CHAPTER 6

INTRODUCTION TO MATRIX METHODS OF STRUCTURAL ANALYSIS

GENERAL

METHODS OF ANALYSIS

There are two distinct approaches to the analysis of indeterminate structures, namely, the force and the displacement methods.

FORCE METHOD

In the force method the redundant retraining forces in the given statically indeterminate structure are removed (or released), resulting in a basic statically determinate system, known as the primary structure. The primary structure is analysed using equilibrium equations of statics and the displacements (deflections and/or slopes) in the direction of the released redundants are determined. The resulting inconsistencies in geometry at the releases are then corrected by introducing additional forces at the releases. These unknown additional forces are determined by imposing geometric compatibility on the structure at the releases. Having thus determined the redundant forces, the forces in the given structure are evaluated by the superimposition of the forces in the primary structure due to given loading and the forces in the primary structure due to the redundants. The above method is known as force method because forces are the independent variables (or the unknowns). From the above explanation, it is understood that equilibrium condition is satisfied at the initial stage of the analysis and the geometric compatibility condition is imposed at a later stage to complete the analysis. The force method is therefore also known as compatibility method. Examples of classical formulations of the force method include the method of consistent deformations, and the method of least work.

DISPLACEMENT METHOD

In the displacement method, artificial restraints are introduced to prevent the displacement of the structure at the joints. The forces required to hold the joints at the restrained position are determined. However, in the restrained structure equilibrium of forces are not satisfied at the joints. Displacements are therefore allowed to occur at the joints until the artificial restraining forces have disappeared. The displacements are then determined by satisfying the joint equilibrium conditions. Once the joint displacements are obtained the forces in the structure are determined by summing up the forces in the restrained structure and the forces due to displacements at the joints. The displacement method is so called because the independent variables (or the unknowns) in the governing equations are displacement quantities. From the above description, it is seen that the geometric compatibility is satisfied at the initial stage of analysis while equilibrium is later imposed at the joints. The displacement method is therefore also known as the equilibrium method. Examples of classical formulations of the displacement method include the slope-deflection and the moment distribution methods.

MATRIX METHODS OF ANALYSIS

The two methods of analysis of indeterminate structures described above in their classical formulations, in general, require the solution of simultaneous equations relating forces to displacements. In the formulations of the force method the analysis results in a number of simultaneous equations equal to the number of redundants (or degree of statical indeterminacy). In the displacement method, the analysis results in a number of simultaneous equations equal to the number of independent displacements (or degree of kinematical indeterminacy).

The analysis of large structures using the hand computation techniques of the classical methods can be both tedious and time consuming. With the availability of the micro-computer, the analysis of large structures is more conveniently carried out using computers, with the aid of software programs based on matrix methods of analysis. Matrix methods are similar to the classical methods in that here also the force-displacement characteristic of a structure is still described using simultaneous equations. However, the matrix methods differ from the classical methods in two important ways. The first is that in the matrix methods all calculations are carried out using matrix algebra, which makes possible a formulation of the solution as a series of matrix operations suitable for a computer. The second is that in the matrix methods, a structure, however complex, is subdivided into a series of discrete or ***finite elements***, and its force-displacement characteristics are assembled from the force-displacement characteristics of the finite elements, thereby greatly simplifying the analysis. Just as there are the classical formulations of the force and the displacement methods, they have their matrix formulations, namely, the flexibility and the stiffness methods.

The flexibility method is a force (or compatibility) method in its matrix formulation. On the other hand, the stiffness method is a displacement (or equilibrium) method in its matrix formulation. Accordingly, in the flexibility method the independent variables are the forces while in the stiffness method the independent variables are the displacements. The flexibility method is so called because the relationship between force and displacement is expressed using flexibility influence coefficients. On the other hand, in the stiffness method, the force-displacement relationship is expressed through the stiffness influence coefficients. The flexibility and the stiffness influence coefficients will be explained a little later in this chapter.

STATICAL AND KINEMATIC INDETERMINACY

STATICAL INDETERMINACY

Recall that a structure is statically indeterminate when there are more reaction components and/or members present than are necessary for the stability of the structure. The degree of indeterminacy is equal to the number of reaction components and/or internal force components present in members, in excess of those necessary for external and/or internal stability. If these excess reaction and/or internal force components are released, the structure is transformed into a statically determinate and yet stable system. These excess reaction and/or internal force components are known as redundants. In the force method it is these redundants that are released to obtain a primary statically determinate structure.

KINEMATIC INDETERMINACY

When a structure is subjected to a system of forces, the overall behaviour of the members of the structure may be defined by the displacement of the joints. The joints undergo displacements in the form of translation and rotation. A system of joint displacements is independent if each displacement can be varied arbitrarily and independently of the other displacements. The number of independent joint displacements (rotation and translation) in a structure necessary to define the deformed shape of the structure when subjected to arbitrary loading is known as the ***number of degrees of freedom***, or the ***degree of kinematic indeterminacy***. If a structure has ***n*** number of independent joint displacements required to describe all possible displacements for any loading conditions, the structure is said to be kinematically indeterminate to the n^{th} degree. When these displacements are set to zero, or to some predetermined values, the structure then becomes kinematically determinate. An example of the case where the displacement value is set to predetermined value is a case where settlement of a support is prescribed. In the displacement method, the possible independent joint displacements are artificially restrained in order to make the structure ***kinematically determinate***.

Examples

Consider the beam shown in Fig.5.1(a). The beam is fixed at A and therefore cannot undergo any displacement at that support. At B, the beam is restrained vertically. Although it is not restrained horizontally at B, axial deformations in beams are small and therefore usually neglected. Therefore the only unknown displacement is the joint rotation θ_B at joint B. Thus, the beam is 1degree kinematically indeterminate. Since the rotation θ_B is a kinematic redundant, if it is removed, i.e., if θ_B is set equal to zero, the resulting primary structure of Fig.5.1(b) is said to be kinematically determinate.

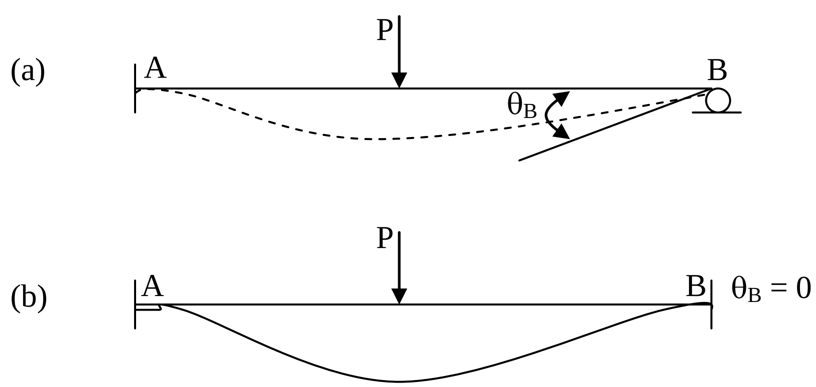


Fig.5.1: (a) 1degree kinematically indeterminate beam;
(b) Kinematically determinate beam.

The beam of Fig.5.1(a) is also 1degree statically indeterminate because there is one unknown force component in excess of the possible number of equilibrium equations. If however the fixed support at A is replaced by a hinge, the degree of statical indeterminacy is reduced by one, making the beam to become statically determinate, since an additional equilibrium condition is introduced. Although the introduction of a hinge at A will reduce the number of statical indeterminacy by one, it will increase the kinematic indeterminacy also by one since an independent rotation at A will become possible. In general, an introduction of a displacement release decreases the statical indeterminacy and increases the kinematic indeterminacy.

As a further illustration of the concept of kinematic indeterminacy (or degrees of freedom) consider the frame of Fig.5.2.

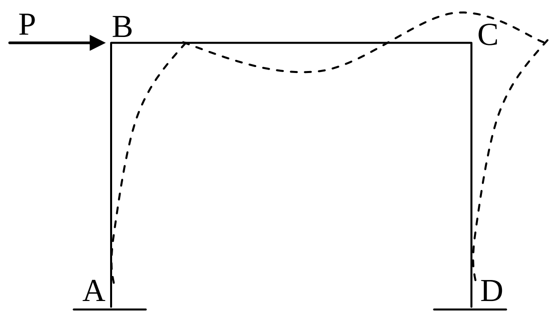


Fig.5.2: 3 degrees kinematically indeterminate frame.

This frame is fixed at supports A and D and therefore there can be no displacements at those points. On the other hand joints B and C are not supported so there can be horizontal and vertical displacements as well as rotation. Thus six displacements (three at each of joints B and C) are altogether needed to completely specify the deformed position of the frame. In general therefore, the frame has six degrees of freedom, or it is said to be 6 degrees kinematically indeterminate. However, if we neglect the axial deformation for each of the columns and assume the horizontal displacements of joints B and C to be equal, then we need just the rotation at each of joints B and C, and the horizontal translation (which is the same for joints B and C) to completely specify the deformed position of the frame. The frame in this case can then be taken to be 3 degrees kinematically indeterminate.

Observe that the above frame is also 3 degrees statically indeterminate. If for instance, the fixed support at D is replaced with a pin, 1more degree of freedom, i.e., rotation, will arise and the frame will become 4 degrees kinematically indeterminate. On the other hand the statical indeterminacy will reduce by 1degree, making it to be 2 degrees statically indeterminate.

Let us consider now an example truss shown in Fig.5.3.

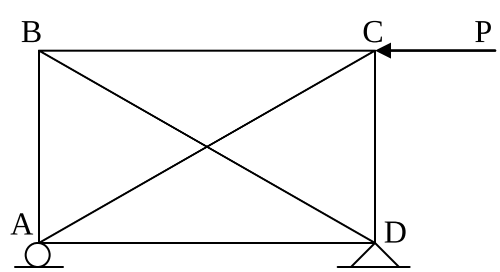


Fig.5.3: 5degrees kinematically indeterminate truss

Each of joints B and C can displace vertically as well as horizontally. Under the assumption of pinned joints, rotation of a joint produces no effect in the members of a truss. Consequently, each of joints B and C has 2 degrees of freedom. Joint A can displace horizontally and so has 1degree of freedom. Joint D can neither displace horizontally nor vertically and so has no degree of freedom. The truss of Fig.5.3 therefore has 5 degrees of freedom, i.e., it is 5 degrees kinematically indeterminate. On the other hand the truss is 1degree statically indeterminate since it has one member in excess of the number required for stability.

CHOICE OF PRIMARY STRUCTURE

In the force method, a primary structure is a statically determinate and stable structure, obtained from a given statically indeterminate structure. On the other hand, in the displacement method, a primary structure is a kinematically determinate structure, obtained from a given kinematically indeterminate structure. The above situations also apply to the flexibility and the stiffness methods since they are force and displacement methods respectively. Regarding the choice of a primary structure (or the lack of choice) when using each of the two methods of analysis, let us consider once again the beam of Fig.5.1(a), now re-presented in Fig.5.4(a).

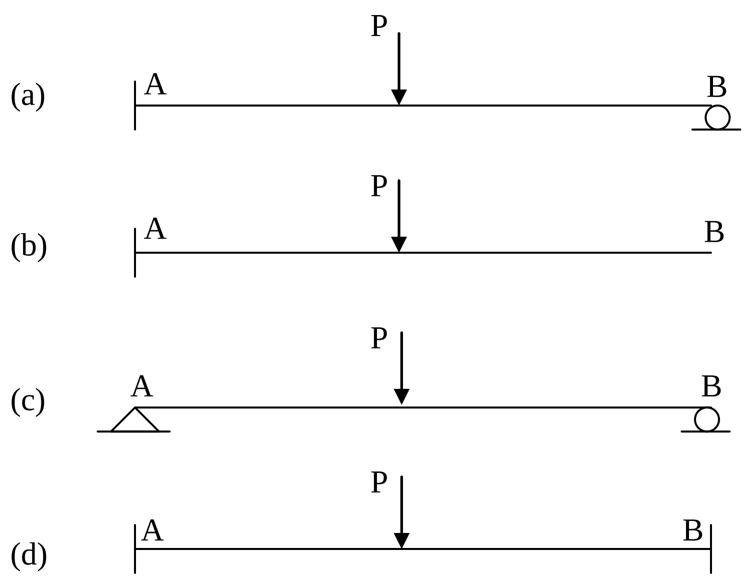


Fig.5.4: (a) Given beam; (b) Statically determinate primary structure with V_B chosen as the redundant; (c) Statically determinate primary structure with M_A chosen as the redundant; (d) Kinematically determinate structure.

In the flexibility method, in order to obtain a primary statically determinate (or primary) structure from a statically indeterminate structure, it is possible to make choice regarding the redundant to choose for release. In the present example it is possible to choose the vertical reaction at support B as the redundant, in which case the primary beam will be as shown in Fig.5.4(b). On the other hand, if the moment at A is chosen as the redundant, the primary beam will be as shown in Fig.5.4(c).

in the stiffness method however, there is no choice in obtaining the primary system since the possible independent displacements are always the same for a given structure. In the present example, the only possible way of making the beam kinematically determinate is to replace the roller at B with a fixed support, as shown in Fig.5.4(d).

COMPARISON OF THE FLEXIBILITY AND THE STIFFNESS METHODS

The method to choose for the analysis of a structure is essentially a matter of computational convenience. When the analysis is performed manually, the most important criterion in selecting out of the two methods is the size of the matrix to be inverted. Thus, when the degree of statical indeterminacy is greater than the degree of kinematic indeterminacy (or number of degrees of freedom), the stiffness method will be preferable because the size of the matrix for inversion will be less than the size of the matrix that will need to be inverted using the flexibility method. On the other hand if the degree of kinematic indeterminacy is greater than the degree of statical indeterminacy, then the flexibility method will be preferable. However, if computer is to be used for the analysis, then the stiffness method is more suitable. This is because in the stiffness method, there is no choice of the unknowns and there is only one possible definite way in which the structure can be restrained. This makes it possible to develop a general computer programme capable of solving all classes of problems. On the other hand, since there is choice regarding the redundants for a structure using the flexibility method, it tends to make a particular programme unique to a particular problem, thereby limiting the effectiveness of use of the possibilities of the computer.

In addition to the foregoing, the stiffness method can be used to analyse both statically determinate and indeterminate structures, using the same procedure. Using the flexibility method however, a different procedure is required for the analysis of each of statically determinate and indeterminate structures.

Since the stiffness method is more systematic and can be more easily implemented on the computer, most commercially available computer programs for structural analysis are based on the stiffness method.

FORMULATIONS OF THE MATRIX METHODS

Depending on whether hand or the computer will be used for the analysis the matrix methods can be presented in two ways. In this text we shall refer to the presentation suitable for manual analysis as the ***basic formulation***, and to the presentation suitable for analysis using the computer as the ***generalised formulation***. In this chapter we shall consider the basic formulation. In this basic formulation, both the flexibility and the stiffness matrices will be developed for the structure at the required coordinates. Although this procedure is not suitable for structures having high degree of indeterminacy and being manually oriented, does not lend itself for computer programming, the understanding of the procedure involved in the formation of the flexibility and the stiffness matrices will be useful for the understanding of the computer-oriented generalised procedure which is employed as a powerful tool in the solution of complex problems. In the generalised formulation, the flexibility and the stiffness matrices of the structure are formed from the matrices of the individual elements that make up the structure.

THE FLEXIBILITY METHOD

Flexibility Influence Coefficient

In the flexibility method, the relationship between the forces and displacements at a structure's coordinates is expressed using flexibility influence coefficients. The flexibility coefficients characterise the behaviour of the structure by specifying the displacement to the applied forces at the structure's coordinates.

To explain the flexibility coefficient consider the cantilever beam shown in Fig.5.5(a). Let it be required to obtain the relationship between force and displacement at cordinate1.

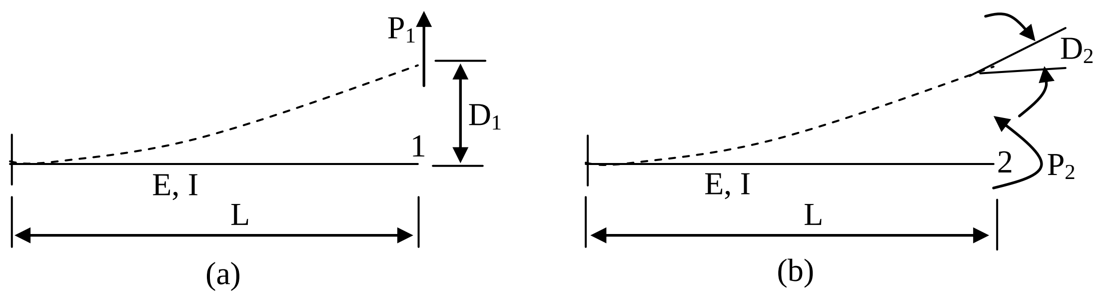


Fig.5.5: (a) Structure with 1coordinate; (b) Structure with 2 coordinates.

If a load P1 is applied at coordinate1, the displacement of the beam at point1 which in this case is a vertical deflection, can be expressed as follows:

$$D_1 = f_{11}P_1 \qquad (5.1)$$

where f_{11} is known as flexibility influence coefficient. It represents the displacement at coordinate1 due to a unit force at coordinate1. The expression for f_{11} can be conveniently obtained using the moment area method for determining displacements. Thus, it can be shown that with $P_1 = 1$,

$$f_{11} = D_1 = \frac{L^3}{3EI}$$

As another illustration, consider the beam of Fig.5.5(b). Let it be required to relate force and displacement at coordinate 2. In this case we will have:

$$D_2 = f_{22}P_2 \qquad (5.2)$$

Again, using moment area method, with $P_2 = 1$,

$$f_{22} = D_2 = \frac{L}{EI}.$$

As before, f_{22}, which is the flexibility influence coefficient, represents the displacement at coordinate 2 due to a unit force at coordinate 2.

Note that here the term force is a generalised force (also known as action) which can be either force or moment. Similarly displacement refers to both deflection and slope. throughout this chapter, these terms shall be employed, denoting force with P and displacement with D. In addition, note that if at a particular point on a structure force-displacement measurements are required for moment and rotation as well as for force and deflection, force and deflection are considered to be applied at the first coordinate (1) while moment and rotation are at the second coordinate (2).

Flexibility Coefficients and Flexibility Matrix

In the cantilever beam illustrations of Fig.5.5, the relationship between force and displacement in each of those cases was required at a single coordinate. Let us now show how such relation ship can be expressed if required for a structure with 2 coordinates. For this purpose let us consider the same cantilever beam, now redrawn and shown in Fig.5.6.

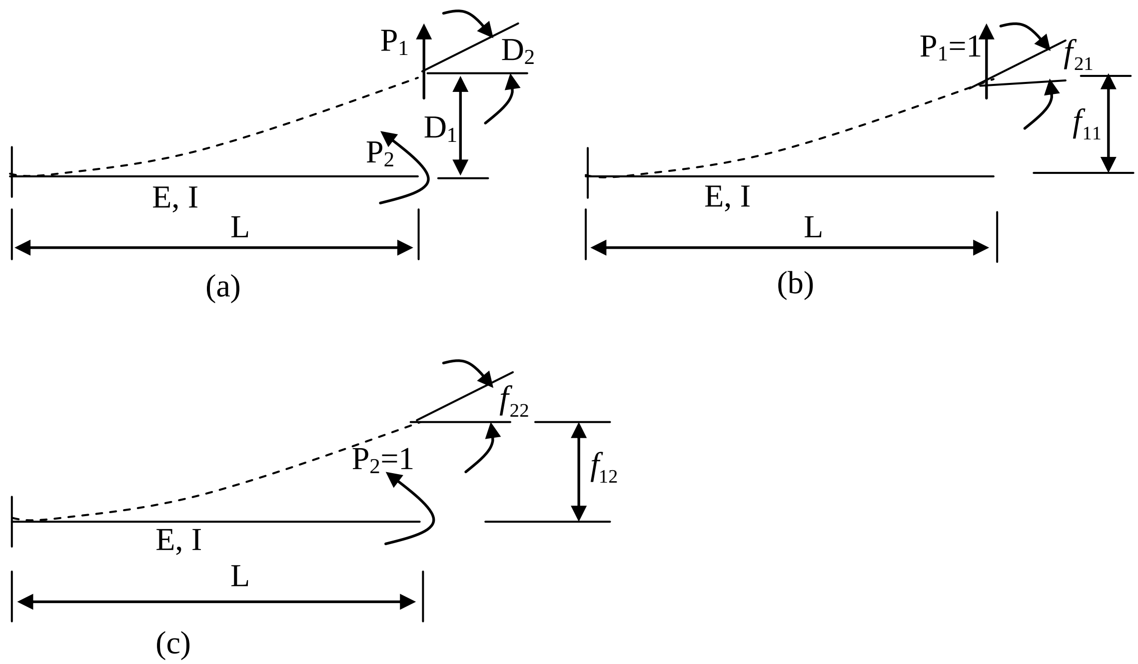


Fig.5.6: (a) Beam with loads and displacements at coordinates 1 and 2; (b) Unit force at coordinate1 only; (c) Unit force at coordinate 2 only.

First, we apply a unit force at coordinate1 only (Fig.5.6(b)) and denote the displacements at coordinates1 and 2 (due to the unit Force) respectively as f_{11} and f_{21}. Next, we apply a unit force at coordinate 2 (due to this force) respectively as f_{12} and f_{22}. When the forces P_1 and P_2 are acting simultaneously (Fig.5.6(a)), the displacements D_1 and D_2 due to these forces can be expressed as follows:

$$D_1 = f_{11}P_1 + f_{12}P_2 \qquad (5.3)$$

and $$D_2 = f_{21}P_1 + f_{22}P_2$$

Expression (5.3) can be written in matrix form as follows:

$$\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \qquad (5.4)$$

In eqn (5.3), f_{11}, f_{12}, f_{21}, and f_{22} are flexibility coefficients. In general for a flexibility coefficient f_{ij}, the first subscript i indicates the coordinate at which the displacement is measured while the second subscript j indicates the coordinate at which the unit force causing the displacement is applied. as explained earlier the flexibility coefficient f_{ij} is defined as the displacement at coordinate i due to a unit force applied at coordinate j.

The matrix

$$F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$ is known as the flexibility matrix for the structure of Fig.5.6. Observe that the elements of the first column of this matrix is obtained by applying a unit force at coordinate1 only while the elements of the second column are obtained by applying a unit force at the second coordinate only. The flexibility coefficients being displacement quantities, can be evaluated using any convenient method for computing displacements. In the case of the cantilever of Fig.5.6, moment area method is particularly convenient.

For a structure with ***n*** coordinates, the force-displacement relationship will be expressed as follows:

$$\begin{aligned} D_1 &= f_{11}P_1 + f_{12}P_2 + \dots\dots\dots\dots + f_{1n}P_n \\ D_2 &= f_{21}P_1 + f_{22}P_2 + \dots\dots\dots\dots + f_{2n}P_n \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ D_n &= f_{n1}P_1 + f_{n2}P_2 + \dots\dots\dots\dots + f_{nn}P_n \end{aligned} \qquad (5.5)$$

In matrix form eqn (5.5) will be:

$$\begin{Bmatrix} D_1 \\ D_2 \\ \dots \\ D_n \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ \dots & \dots & \dots & \dots \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ \dots \\ P_n \end{Bmatrix} \qquad (5.6)$$

or $$D = FP \qquad (5.7)$$

D is the displacement vector and P is the force vector.

The elements of the force vector can be obtained by solving eqn (5.7). Thus:

$$P = F^{-1}D \qquad (5.8)$$

From eqn (5.6), it can be observed that a flexibility matrix F will always have ***n*** rows and ***n*** columns, forming a square matrix of the order $n \times n$.

It should be noted that according to Maxwell's law of reciprocal displacements (i.e., for a linearly elastic structure, the displacement of a point *i* due to a unit load acting at a point *j* is equal to the displacement of point *j* due to a unit load acting at point *i*), flexibility coefficients for linear elastic behaviour have the property of reciprocity. Thus:

$$f_{ij} = f_{ji} \qquad (5.9)$$

Put in other words, the flexibility matrix is a symmetric matrix.

We now show how to generate flexibility matrices by means of the following example.

Example 5.1

Generate the flexibility matrix F for the cantilever beam shown in Fig.5.7(a), in terms of coordinates 1, 2, 3 and 4.

SOLUTION

The moment area method will be used for the computation of the displacements (or flexibility coefficients). Thus, to obtain the displacements in the first column of the matrix F, we apply a unit force at coordinate 1 only (Fig.5.7(b)) and compute the displacements at the coordinates as follows:

$$f_{11} = \frac{1}{2} \times \frac{2L}{EI} \times 2L \times \frac{2}{3} \times 2L = \frac{8L^3}{3EI}$$

$$f_{21} = \frac{1}{2} \times \frac{2L}{EI} \times 2L = \frac{2L^2}{EI}$$

$$f_{31} = \frac{L}{EI} \times L \times \frac{L}{2} + \frac{1}{2} \times \frac{L}{EI} \times L \times \frac{2L}{3} = \frac{5L^3}{6EI}$$

$$f_{41} = \frac{L^2}{EI} + \frac{L^2}{2EI} = \frac{3L^2}{2EI}.$$

Similarly, to obtain the displacements in the second column of the matrix, we apply a unit force (in this case a unit couple) at coordinate 2 only (Fig.5.7(c)) and compute the displacements at the coordinates as follows:

$$f_{12} = \frac{1}{EI} \times 2L \times L = \frac{2L^2}{EI} \qquad \text{(Notice that } f_{12} = f_{21}\text{)}$$

$$f_{22} = \frac{2L}{EI}$$

$$f_{32} = \frac{1}{EI} \times L \times \frac{L}{2} = \frac{L^2}{2EI}$$

$$f_{42} = \frac{L}{EI}$$

In the same manner, to obtain the displacements in the third column of the matrix a unit force is applied at coordinate 3 only (Fig.5.7(d)) and the displacements are computed. To generate the displacement in the fourth column, a unit force is applied at coordinate 4 only (Fig.5.7(e)) and the displacements are computed.

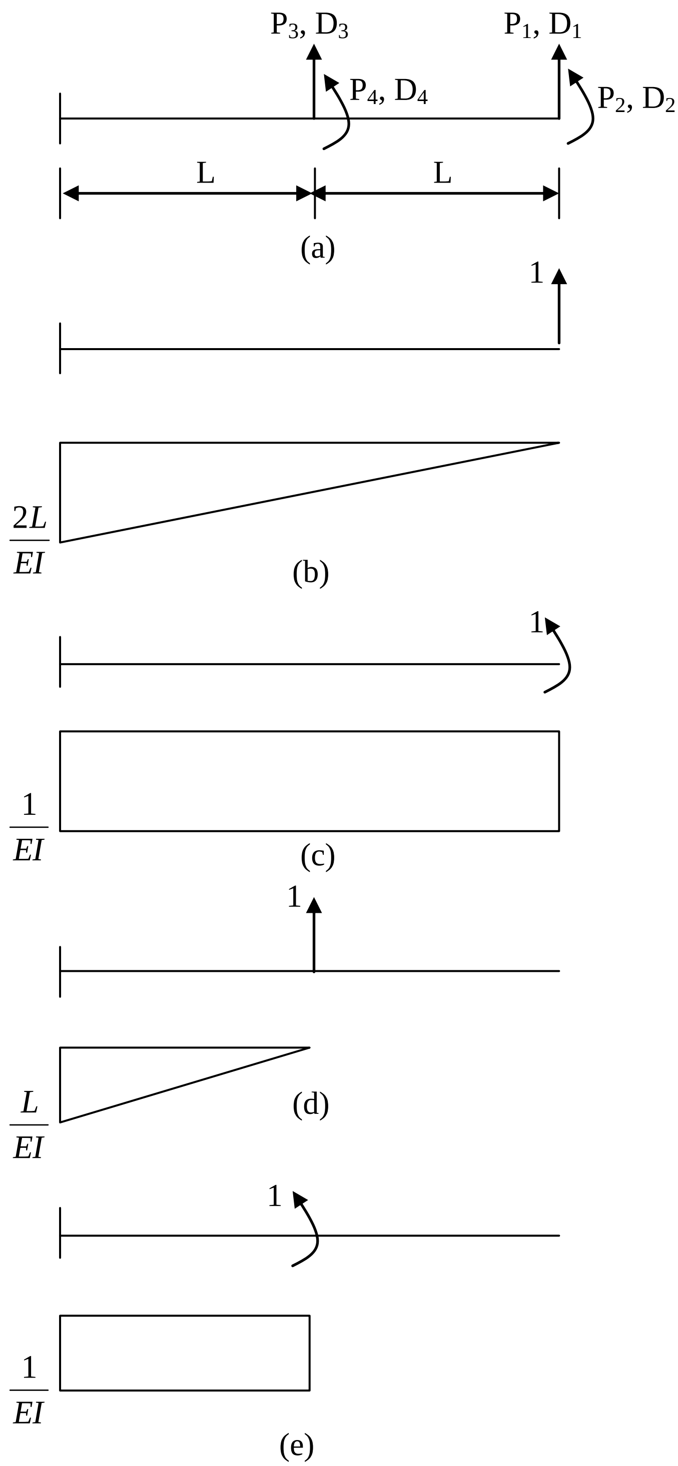


Fig.5.7: (a) Given beam with coordinates; (b) Unit force at coordinate 1, with M/EI diagram; (c) Unit couple at coordinate 2, with M/EI diagram; (d) Unit force at coordinate 3, with M/EI diagram; (e) Unit couple at coordinate 4, with M/EI diagram.

The resulting flexibility matrix is the middle matrix contained in the force-displacement relationship of eqn (5.10).

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} = \frac{1}{EI} \begin{bmatrix} 8L^3/3 & 2L^2 & 5L^3/6 & 3L^2/2 \\ 2L^2 & 2L & L^2/2 & L \\ 5L^3/6 & L^2/2 & L^3/3 & L^2/2 \\ 3L^2/2 & L & L^2/2 & L \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} \qquad (5.10)$$

Observe that in the above flexibility matrix (the middle matrix),

$f_{12} = f_{21}$, $f_{23} = f_{32}$, $f_{34} = f_{43}$, as explained earlier.

THE FLEXIBILITY METHOD: BASIC FORMULATION

In its basic formulation, the flexibility method is essentially the method of consistent deformations cast in matrix form. Although it can be used to analyse both statically determinate and indeterminate structures, it shall be used here only for the analysis of indeterminate structures since determinate structures can more simply be solved using only equilibrium equations of statics.

Procedure for Analysis

Just as in the method of consistent deformations, the following 5 steps are involved in the analysis:

Step 1

The degree of statical indeterminacy is determined. Once this is done, the redundants are released to obtain a statically determinate and yet stable primary structure. It is also necessary to exercise care in selecting the redundants since the ease or otherwise with which the displacements are determined often depends on the choice of redundants.

Step 2

The displacements of the primary structure due to given loading are evaluated at each release coordinate and in the direction of the release. Any convenient method for determining displacements may be employed for this purpose. The displacements so obtained form the displacement vector D.

Step 3

The displacements at the release coordinates due to unit values of the redundants are determined. The unit values of the redundants are applied at all the coordinates, one only at a time, in turn. These displacements form the flexibility matrix F for the structure.

Step 4

The magnitudes of the redundant forces necessary to ensure geometric continuity of the structure are determined by the following equation:

$$\{D_P\} + [F]\{R\} = 0 \ , \qquad (5.11)$$

where:

$D_P =$ Displacements at the coordinates corresponding to the releases, due to given loading on the primary structure,
$F =$ Structure flexibility matrix,
$R =$ Unknown redundant forces on the primary structure.

From eqn (5.11), the redundant forces are determined as follows:

$$\{R\} = [F]^{-1}\{-D_P\} \qquad (5.12)$$

Step 5
The forces on the original indeterminate structure are obtained as the sum of the redundant forces and the forces on the released structure due to given loading. Thus, the force at any coordinate '***i***' can be obtained as follows:

$$P_i = P_{oi} + \{f_{i1}R_1 + f_{i2}R_2 + \dots\dots\dots + f_{in}R_n\} \qquad (5.13)$$

or in matrix form, the forces at all the coordinates can be expressed as:

$$\{P\} = \{P_o\} + [F]\{R\} \qquad (5.14)$$

where:
$P_o =$ Value of P on the primary structure due to the given loading,
$f_{ij} =$ Value of P at coordinate i of the primary structure when a unit value of R is applied at coordinate *j* of the structure.

Example5.2
Determine the support reactions and the moment at support A for the beam shown in Fig.5.8(a). EI is constant throughout.

SOLUTION
Redundants
The beam is 2 degrees indeterminate. Several alternatives exist regarding the choice of redundants. For example, the vertical reactions R_B and R_C, R_B and M_A, R_C and M_A, or M_A and M_B can be chosen as the redundants. In this example let us choose M_A and M_B as the redundants. Accordingly the primary structure and the structure coordinates are shown in Fig.5.8(b). Thus the primary structure consists of two simple beams AB and BC.

Computation of Displacements
The displacements D_1 and D_2 in the primary structure,due to given loading, together with the bending moment diagrams caused by these loads, are shown in Fig.5.8(c). The displacements will be computed using the moment area theorems. Thus, referring to Fig.5.8(c) the displacements are obtained as follows:

$$D_{1P} = \frac{t_{BA}}{5}$$

Here t_{BA} is the tangential deviation of point B from the tangent on the elastic curve from point A. According to the second moment area theorem,

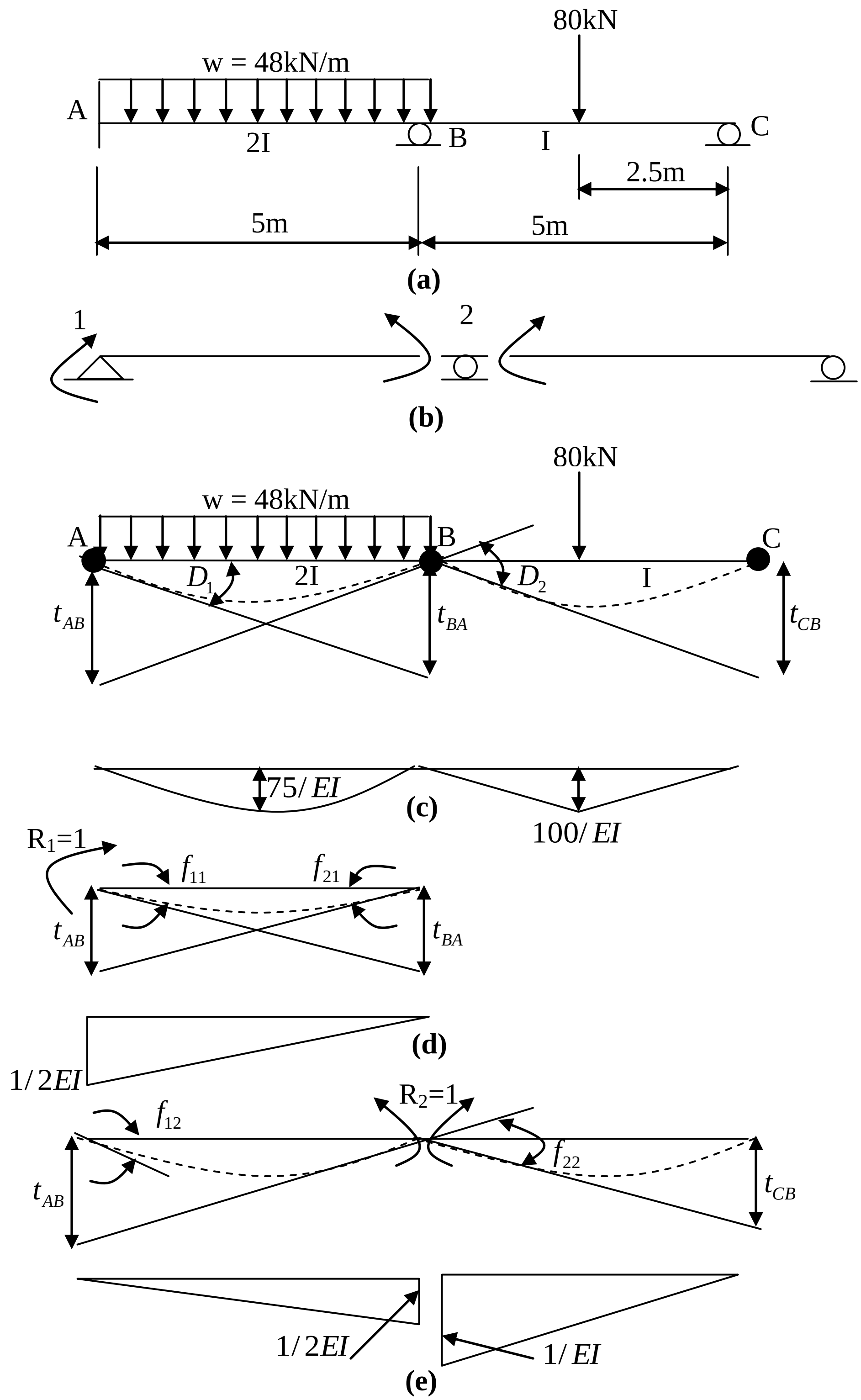


Fig.5.8: (a) Given beam and loading; (b) Primary structure and coordinates;
(c) Displacements and M/EI diagrams of the primary structure due to given loading;
(d) Displacements and M/EI diagram due to unit couple at coordinate 1;
(e) Displacements and M/EI diagrams due to unit couples at coordinate 2.

$$t_{BA} = \frac{2}{3} \times \frac{75}{EI} \times 5 \times 2.5 = \frac{625}{EI}$$

$$\therefore D_{1P} = \frac{625}{5EI} = \frac{125}{EI}$$

$$D_{2P} = \frac{t_{AB}(= t_{BA})}{5} + \frac{t_{CB}}{5}$$

$$t_{CB} = \frac{1}{2} \times \frac{100}{EI} \times 5 \times 2.5 = \frac{625}{EI}$$

$$\therefore D_{2P} = \frac{625}{5EI} + \frac{625}{5EI} = \frac{250}{EI}$$

The displacements due to unit values of the redundants at A (or R_1) and at B (or R_2), together with their moment diagrams are shown in Figs.5.8 (d) and (e) respectively.
Referring to Fig.5.8(d), we have:

$$f_{11} = \frac{t_{BA}}{5} = \frac{1}{2} \times \frac{1}{2EI} \times 5 \times \frac{2}{3} \times 5 \times \frac{1}{5} = \frac{5}{6EI}$$

$$f_{21} = \frac{t_{AB}}{5} = \frac{1}{2} \times \frac{1}{2EI} \times 5 \times \frac{1}{3} \times 5 \times \frac{1}{5} = \frac{5}{12EI}$$

Referring to Fig.5.8(e), we have:

$$f_{22} = \frac{t_{AB}}{5} + \frac{t_{CB}}{5} = (\frac{1}{2} \times \frac{1}{2EI} \times 5 \times \frac{2}{3} \times 5 \times \frac{1}{5}) + (\frac{1}{2} \times \frac{1}{EI} \times 5 \times \frac{2}{3} \times 5 \times \frac{1}{5}) = \frac{5}{6EI} + \frac{5}{3EI}$$

or $\quad f_{22} = \frac{15}{6EI} = \frac{2.5}{EI}$

The flexibility matrix for the chosen primary structure will be:

$$[F] = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 5/6 & 5/12 \\ 5/12 & 2.5 \end{bmatrix}$$

The determinant S of the above matrix is $S = 275/144$
The redundant forces are evaluated using eqn (5.12):

$$\{R\} = [F]^{-1}\{-D_P\}$$

$$[F]^{-1} = \frac{144EI}{275}\begin{bmatrix} 2.5 & -5/12 \\ -5/12 & 5/6 \end{bmatrix}$$

$$\therefore \{R\} = \frac{144EI}{275}\begin{bmatrix} 2.5 & -5/12 \\ -5/12 & 5/6 \end{bmatrix}\frac{1}{EI}\begin{Bmatrix} -125 \\ -250 \end{Bmatrix}$$

or $\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} -109.09 \\ -81.82 \end{Bmatrix} kNm$

The negative signs for the redundants indicate that their actual directions are opposite the directions assumed at the coordinates. Compare the values for the redundants with the analysis results of Examples 1.4 and 2.2.

From statics the vertical reactions are obtained as follows:

$$R_A = \frac{-81.82 + 109.09}{5} + \frac{48 \times 5}{2} = 125.45 kN$$

$$R_B = \left(\frac{81.82}{5} + \frac{80}{2}\right) + 114.55 = 170.91 kN$$

$$R_C - 80 + 170.91 + 125.45 - 240 = 0$$

or $R_C = 23.64 kN$

Checks

$M_B = 23.64(5) - 80(2.5) = -81.8 kNm \quad (Satisfied)$

$M_A = 23.64(10) - 80(7.5) + 170.91(5) - 48 \times 5 \times 2.5 = -109.05 kNm \quad (Satisfied)$

Example 5.3

Determine the support reactions and the moment at support A for the beam shown in Fig.5.9(a). EI is constant throughout.

SOLUTION

Observe that this beam was analysed earlier in Examples 1.2 and 2.3.

Redundants

The beam is 2 degrees statically indeterminate. Although the support moments can be chosen as redundants as was done in Example5.2, in order to show how the choice of redundants can affect the amount of computation effort required, let us here choose the vertical reactions R_B and R_C as the redundants. Accordingly, the primary structure and the structure coordinates are shown in Fig.5.9(b).

Computation of Displacements

The displacements D_1 and D_2 will be computed using the moment area theorems. Accordingly, the displacements in the primary structure, due to given loading, together with the bending moment diagrams caused by this loading, are shown in Fig.5.9(c). The displacements due to unit values of the redundants R_1 and R_2, together with their moment diagrams are shown in Figs.5.9(d) and (e) respectively.

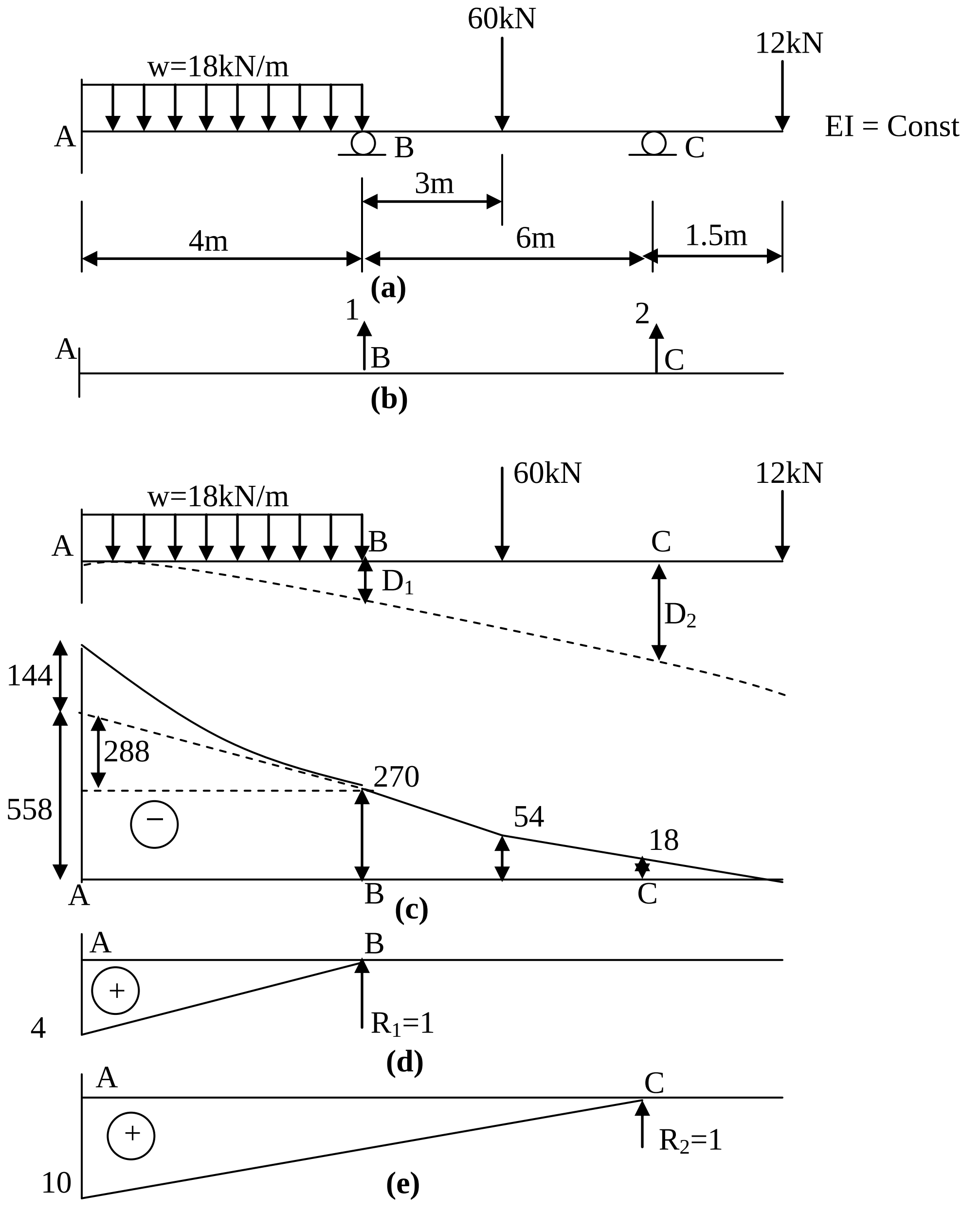


Fig.5.9: (a) Given beam and loading; (b) Primary structure and coordinates;
(c) Displacements and moment diagram of the primary structure due to given loading;
(d) Displacement D_1 and moment diagram due to unit value of the redundant at coordinate 1; (e) displacement D_2 and moment diagram due to unit value of the redundant at coordinate 2.

Notice that in Fig.5.9(c) the bending moment diagram within the beam segment AB has been broken into separate parts for convenience in obtaining the areas of moment diagrams necessary in using the moment area theorems within that segment. To this end the moment at support A is shown to consist of the sum of the moments given by the point loads of 12kN and 60kN (i.e. 558kNm) and the moment given by the uniform load of 18kN/m (i.e. 144kNm).

Referring now to Fig.5.9(c), we have:

$$D_{1P} = -\frac{1}{EI}\left(270\times 4\times 2+\frac{1}{2}\times 288\times 4\times\frac{2}{3}\times 4+\frac{1}{3}\times 144\times 4\times\frac{3}{4}\times 4\right) \qquad \text{(a)}$$

or $D_{1P} = -\dfrac{4272}{EI}$

Observe that the third component on the right hand side of expression (a) is the displacement due to the uniform load as represented by the parabola of span AB. The area of this second-degree parabola $A=\frac{1}{3}aL$ and the centroid measured from point B along the beam axis is obtained from the formula $\bar{x}=\frac{3}{4}L$. In Fig.5.9(c), $a=144$ and $L=4$. Since we still need both a and $\bar{x}$ in order to calculate D_{2P}, we evaluate them to obtain: $A=192$ and $\bar{x}=3$.

$$D_{2P} = -\frac{1}{EI}\left[270\times 4\times 8+\frac{1}{2}\times 288\times 4\left(\frac{2}{3}\times 4+6\right)+192(3+6)+54\times 3\times 4.5\right]-$$

$$-\frac{1}{EI}\left[\frac{1}{2}\times 216\times 3\left(\frac{2}{3}\times 3+3\right)+18\times 3\times 1.5+\frac{1}{2}\times 36\times 3\times\frac{2}{3}\times 3\right]$$

or $D_{2P} = -\dfrac{17898}{EI}$

Referring to Fig.5.9(d), we have:

$$f_{11} = \frac{1}{EI}\times\frac{1}{2}\times 4\times 4\times\frac{2}{3}\times 4 = \frac{21.33}{EI}$$

$$f_{21} = \frac{1}{EI}\times\frac{1}{2}\times 4\times 4\left(\frac{2}{3}\times 4+6\right) = \frac{69.33}{EI}$$

Referring now to Fig.5.9(e) we have:

$$f_{22} = \frac{1}{EI}\times\frac{1}{2}\times 10\times 10\times\frac{2}{3}\times 10 = \frac{333.33}{EI}$$

The structure flexibility matrix for the chosen primary structure will be:

$$[F] = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 21.33 & 69.33 \\ 69.33 & 333.33 \end{bmatrix}$$

The redundant forces are evaluated using eqn (5.12):

$$\{R\} = [F]^{-1}\{-D_P\}$$

The determinant *S* for the above matrix is $S = 2303.28$.

$$[F]^{-1} = \frac{EI}{2303.28}\begin{bmatrix} 333.33 & -69.33 \\ -69.33 & 21.33 \end{bmatrix}$$

$$\therefore \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{EI}{2303.28}\begin{bmatrix} 333.33 & -69.33 \\ -69.33 & 21.33 \end{bmatrix}\frac{1}{EI}\begin{Bmatrix} 4272 \\ 17898 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} 79.5 \\ 37.16 \end{Bmatrix} kN$$

With the redundants determined, the reaction at A and the moments at B and C can be obtained from statics. Thus:

$$M_C = -12(1.5) = -18kNm$$
$$M_B = -12(7.5) + 37.16(6) - 60(3) = -47.04kNm$$
$$M_A = -12(11.5) + 37.16(10) - 60(7) + 79.5(4) - 18 \times 4 \times 2 = -12.4kNm$$
$$R_A = 18(4) + 60 + 12 - 79.5 - 37.16 = 27.34kN.$$

The values for the support moments obtained above are the same with those obtained in Example 1.2, ignoring rounding-off errors.

Example 5.4

Determine the support reactions and the moments at A and B for the frame shown in Fig.5.10(a).

SOLUTION

Redundants

The frame is 2 degrees statically indeterminate. Let us choose the horizontal and the vertical reactions at support D as the redundants. The primary frame with the positive directions of the coordinates is shown in Fig.5.10(b).

Computation of Displacements

The virtual work method will be used for the evaluation of displacements. Fig.5.10(c) shows the primary structure subjected to the given loading, while Figs.5.10(d) and (e) respectively show the primary structure subjected to unit values of redundants R_1 and R_2.

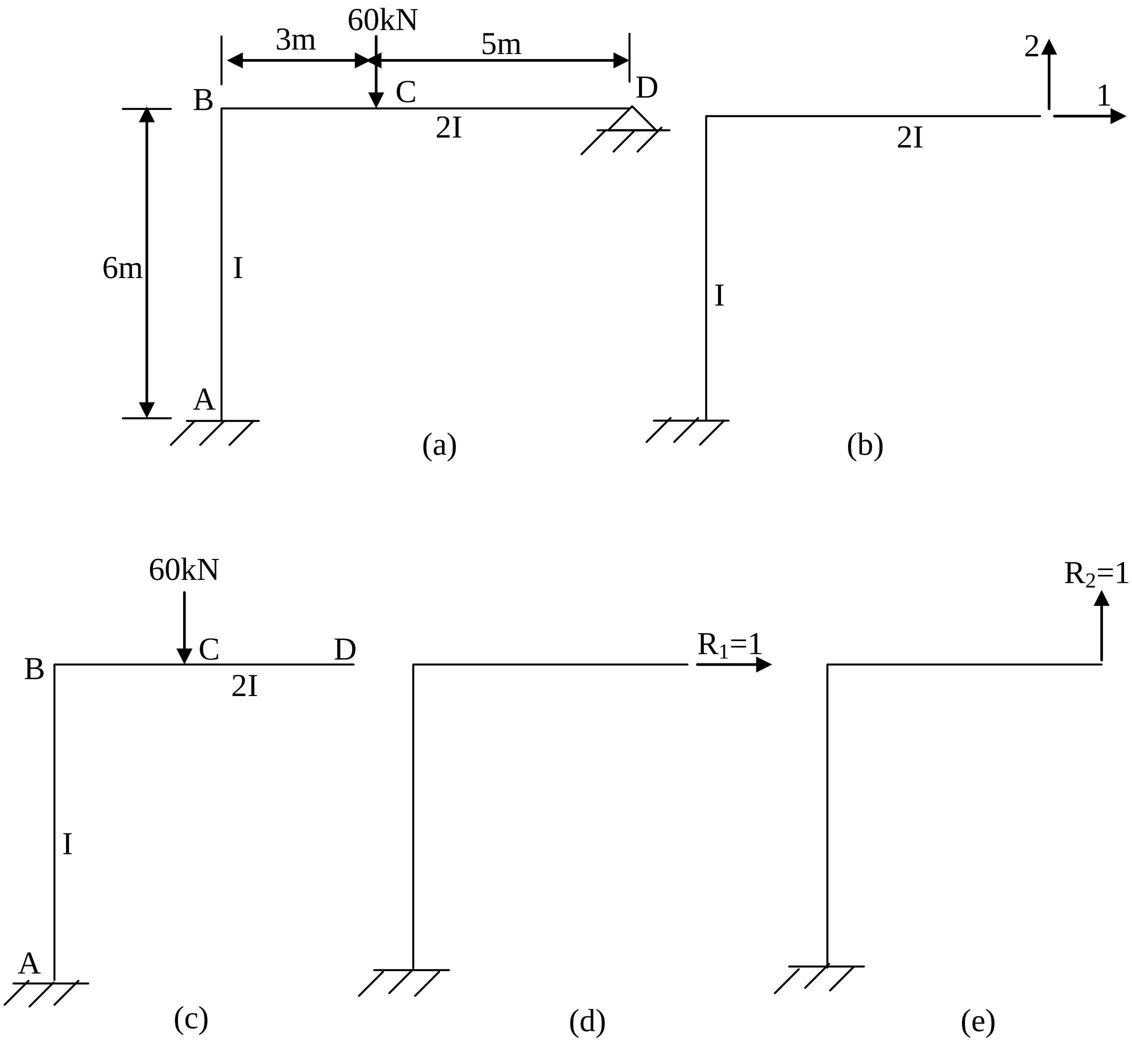


Fig.5.10: (a) Given frame and loading; (b) Primary structure with coordinates;
(c) Primary structure subjected to given loading; (d) Primary structure subjected to unit value of R_1; (e) Primary structure subjected to unit value of R_2.

The various parameters necessary for the use of virtual work expression are shown in Table 5.1. The displacement quantities are now computed as follows:

$$f_{11} = \int_0^6 \frac{(-x)^2 dx}{EI} = \frac{1}{EI}\left[\frac{x^3}{3}\right]_0^6$$

or $f_{11} = \dfrac{72}{EI}$

$$f_{12} = \int_0^6 \frac{8(-x)dx}{EI} = -\frac{1}{EI}\left[4x^2\right]_0^6$$

or $f_{12} = -\dfrac{144}{EI}$

$$f_{22} = \int_0^5 \frac{x^2 dx}{2EI} + \int_5^8 \frac{x^2 dx}{2EI} + \int_0^6 \frac{64}{EI}dx$$

$$= \frac{1}{EI}\left[\frac{x^3}{6}\right]_0^5 + \frac{1}{EI}\left[\frac{x^3}{6}\right]_5^8 + \frac{1}{EI}\left[64x\right]_0^6$$

or $f_{22} = \dfrac{469.33}{EI}$

$$D_{1P} = \int_0^6 \frac{-180(-x)}{EI}dx = \frac{1}{EI}\left[90x^2\right]_0^6$$

or $D_{1P} = \dfrac{3240}{EI}$

$$D_{2P} = \int_5^8 \frac{(-60x+300)x}{2EI}dx + \int_0^6 \frac{-180(8)}{EI}dx$$

$$= \frac{1}{EI}\left[75x^2 - 10x^3\right]_5^8 - \frac{1}{EI}\left[1440x\right]_0^6$$

or $D_{2P} = -\dfrac{9585}{EI}$.

Table 5.1

Segment	Origin	Limits	M_P	m_1	m_2
CD	D	0 - 5	0	0	x
BC	D	5 - 8	$-60(x-5)$	0	x
AB	D	0 - 6	$-60(3)$	$-x$	8

$$[F] = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 72 & -144 \\ -144 & 469.33 \end{bmatrix}$$

The determinant $S = 469.33(72) - (-144)^2 = 13055.76$

$$[F]^{-1} = \frac{EI}{1305.576}\begin{bmatrix} 469.33 & 144 \\ 144 & 72 \end{bmatrix}$$

$$\therefore \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{EI}{1305.576}\begin{bmatrix} 469.33 & 144 \\ 144 & 72 \end{bmatrix}\frac{1}{EI}\begin{Bmatrix} -3240 \\ 9585 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} -10.75 \\ 17.12 \end{Bmatrix} kN$$

The negative sign for R_1 indicates that its correct direction is to the left and not to the right as earlier assumed (Fig.5.10(d)). The remaining reactions can be obtained from statics. They are:

$$H_A = 10.75 kN \rightarrow;$$
$$V_A = 42.88 kN \uparrow;$$
$$M_A = 21.52 kNm;$$
$$M_B = -43 kNm.$$

Example 5.5

Determine the member forces in the truss shown in Fig.5.11(a). EA is constant for all members.

SOLUTION

Redundants

The truss is 1 degree statically indeterminate. Let us choose the horizontal reaction at support C as the redundant. The primary structure with the given loading and the resulting support reactions are shown in Fig.5.11(b) while the primary structure subjected to a unit value of the redundant together with the resulting support reactions, are shown in Fig.5.11(c).

Computation of Displacements

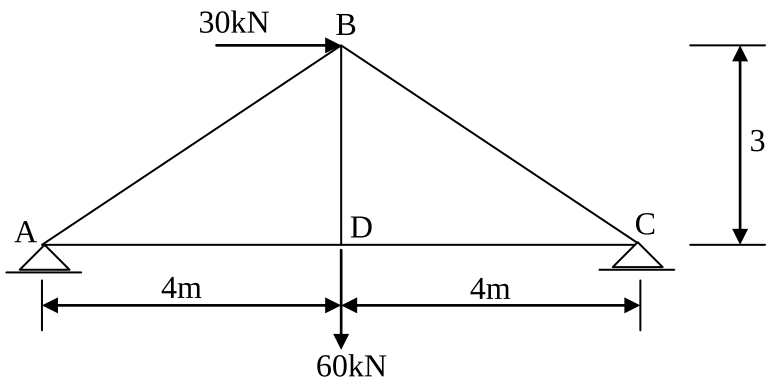
Displacements will be computed using virtual work method. Thus:

$$D_{1P} = \sum \frac{P_o pL}{EA}$$

and $$f_{11} = \sum \frac{p^2 L}{EA}$$

Here, $P_o =$ force in a member in the primary structure due to given loading;

$p =$ force in a member in the primary structure due to unit value of the redundant.



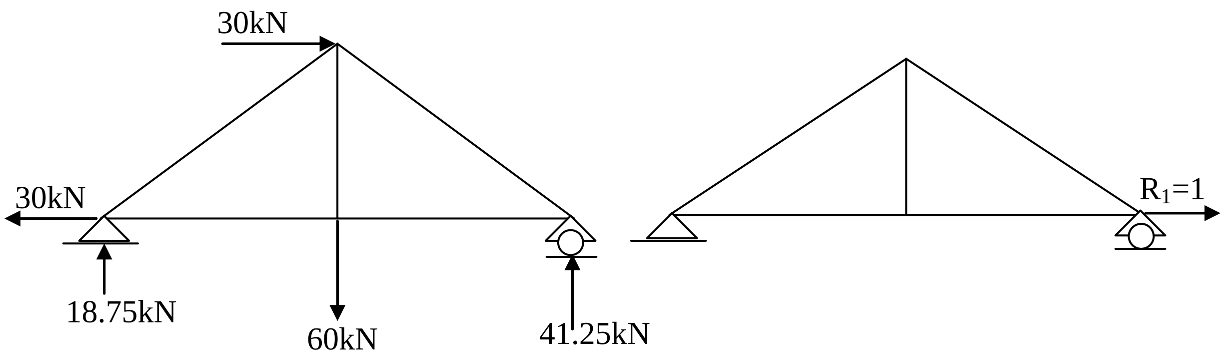


Fig.5.11: (a) Given truss and loading; (b) Primary truss subjected to given loading; (c) Primary truss subjected to a unit value of the redundant R_1.

Table 5.2 shall be employed in the computations for convenience.

Table 5.2

Member	P_0(kN)	p(kN)	L(m)	P_0pL	p^2L	$P = P_0 + R_1(p)$
AB	-31.25	0	5	0	0	-31.25
AD	55	1	4	220	4	0
BD	60	0	3	0	0	60
BC	-68.75	0	5	0	0	-68.75
CD	55	1	4	220	4	0
			Σ	440	Σ 8	

From Table 5.2, we have:

$$D_{1P} = \frac{440}{EA}; \qquad f_{11} = \frac{8}{EA}.$$

The compatibility equation is:

$$D_{1P} + f_{11}R_1 = 0,$$

from where,

$$R_1 = f_{11}^{-1}\left(-D_{1P}\right)$$

$$= \frac{EA}{8}\left(-\frac{440}{EA}\right)$$

or $\quad R_1 = -55kN.$

The negative sign for the redundant indicates that it is acting to the left and not to the right as earlier assumed (Fig.5.11(c)).

The member forces in the given truss are obtained by the superposition of the forces due to given loading and the forces due to the redundant. In matrix form the forces can be obtained as:

$$\{P\} = \{P_0\} + \{R_1\}\{p\}$$

The member forces obtained using the above formula are shown in the last column of Table 5.2. In these final forces, negative sign indicates compression and positive sign indicates tension in the member.

THE STIFFNESS METHOD

Stiffness Influence Coefficient

In the flexibility method of analysis, we saw that a flexibility coefficient can be used to relate force and displacement at a coordinate. An alternative way of relating force and displacement at a coordinate is by using the stiffness influence coefficient.

To explain the concept of the stiffness coefficient consider the beam shown in Fig5.12. Let it be required to obtain the relationship between force and displacement at coordinate 1.

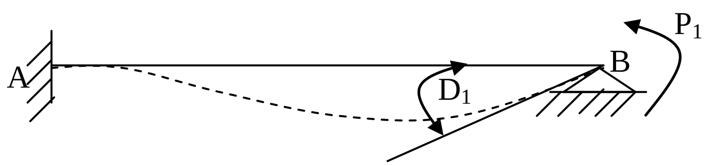


Fig.5.12. Beam with force and displacement at coordinate 1.

If a displacement D_1 is applied at coordinate 1, the force at coordinate 1, required to keep the beam in the displaced position can be expressed as follows:

$$P_1 = k_{11}D_1 \qquad (5.15)$$

where k_{11} is known as stiffness influence coefficient. It represents the force at coordinate 1, required to produce a unit displacement at coordinate 1.

Assume now that in addition to permitting the beam to rotate at coordinate 1 (Fig.5.12), that it is also permitted to rotate at coordinate 2 (Fig.5.13(a)).

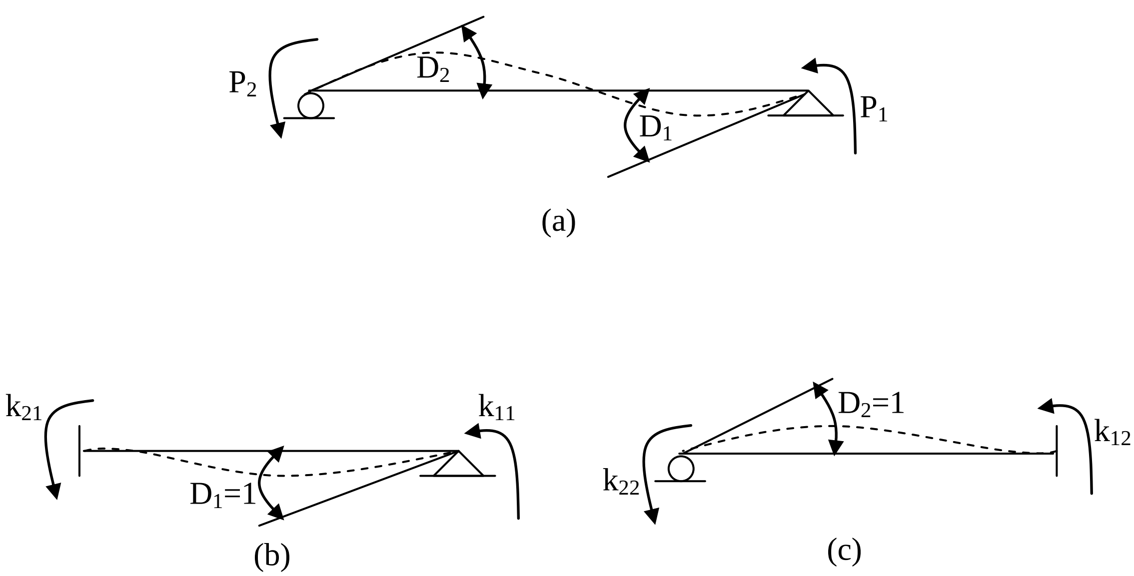


Fig.5.13: (a) Beam with loads and displacements at coordinates 1 and 2; (b) Unit displacement at coordinate 1 only; (c) Unit displacement at coordinate 2 only.

The forces will in this case be obtained by superposition of displacements as follows:

First we apply a unit displacement at coordinate 1 only (Fig.5.13(b)) and denote the forces at coordinates 1 and 2 (required to produce the unit displacement) as k_{11} and k_{21}. Next, we apply a unit displacement at coordinate 2 only (Fig.5.13(c)) and denote the forces at coordinates 1 and 2 (required to produce this displacement) respectively as k_{12} and k_{22}.

The forces required to produce displacements D_1 and D_2 at the same time can then be obtained as:

$$\begin{aligned} P_1 &= k_{11}D_1 + k_{12}D_2 \\ P_2 &= k_{21}D_1 + k_{22}D_2 \end{aligned} \qquad (5.16)$$

In eqn (5.16), k_{11}, k_{12}, k_{21} and k_{22} are stiffness coefficients. In general for a stiffness coefficient k_{ij}, the first subscript *i* indicates the coordinate at which the force is measured, while the second subscript *j* indicates the coordinate at which the displacement is applied. The stiffness coefficient k_{ij} is therefore defined as the force at coordinate *i* required to produce a unit displacement at coordinate *j*.

The matrix $K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$ containing the stiffness influence coefficients, is known as the stiffness matrix. The stiffness coefficients being force quantities, can be obtained from the slope-deflection equation.

For a structure with ***n*** coordinates, the force-displacement relationship will be expressed as follows:

$$\begin{aligned} P_1 &= k_{11}D_1 + k_{12}D_2 + \ldots\ldots\ldots\ldots + k_{1n}D_n \\ P_2 &= k_{21}D_1 + k_{22}D_2 + \ldots\ldots\ldots\ldots + k_{2n}D_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ P_n &= k_{n1}D_1 + k_{n2}D_2 + \ldots\ldots\ldots\ldots + k_{nn}D_n \end{aligned} \qquad (5.17)$$

In matrix form eqn (5.17) will be:

$$\begin{Bmatrix} P_1 \\ P_2 \\ \ldots \\ P_n \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \ldots & k_{1n} \\ k_{21} & k_{22} & \ldots & k_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ k_{n1} & k_{n2} & \ldots & k_{nn} \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ \ldots \\ D_n \end{Bmatrix} \qquad (5.18)$$

or
$$P = KD \qquad (5.19)$$

P is the force vector and D is the displacement vector.
The elements of the displacement vector can be obtained by solving eqn (5.19). This is achieved if both sides of the equation are pre-multiplied by K^{-1}. Thus:

$$D = K^{-1}P \qquad (5.20)$$

From eqn (5.18), it can be observed that a stiffness matrix K will always have ***n*** rows and ***n*** columns, forming a square matrix of the order $n \times n$.

Like the flexibility matrix, the stiffness matrix is also symmetric.

Comparison of equations (5.20) and (5.7) shows that:

$$F = K^{-1} \qquad (5.21)$$

In other words, the flexibility matrix of a structure is the inverse of the stiffness matrix, and vice versa.

We now show how to generate stiffness matrices by means of the following example.

Example 5.6

Generate the stiffness matrix for the frame with coordinates as shown in Fig.5.14. EI is constant throughout.

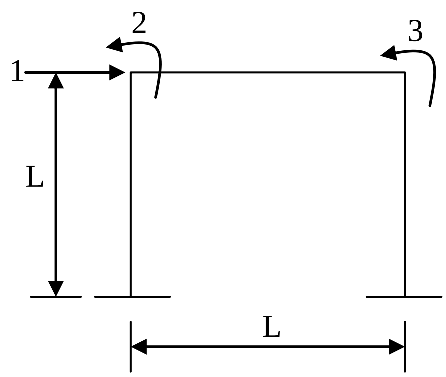


Fig.5.14. Given frame and coordinates.

SOLUTION

Coordinate 1 is given a unit displacement (Fig.5.15(a)) and the forces required at all the coordinates to hold the frame in the deformed position are evaluated. The forces on the free-body diagram of Fig.5.15(b) are:

$$k_{11} - \frac{12EI}{L^3} - \frac{12EI}{L^3} = 0$$

or $$k_{11} = \frac{24EI}{L^3}$$

$$k_{21} - \frac{6EI}{L^2} = 0$$

or $$k_{21} = \frac{6EI}{L^2}$$

$$k_{31} - \frac{6EI}{L^2} = 0$$

or $$k_{31} = \frac{6EI}{L^2}.$$

The above stiffness coefficients form the first column of the stiffness matrix of the structure.

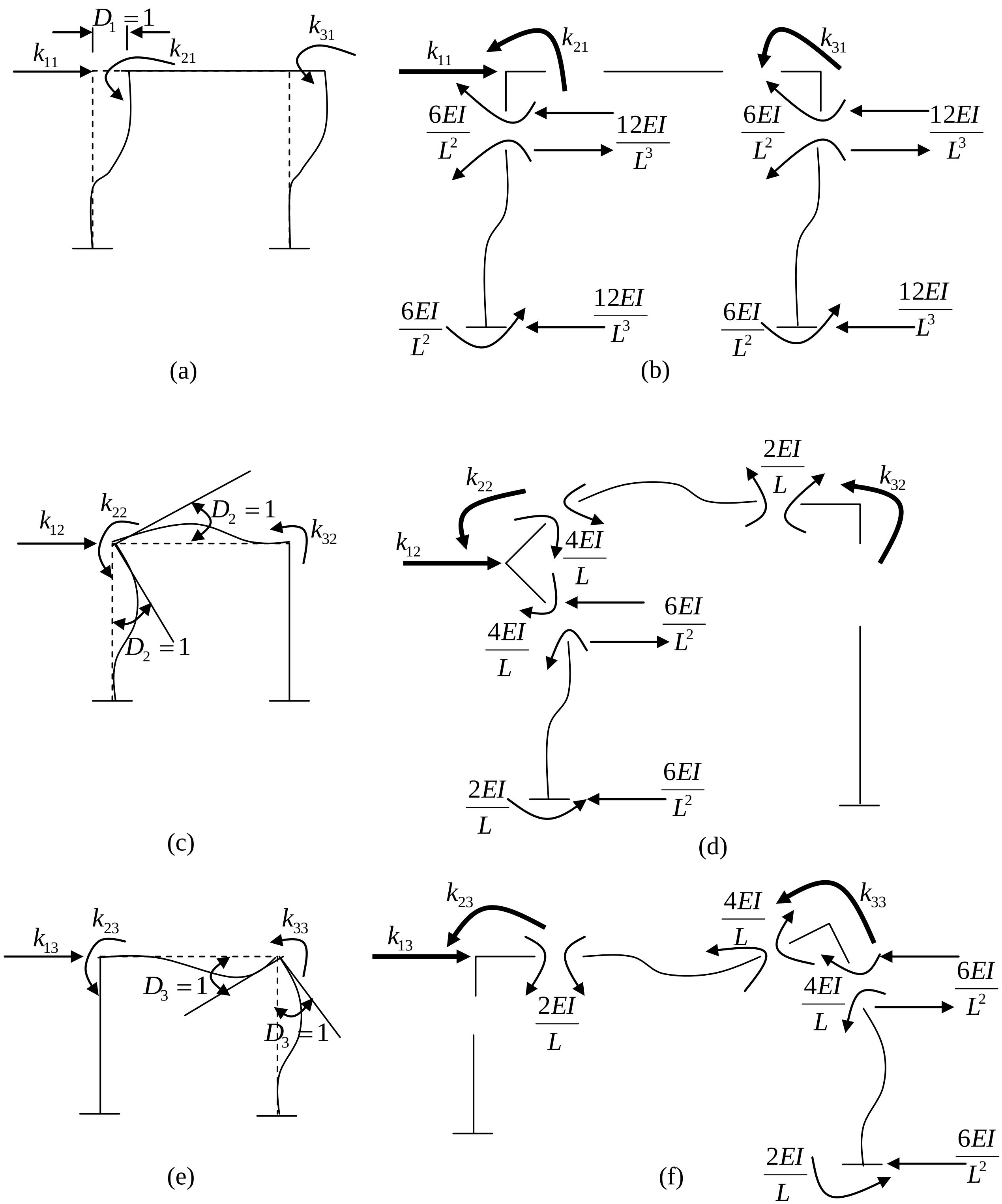


Fig.5.15: (a) Unit displacement at coordinate 1; (b) Free-body diagrams for the computation of k_{11}, k_{21} and k_{31}; (c) Unit displacement at coordinate 2; (d) Free-body diagrams for the computation of k_{12}, k_{22} and k_{32}; (e) Unit displacement at coordinate 3; (f) Free-body diagrams for the computation of k_{13}, k_{23} and k_{33}.

Similarly, to obtain the elements of the second column of the matrix, we impose a unit displacement at coordinate 2 (Fig.5.15(c)) and compute the forces required at all the coordinates to keep the frame in the displaced position. Those forces are obtained from the free-body diagram of Fig.5.15(d) as follows:

$$k_{12} - \frac{6EI}{L^2} = 0$$

or $$k_{12} = \frac{6EI}{L^2}$$

$$k_{22} - \frac{4EI}{L} - \frac{4EI}{L} = 0$$

or $$k_{22} = \frac{8EI}{L}$$

$$k_{32} - \frac{2EI}{L} = 0$$

or $$k_{32} = \frac{2EI}{L}$$

Also to obtain the elements of the third column of the matrix we impose a unit displacement at coordinate 3 only (Fig.5.15(e)) and determine the forces required for the frame to maintain its deformed configuration. These forces are obtained from the free-body diagram of Fig.5.15(f) as follows:

$$k_{13} - \frac{6EI}{L^2} = 0$$

or $$k_{13} = \frac{6EI}{L^2}$$

$$k_{23} - \frac{2EI}{L} = 0$$

or $$k_{23} = \frac{2EI}{L}$$

$$k_{33} - \frac{4EI}{L} - \frac{4EI}{L} = 0$$

or $$k_{33} = \frac{8EI}{L}$$

Thus the stiffness matrix K will be:

$$K = \frac{EI}{L}\begin{bmatrix} 24/L^2 & 6/L & 6/L \\ 6/L & 8 & 2 \\ 6/L & 2 & 8 \end{bmatrix}$$

THE STIFFNESS METHOD: BASIC FORMULATION

As explained earlier, in the basic formulation of the stiffness method, the stiffness matrix $[K]$ will be developed direct for the structure at the required coordinates. This formulation is suitable only for manual computation and since it does not lend itself to computer programming, it is not suitable for structures with high degree of kinematic indeterminacy.

Procedure for Analysis

Step 1

The degree of kinematic indeterminacy is determined. Next, the coordinates at which joint displacements are possible are identified. Restraining forces are applied at the coordinates to prevent joint displacements.

Step 2

The restraining forces are computed as sum of fixed-end forces for the members meeting at a joint.

Step 3

The forces required to hold the restrained structure with a unit displacement at one of the coordinates only (with all other coordinates having zero displacement) are determined. Each of the coordinates in turn is given a unit displacement and the force required to hold the structure in the displaced state is determined at each of the coordinates. These forces are the elements of the stiffness matrix $[K]$.

Step 4

The values of the displacements required to ensure joint equilibrium of the structure are determined from the equation:

$$\{P\} + [K]\{D\} = 0 \qquad (5.22)$$

where $\{P\} =$ restraining force at the joints;

$[K] =$ stiffness matrix, and

$\{D\} =$ unknown displacements at the coordinates.

Displacements are then obtained by solving eqn (5.22) as follows:

$$\{D\} = [K]^{-1}\{-P\} \qquad (5.23)$$

Step 5

The forces in the given structure are determined by summation of the forces on the restrained structure and the forces due to joint displacements. The procedure involved will now be illustrated by the following examples.

Example 5.7

Determine the member-end moments for the beam shown in Fig.5.16(a).

SOLUTION

Observe that this beam was analysed in Example 5.2 by the flexibility method. The beam is 2 degrees kinematically indeterminate. The degrees of freedom are the independent rotations at B and C. Accordingly, the coordinates, 1 and 2, are as shown in Fig.5.16(b).

Next, the restraining forces, which are equal to the sum of the end forces, are evaluated. These restraining forces are considered positive when their directions are the same as those of the coordinates. The end forces are shown in Fig.5.16(c).

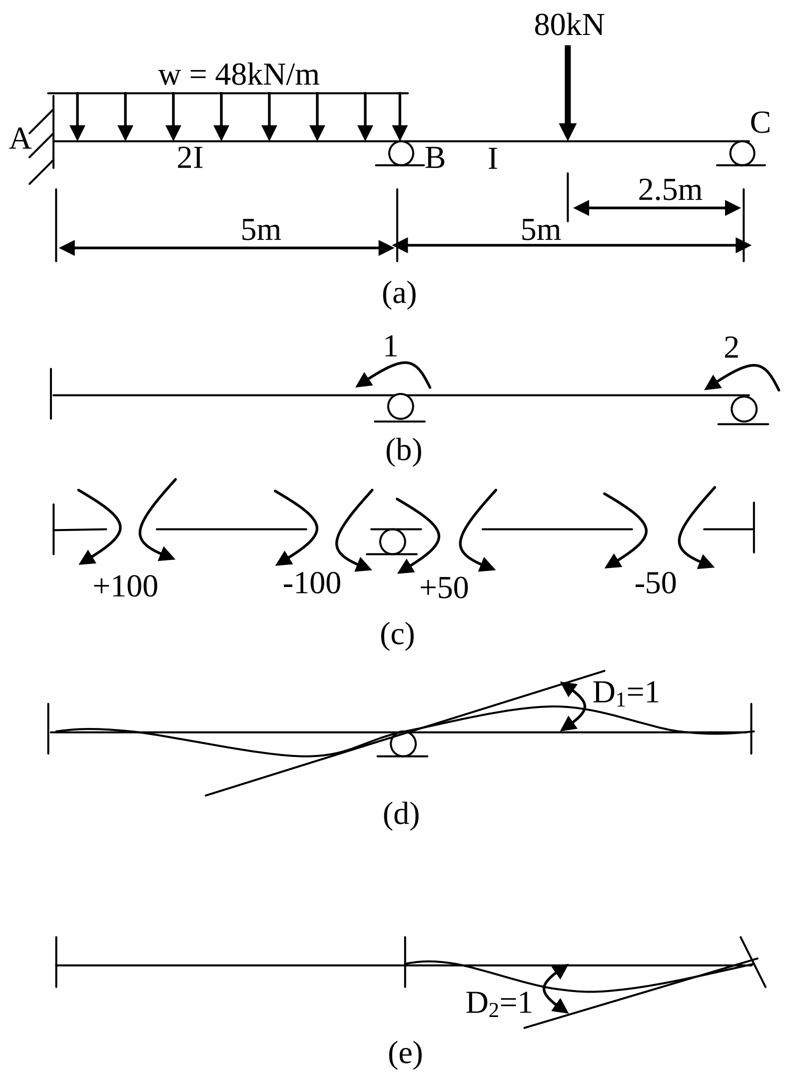


Fig.5.16: (a) Given beam with loading; (b) Coordinates;
(c) Joint end forces: (d) Unit displacement at coordinate 1;
(e) Unit displacement at coordinate 2.

The restraining forces in this example are:

$$\{P\} = \begin{Bmatrix} -50 \\ -50 \end{Bmatrix} kNm$$

The elements of the first column of the stiffness matrix are the forces required at each of the coordinates to hold the beam in the deformed position as shown in Fig.5.16(d), obtained by imposing unit displacement at the first coordinate (D_1=1) while D_2=0.
Thus:

$$k_{11} = \frac{8EI}{5} + \frac{4EI}{5} = \frac{12EI}{5};$$

$$k_{21} = \frac{2EI}{5}.$$

Similarly, the elements of the second column of the matrix are the forces required at each of the coordinates to hold the beam in the deformed position as shown in Fig.5.16(e), obtained by imposing unit displacement at the second coordinate (D_2=1) while D_1=0.
Thus:

$$k_{12} = \frac{2EI}{5};$$

$$k_{22} = \frac{4EI}{5}.$$

Therefore the stiffness matrix $[K]$ will be:

$$[K] = \frac{EI}{5}\begin{bmatrix} 12 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\therefore [K]^{-1} = \frac{5}{44EI}\begin{bmatrix} 4 & -2 \\ -2 & 12 \end{bmatrix}$$

Substitution into eqn (5.23) gives:

$$\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \frac{5}{44EI}\begin{bmatrix} 4 & -2 \\ -2 & 12 \end{bmatrix}\begin{Bmatrix} +50 \\ +50 \end{Bmatrix}$$

or $$\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \frac{1}{EI}\begin{Bmatrix} 11.36 \\ 56.82 \end{Bmatrix}$$

The final end moments are obtained by substituting the above values of D_1 and D_2 into the slope-defection equation as follows:

$$M_{AB} = \frac{4EI}{5}\left[0 + \frac{11.36}{EI}\right] + 100 = 109.09kNm;$$

$$M_{BA} = \frac{4EI}{5}\left[2\times\frac{11.36}{EI}+0\right]-100 = -81.82kNm;$$

$$M_{BC} = \frac{2EI}{5}\left[2\times\frac{11.36}{EI}+\frac{56.82}{EI}\right]+50 = 81.82kNm;$$

$$M_{CB} = \frac{2EI}{5}\left[2\times\frac{56.82}{EI}+\frac{11.36}{EI}\right]-50 = 0.$$

Example 5.8

Analyse the beam shown in Fig.5.17(a) for member-end moments.

SOLUTION

This beam was previously analysed by the rotation contribution method in Example 3.1. The beam is 2 degrees kinematically indeterminate with the independent rotations at B and C as the degrees of freedom. Accordingly the two coordinates, 1 and 2, are shown in Fig.5.17(b). The joint end forces are shown in Fig.5.17(c) for the purpose of obtaining the restraining forces. The restraining forces are:

$$\{P\} = \begin{Bmatrix} 50 \\ 22.5 \end{Bmatrix} kNm$$

The elements of the first column of the stiffness matrix are obtained by determining the forces at the coordinates due to D_1=1 and D_2=0 (Fig.5.17(d)). Thus:

$$k_{11} = \frac{12EI}{6}+\frac{4EI}{4} = 3EI;$$
$$k_{21} = \frac{6EI}{6} = EI.$$

Similarly, the elements of the second column of the matrix are obtained by determining the forces at the coordinates due to D_2=1 and D_1=0 (Fig.5.17(e)) as follows:

$$k_{12} = \frac{6EI}{6} = EI;$$
$$k_{22} = \frac{12EI}{6}+\frac{8EI}{5} = \frac{18EI}{5}.$$

The stiffness matrix $[K]$ will be:

$$[K] = \frac{EI}{5}\begin{bmatrix} 15 & 5 \\ 5 & 18 \end{bmatrix}$$

$$\therefore [K]^{-1} = \frac{5}{245EI}\begin{bmatrix} 18 & -5 \\ -5 & 15 \end{bmatrix}$$

Substitution into eqn (5.23) gives:

$$\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \frac{5}{245EI}\begin{bmatrix} 18 & -5 \\ -5 & 15 \end{bmatrix}\begin{Bmatrix} -50 \\ -22.5 \end{Bmatrix}$$

or $$\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \frac{1}{EI}\begin{Bmatrix} -16.07 \\ -1.79 \end{Bmatrix}$$

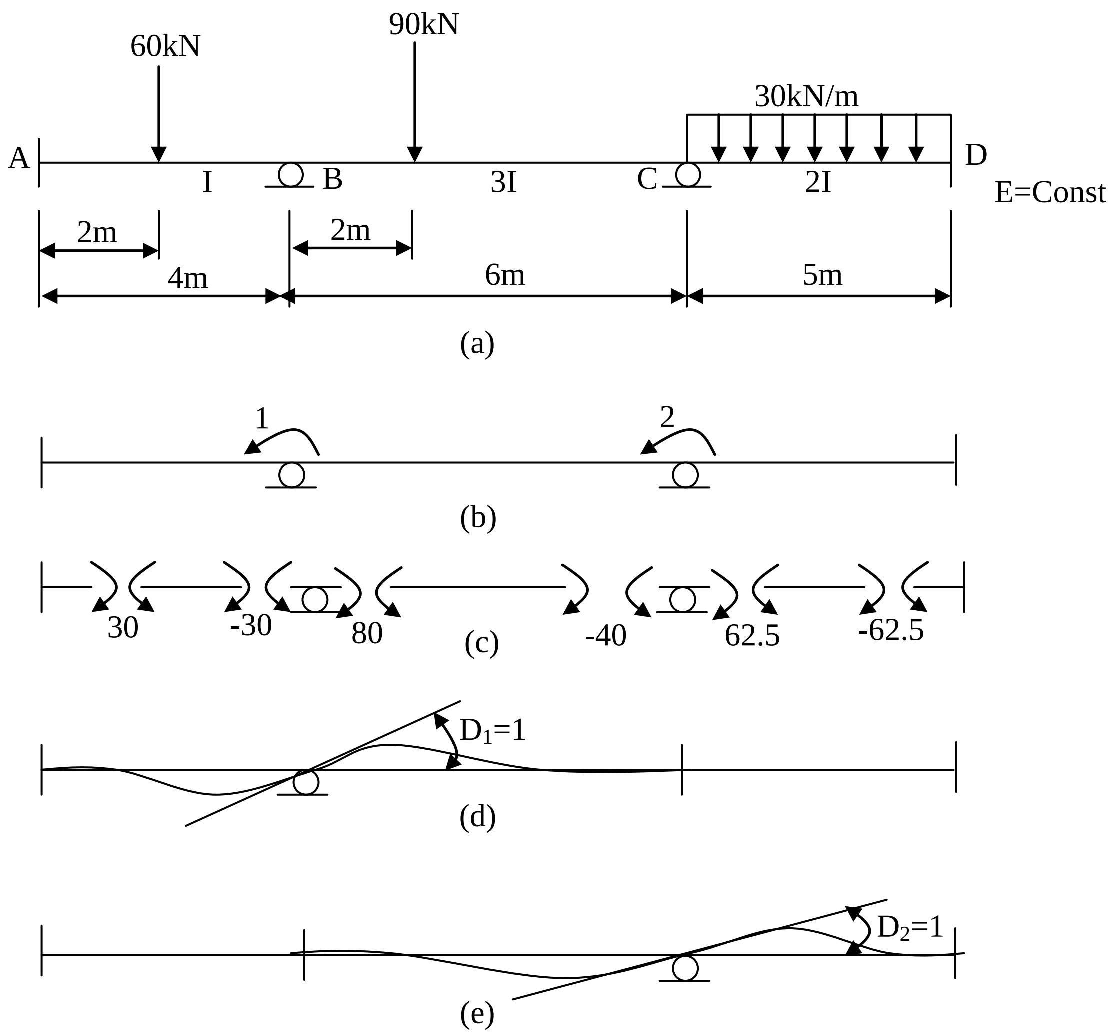


Fig.5.17: (a) Given beam with loading; (b) Coordinates;
(c) Joint end forces: (d) Unit displacement at coordinate 1;
(e) Unit displacement at coordinate 2.

The final end moments are obtained by substitution of D_1 and D_2 into the slope-deflection equation as follows:

$$M_{AB} = \frac{2EI}{4}\left[0 - \frac{16.07}{EI}\right] + 30 = 21.96kNm;$$

$$M_{BA} = \frac{2EI}{4}\left[\frac{2(-16.07)}{EI} + 0\right] - 30 = -46.07kNm;$$

$$M_{BC} = \frac{6EI}{6}\left[\frac{2(-16.07)}{EI} - \frac{1.79}{EI}\right] + 80 = 46.07kNm;$$

$$M_{CB} = \frac{6EI}{6}\left[\frac{2(-1.79)}{EI} - \frac{16.07}{EI}\right] - 40 = -59.65kNm;$$

$$M_{CD} = \frac{4EI}{5}\left[\frac{2(-1.79)}{EI} + 0\right] + 62.5 = 59.64kNm;$$

$$M_{AB} = \frac{4EI}{5}\left[0 - \frac{1.79}{EI}\right] - 62.5 = -63.93kNm.$$

Example 5.9

Analyse the frame shown in Fig.5.18(a) for member-end moments.

SOLUTION

Observe that this frame was previously analysed in Example 1.6 by the slope-deflection method. It is 2 degrees kinematically indeterminate with the independent rotations at B and D as the degrees of freedom. Accordingly, the 2 coordinates , 1 and 2, are shown in Fig.5.18(b). The fixed-end forces which had earlier been calculated in Example 1.6, are listed below:

$$FEM_{AB} = \frac{100 \times 8}{8} = 100kNm;$$

$$FEM_{BA} = -100kNm;$$

$$FEM_{BC} = FEM_{CB} = 0;$$

$$FEM_{BD} = \frac{15 \times 6^2}{12} = 45kNm;$$

$$FEM_{DB} = -45kNm;$$

$$FEM_{DE} = \frac{75 \times 6}{8} = 56.25kNm;$$

$$FEM_{ED} = -56.25kNm..$$

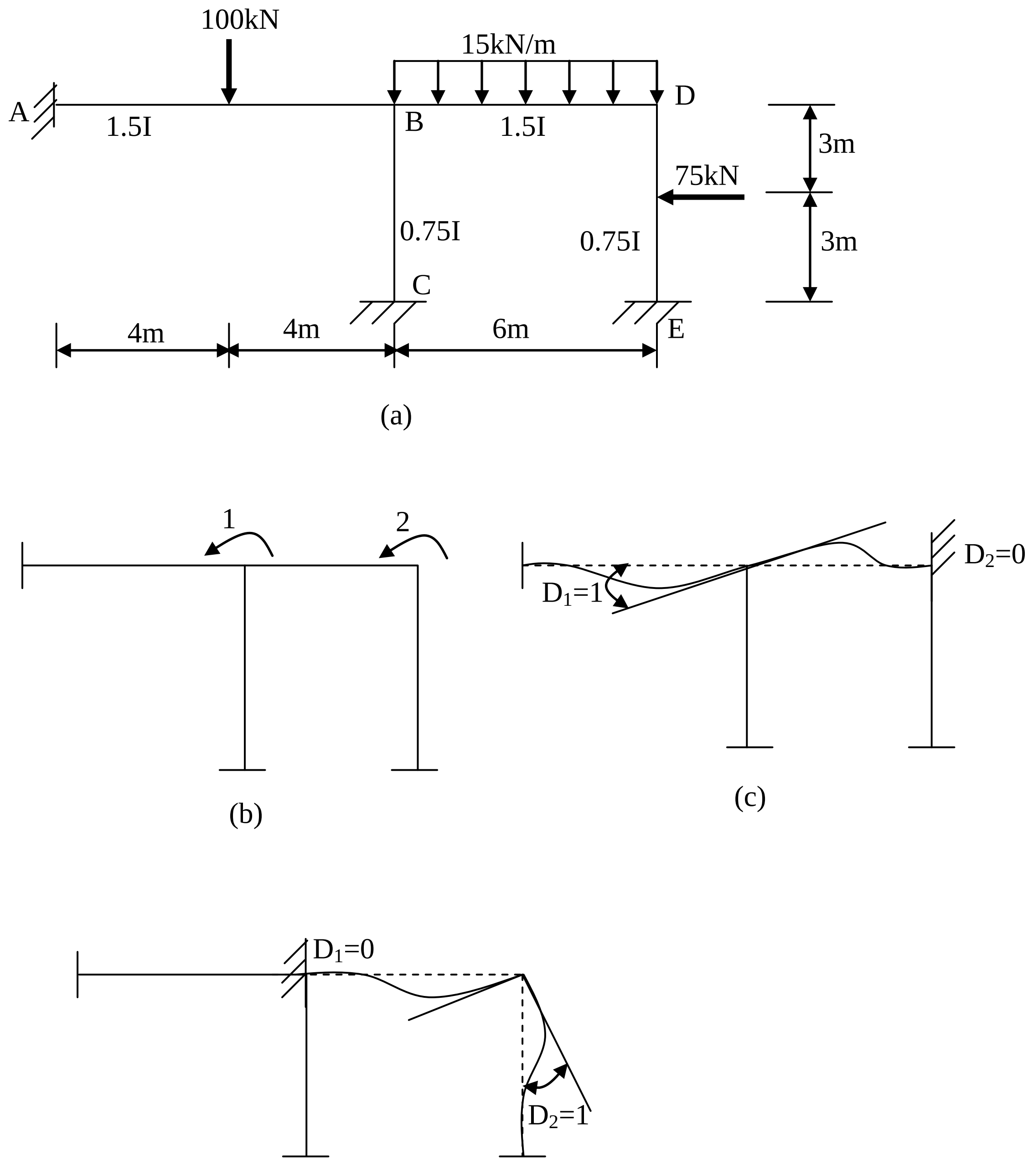


Fig.5.18: (a) Given frame with loading; (b) Coordinates;
(c) Unit displacement at coordinate 1; (d) Unit displacement at coordinate 2.

The elements of the first column of the stiffness matrix are obtained by determining the forces at the coordinates due to $D_1=1$ and $D_2=0$ (Fig.5.18(c)). Thus:

$$k_{11} = \frac{4(1.5EI)}{8} + \frac{4(0.75EI)}{6} + \frac{4(1.5EI)}{6} = 2.25EI$$

$$k_{21} = \frac{2(1.5EI)}{6} = 0.5EI$$

Similarly, the elements of the second column of the stiffness matrix are obtained by determining the forces at the coordinates due to D_2=1 and D_1=0 (Fig.5.18(d)) as follows:

$$k_{12} = k_{21} = 0.5EI \ ;$$

$$k_{22} = \frac{4(1.5EI)}{6} + \frac{4(0.75EI)}{6} = 1.5EI \ .$$

The vector of restraining forces at joints B and D will be:

$$\{P\} = \begin{Bmatrix} -55 \\ 11.25 \end{Bmatrix} kNm$$

The stiffness matrix $[K]$ will be:

$$[K] = EI\begin{bmatrix} 2.25 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$$

$$\therefore [K]^{-1} = \frac{1}{3.125EI}\begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 2.25 \end{bmatrix}$$

Substitution into eqn (5.23) gives:

$$\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \frac{1}{3.125EI}\begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 2.25 \end{bmatrix}\begin{Bmatrix} 55 \\ -11.25 \end{Bmatrix}$$

or $$\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \frac{1}{EI}\begin{Bmatrix} 28.2 \\ -16.9 \end{Bmatrix}$$

Member-end moments are obtained by substituting the above displacement quantities into the slope-deflection equation as follows:

$$M_{AB} = \frac{2(1.5EI)}{8}\left[0 + \frac{28.2}{EI}\right] + 100 = 110.58kNm \, ;$$

$$M_{BA} = \frac{2(1.5EI)}{8}\left[\frac{2(28.2)}{EI} + 0\right] - 100 = -78.85kNm \, ;$$

$$M_{BC} = \frac{2(0.75EI)}{6}\left[\frac{2(28.2)}{EI} + 0\right] + 0 = 14.1kNm;$$

$$M_{CB} = \frac{2(0.75EI)}{6}\left[0 + \frac{28.2}{EI}\right] + 0 = 7.05kNm \quad ;$$

$$M_{BD} = \frac{2(1.5EI)}{6}\left[\frac{2(28.2)}{EI} - \frac{16.9}{EI}\right] + 45 = 64.75kNm;$$

$$M_{DB} = \frac{2(1.5EI)}{6}\left[\frac{2(-16.9)}{EI} + \frac{28.2}{EI}\right] - 45 = -47.8kNm;$$

$$M_{DE} = \frac{2(0.75EI)}{6}\left[\frac{2(-16.9)}{EI} + 0\right] + 56.25 = 47.8kNm;$$

$$M_{ED} = \frac{2(0.75EI)}{6}\left[0 - \frac{16.9}{EI}\right] - 56.25 = -60.48kNm.$$