural resistance may be higher than the true value.

The yield line phenomenon involves:

- a slab under increasing loads where cracking and reinforcement yielding occur in the most highly stressed zone (i.e. around maximum moment)
- the highly stressed zone normally acts as a plastic hinge where the subsequent loads are distributed to other region of the slab
- cracks develop forming patterns of yield lines until a mechanism is formed,
- collapse is then indicated by increasing deflection under constant load

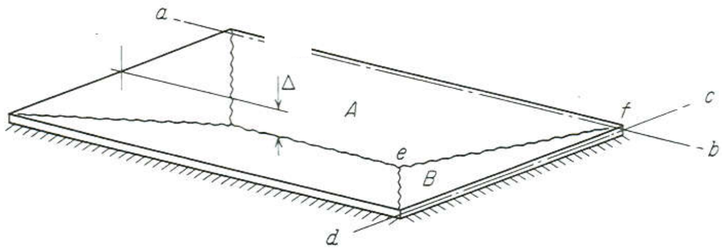


Fig. 2.1 Deformation of slab with yield lines

<u>Sign convention</u>

positive yield line
negative yield line
axis of rotation
unsupported edge

xxxxxxxxx built in edge
simply supported edg
point load
column support

Characteristic features of yield lines:

- yield lines are generally straight and end at a slab boundary or the intersection of other yield lines
- axes of rotation generally lie along lines of support
- Axes of rotation pass over column supports
- For a mechanism to develop, the yield line may pass through the intersection of the axes of rotation of adjacent segments

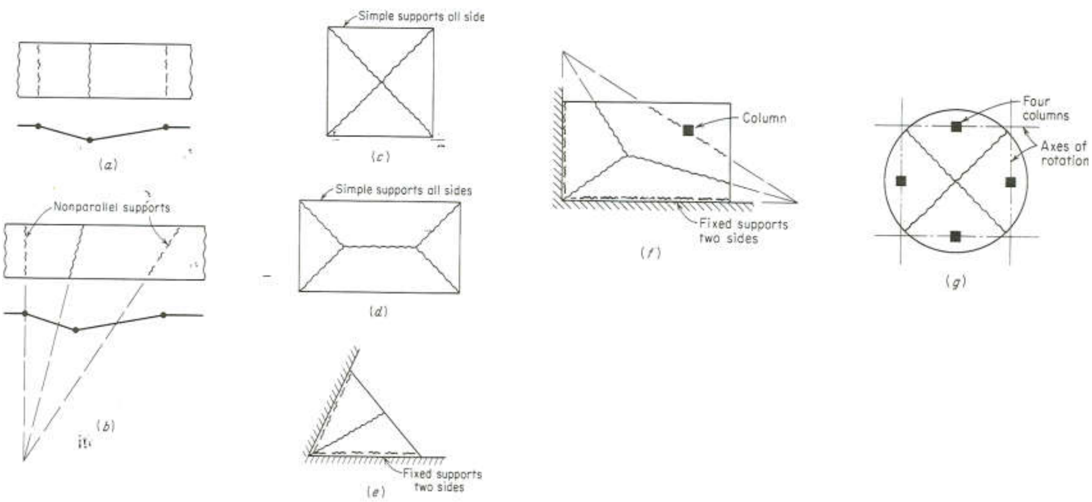


Fig. 2.2 Typical Yield line patterns

2.3 Method of yield line analysis

There are two methods of yield line analysis of slabs:

(1) the equilibrium method
(2) the virtual work method

In either method, a YL pattern is assumed so that a collapse mechanism is produced. Then for that failure mechanism, the geometric parameters that define the exact location and orientation of the yield lines are determined and also the relation between applied loads and resisting moments is solved.

It is necessary to investigate all possible mechanisms for any slab to confirm that the correct solution, giving the lowest failure load, has been found. For example the following rectangular slab may fail by either of the two mechanisms shown.

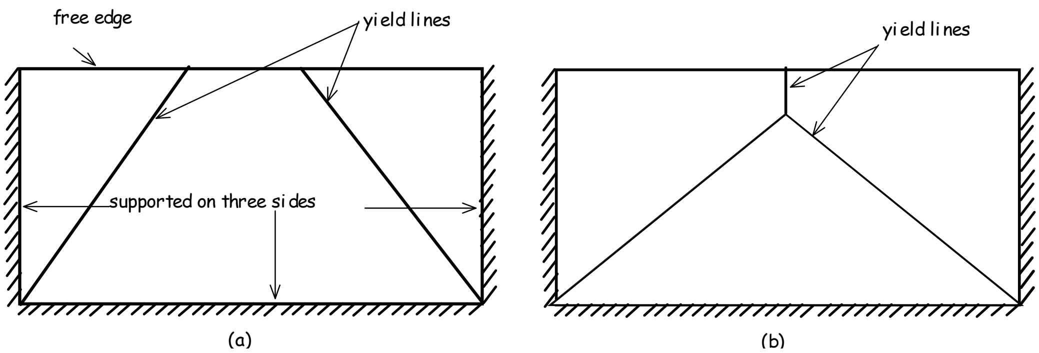


Fig. 2.3 Alterative mechanisms for a slab supported on three sides

(a) Equilibrium method of Analysis

It makes use of the equilibrium equations for individual segment to obtain the collapse load.

The FBD represented by each collapsing segment is in equilibrium under

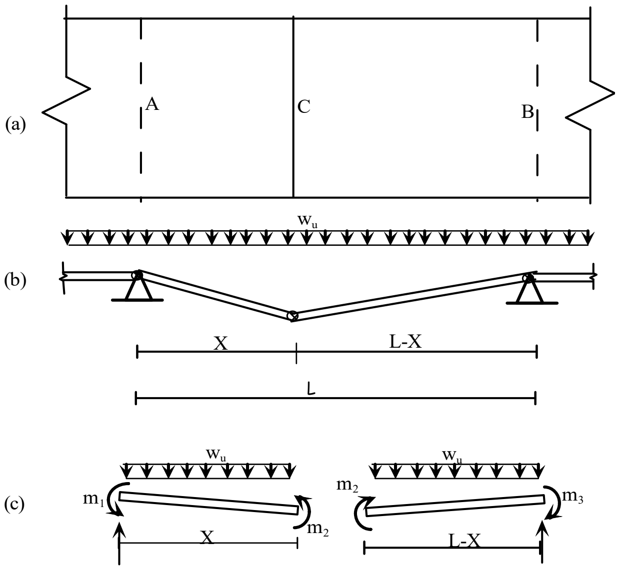
- applied loads,
- yield moments and
- Reactions or shears along support lines.

The method of segment equilibrium should not be confused with a true equilibrium method such as then strip method. A true equilibrium method is a lower bound method of analysis.

Essentially, the yield lines form at lines of maximum moment where neither shear nor torsion is typically present at positive yield lines.

For demonstration purpose consider the one way slab uniformly loaded and is continuous as shown in Fig. below.

Let the slab with span L is reinforced to provide resistance of m_2 KN.m per m through the span and m_1 and m_3 KN.m per m at the two supports. Suppose it is desired to determine the collapse load w_u.



For a known yield moments m_1, m_2, m_3, a trail location of the positive yield line is assumed.

Normally for a given loading and correct collapse pattern the solution is unique. However if a different pattern is assumed, this solution can describe nothing which pattern is the governing one. Hence, it becomes essential to use the energy approach in completed problem for further verification.

For the problem posed, consider the FBD in Figure (c);

From left segment:

$$\sum M_A = 0 \quad \Rightarrow m_1 + m_2 - {w_u x^2}/{2} = 0$$

From right segment:

$$\sum MB = 0 \quad \Rightarrow m_2 + m_3 - {w_u (l-x)^2}/{2} = 0$$

Solving for w_u from the two expressions and equating, one may obtain a practical solution for x as:

$$x = \frac{m_1 + m_2}{m_3 - m_1} L \left\{ -1 + \sqrt{1 + \frac{m_3 - m_1}{m_1 + m_2}} \right\}$$

For instance, when L = 3m, $m_1 = m_2$ = 7KN.m/m and m_3 = 10KN.m/m

X = 1.427m and w_u = 13.75 KN/m^2

When $m_1 = m_3$ = 10KN.m/m and m_2 = 7KN.m/m, L = 3m

X = 1.5m and w_u = 15.11 KN/m^2

(b) Virtual work method of Analysis

Based on principle that work done by external forces in undergoing a small virtual displacements is equal to the internal virtual work done in rotations along yield lines, the ultimate load which the slab can sustain is determined. In other words, the work during small motion of collapse mechanism is equal to the work absorbed by the plastic hinges formed along the yield lines. In here, the segment of the slab with in the yield lines is assumed to go through rigid body displacement with the collapse load acting on the structure.

$$W_E = W_I$$

i.e. Work done by external forces = Energy absorbed by the hinges (internal work)

To develop suitable expression for each work, let w_u be the uniform distributed load,

$$W_E = \iint w_u dxdy\delta_{xy} \quad = \sum W_u \Delta$$

where: δ_{xy} = virtual displacement at load point considered
W_u = resultant of the load on each segment
Δ = the corresponding displacement at centroid of the load in each segment

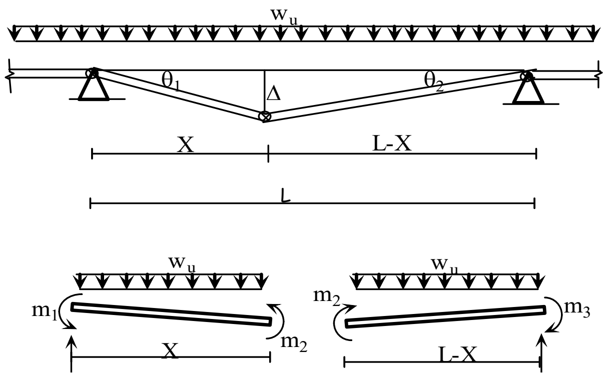
Since the relative rotation of the surface takes place about yield line, one may obtain

$$W_I = \sum m_{un}\theta_n L_y$$

Where: m_{un} = the ultimate moment across any yield line.
L_y = length of yield line
Q_n = relative rotation of the two adjacent plates perpendicular to the yield line

Thus, $W_E = W_I \quad \Rightarrow \quad \sum W_u \Delta \quad = \quad \sum m_{un}\theta_n L_y$

For demonstration purpose, consider the one-way slab of the previous example.



The slab is reinforced at left and right supports, and in span in such a way that the capacities of the respective sections are m_1, m_3 and m_2.

Let Δ be the plastic deflection at x-distance from left hand support to positive y_L

External work $W_E = \frac{wx\Delta}{2} + \frac{w(L-x)\Delta}{2} = \frac{1}{2}w_u \Delta L$ per m strip

Internal work $W_I = \sum m\theta L_y = (m_1 + m_2)\theta_1 + (m_2 + m_3)\theta_2$ per m strip

For small deflection $\theta_1 = \frac{\Delta}{x}$ and $\theta_2 = \frac{\Delta}{L-x}$

$$W_I = (m_1 + m_2)\frac{\Delta}{x} + (m_2 + m_3)\frac{\Delta}{L-x}$$

By principle of virtual work; $W_E = W_I$ which simplifies to give:

$$w_u = \frac{2\left[L(m_1 + m_2) - x(m_1 - m_3)\right]}{xL(L-x)}$$

To obtain the minimum collapse load $\frac{dw_u}{dx} = 0$

$\Rightarrow \quad -(m_1 - m_3)xL(L-x) - \left[L(m_1 + m_2) - x(m_1 - m_3)\right](L^2 - 2xL) = 0$ This gives the practical values of x such that:

$x = \frac{m_1 + m_2}{m_3 - m_1} L \left[-1 + \sqrt{1 + \frac{m_3 - m_1}{m_1 + m_2}} \right]$ Thus, identical solution to the equilibrium methods is obtained

2.4 Moments along Skewed yield lines

Consider a two way slab orthogonally reinforced where the yield lines are inclined at an angle α with one of the principal axes.

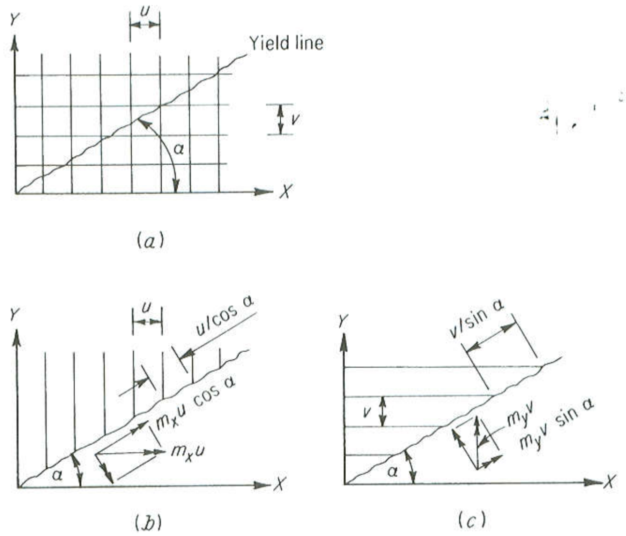


Fig. 2.4

The above Fig. shows an orthogonal gird of reinforcement having moment resistance m_y per unit length about the y axis and moment resistance m_x per unit length about x axis.

From Fig. 3.3b:

- the resisting moment per unit length along the α axis provided by the y direction bars is:

$$m_{\alpha y} = \frac{m_x u \cos\alpha}{u/\cos\alpha} = m_x \cos^2\alpha$$

- the resisting moment per unit length perpendicular to α axis provided by the y direction bars is:

$$m_{\perp\alpha y} = \frac{m_x u \sin\alpha}{u/\cos\alpha} = m_x \cos\alpha \sin\alpha$$

From Fig. 3.3c:

- the resisting moment per unit length along the α axis provided by the x direction bars is:

$$m_{\alpha x} = \frac{m_y v \sin\alpha}{v/\sin\alpha} = m_y \sin^2\alpha$$

- the resisting moment per unit length perpendicular to α axis provided by the x direction bars is:

$$m_{\perp\alpha x} = \frac{m_y v \cos\alpha}{v / \sin\alpha} = m_y \cos\alpha \sin\alpha$$

Thus for the combined set of bars:

- the resisting normal moment per unit length measured along the α axis is:

$$m_\alpha = m_x \cos^2\alpha + m_y \sin^2\alpha \qquad (1^*)$$

- the resisting torsional moment per unit length measured along the α axis is:

$$m_{\perp\alpha} = m_x \cos\alpha \sin\alpha - m_y \cos\alpha \sin\alpha \qquad (2^*)$$

Definition:

An isotropically reinforced slab is one in which the ultimate moment per unit length of the slab is the same in two orthogonal directions

An orthotropically reinforced slab is one in which the ultimate moment per unit length of the slab is different in the two orthogonal directions

If a slab is isothropically reinforced with $m_x = m_y = m$, eqns(1*) and (2*) become:

$$m_\alpha = m$$

$$m_{\perp\alpha} = 0$$

Therefore the ultimate moment resistance in an isotropically reinforced slabs in any direction is the same.

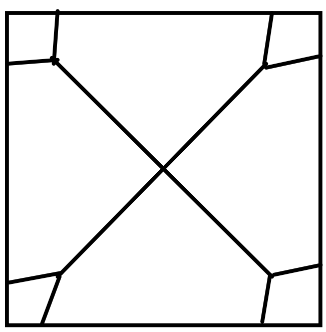
If a slab is orthothropically reinforced with $m_x = m$ and $m_y = \mu m$, eqns(1*) and (2*) become:

$$m_\alpha = m(\cos^2\alpha + \eta \sin^2\alpha)$$

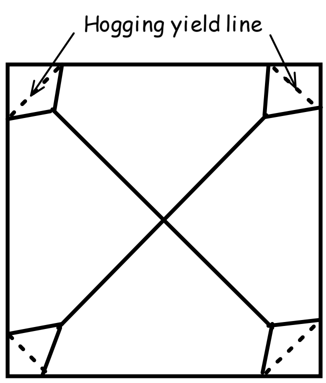
$$m_{\perp\alpha} = m(1-\eta)\sin\alpha\cos\alpha$$

2.5 Effects of restrained corners

Corner lever is the effect of forking of the yield line before reaching the corner. To this effect the following schematic sketches the resulting yield pattern at the corners.



Uplift when the corners are not held or fastened

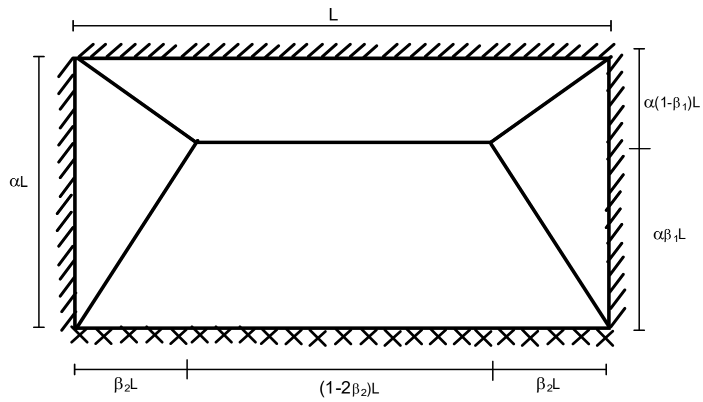


Corners held down but not well reinforced for negative moment

The triangular segment fails to form when the negative reinforcement is large and hence, the simple diagonal yield line in to the corner is correct with out modification.

2.6 Slabs with more than one variable

So far the slabs consider have only one variable dimension which defines the yield line mechanism. When the slab has more than one variable, the work equation, together with equations obtained by differentiating with respect to each unknown, give the necessary expressions to obtain solution. This can be illustrated using the following example.



In this case first develop suitable expression from the work relation for w_u in terms of β_1 & β_2. Then

$$\frac{\partial w_u}{\partial \beta_1} = 0$$

$$\frac{\partial w_u}{\partial \beta_2} = 0$$

Will provide two additional equations to make the problem solvable.