CHAPTER-II
INTRODUCTION TO MODELLING

Vibration Analysis Procedure

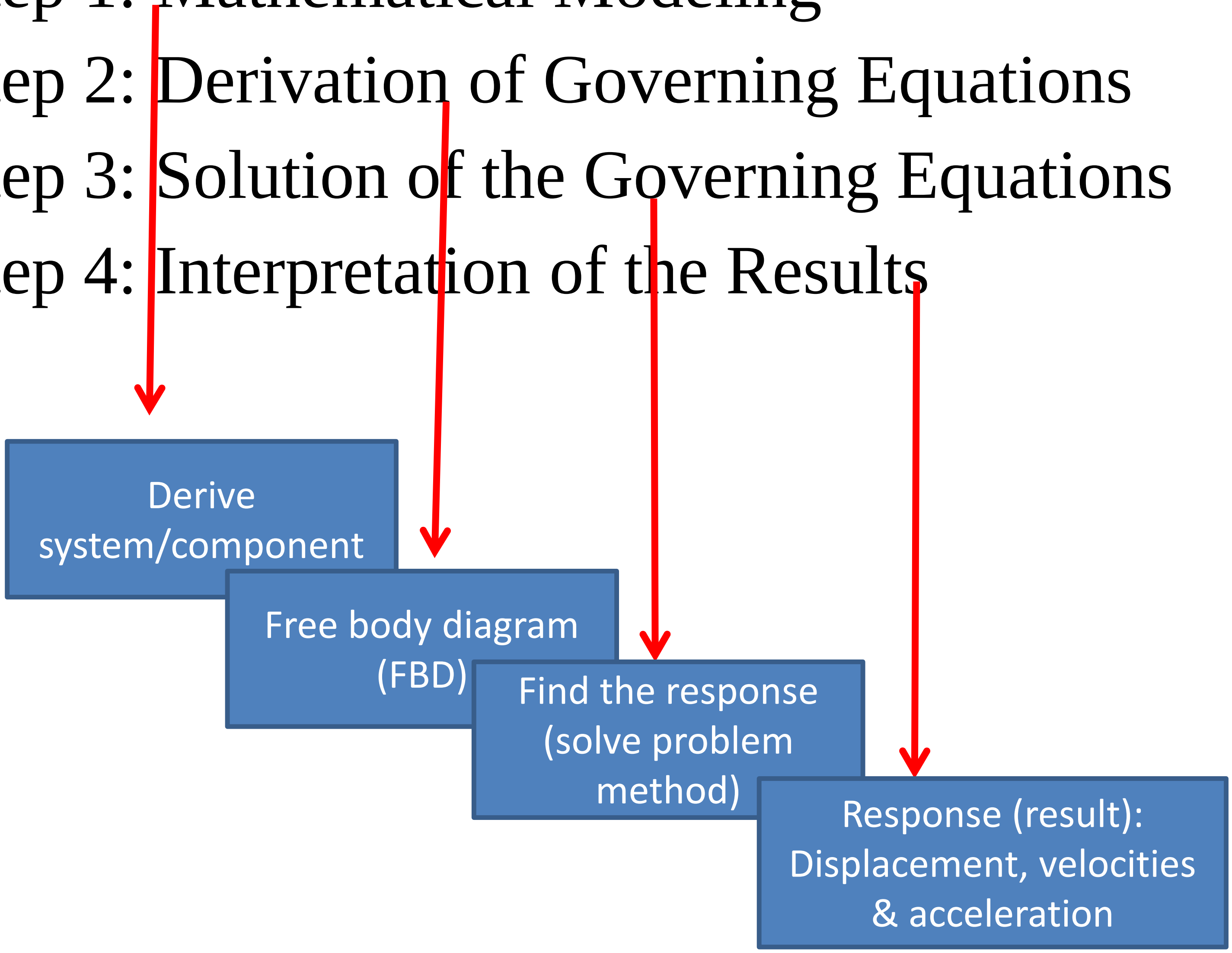
- vibratory system is a dynamic one for which the variables such as the excitations (inputs) and responses (outputs) are time dependent.
- The response of a vibrating system generally depends on the initial conditions as well as the external excitations.
- **Most practical vibrating systems are very complex**, and it is impossible to consider all the details for a mathematical analysis.
- **Only the most important features are considered in the analysis** to predict the behavior of the system under specified input conditions.
- Thus, the analysis of a vibrating system usually involves:
  - **mathematical modeling,**
  - **derivation of the governing equations,**
  - **solution of the equations, and**
  - **interpretation of the results.**

Step 1: Mathematical Modeling

Step 2: Derivation of Governing Equations

Step 3: Solution of the Governing Equations

Step 4: Interpretation of the Results



Step 1: Mathematical Modelling

- The mathematical model should include enough details to allow describing the system in terms of equations without making it too complex.
- The mathematical model may be linear or nonlinear, depending on the behaviour of the systems components.
- Linear models permit quick solutions and are simple to handle;
- However, nonlinear models sometimes reveal certain characteristics of the system that cannot be predicted using linear models.

Consider the forging hammer shown in Fig.

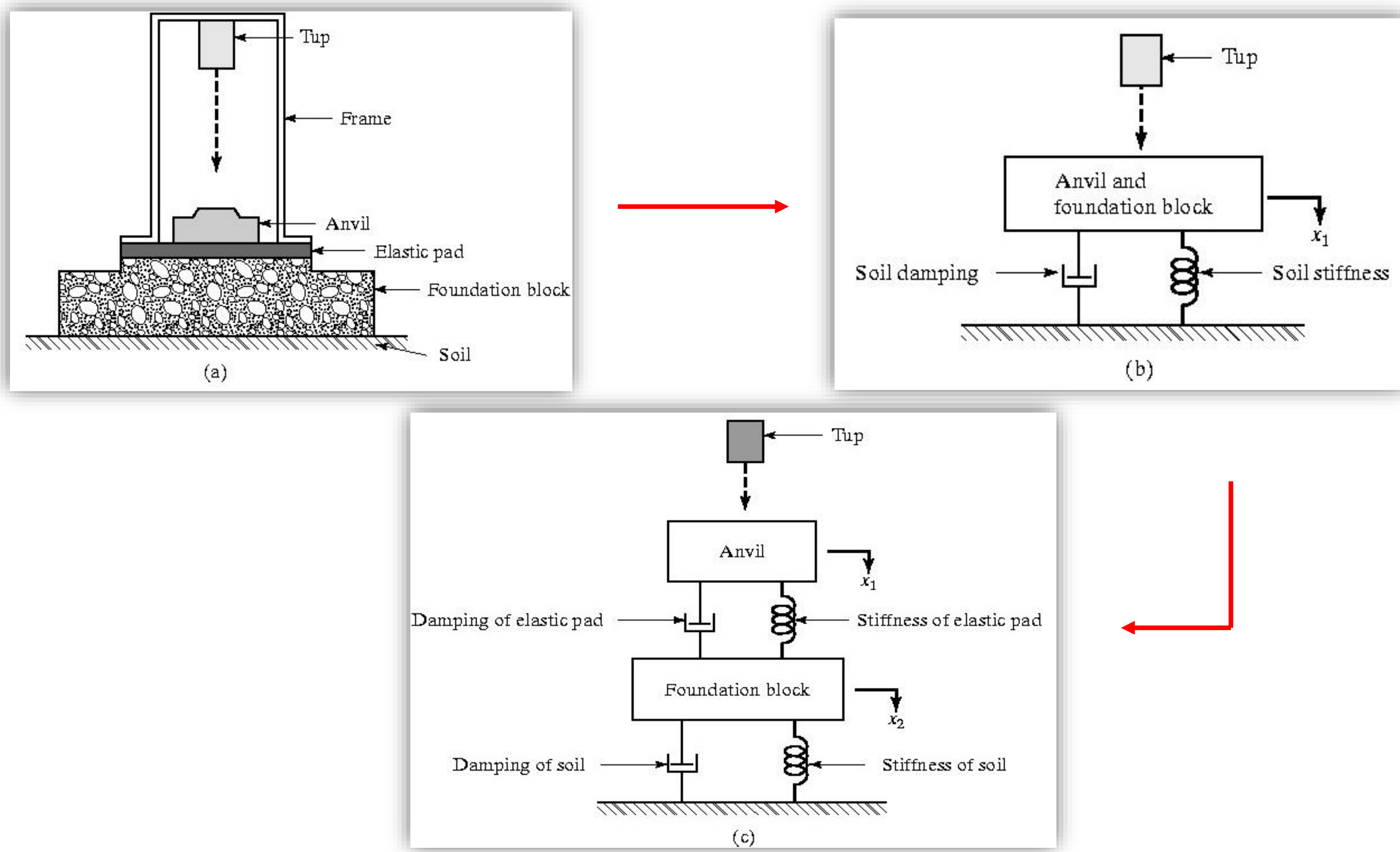


Fig. Modelling of a forging hammer

Step 2: Derivation of Governing Equations

- Once the mathematical model is available, we use the **principles of dynamics** and derive the equations that describe the vibration of the system.
- The **equations of motion** can be derived conveniently by drawing the **free-body diagrams** of all the masses involved.
- The free-body diagram of a mass can be obtained by isolating the mass and indicating **all externally applied forces, the reactive forces, and the inertia forces.**

Step 3: Solution of the Governing Equations

- The equations of motion must be solved to find the response of the vibrating system.
- Depending on the nature of the problem, we can use one of the following techniques for finding the solution:
  - ✓ **standard methods of solving differential equations,**
  - ✓ **Laplace transform methods, matrix methods, and**
  - ✓ **numerical methods.**
- Furthermore, the solution of partial differential equations is far more involved than that of ordinary differential equations.
- **Numerical methods involving computers** can be used to solve the equations. However, it will be difficult to draw general conclusions about the behaviour of the system using computer results.

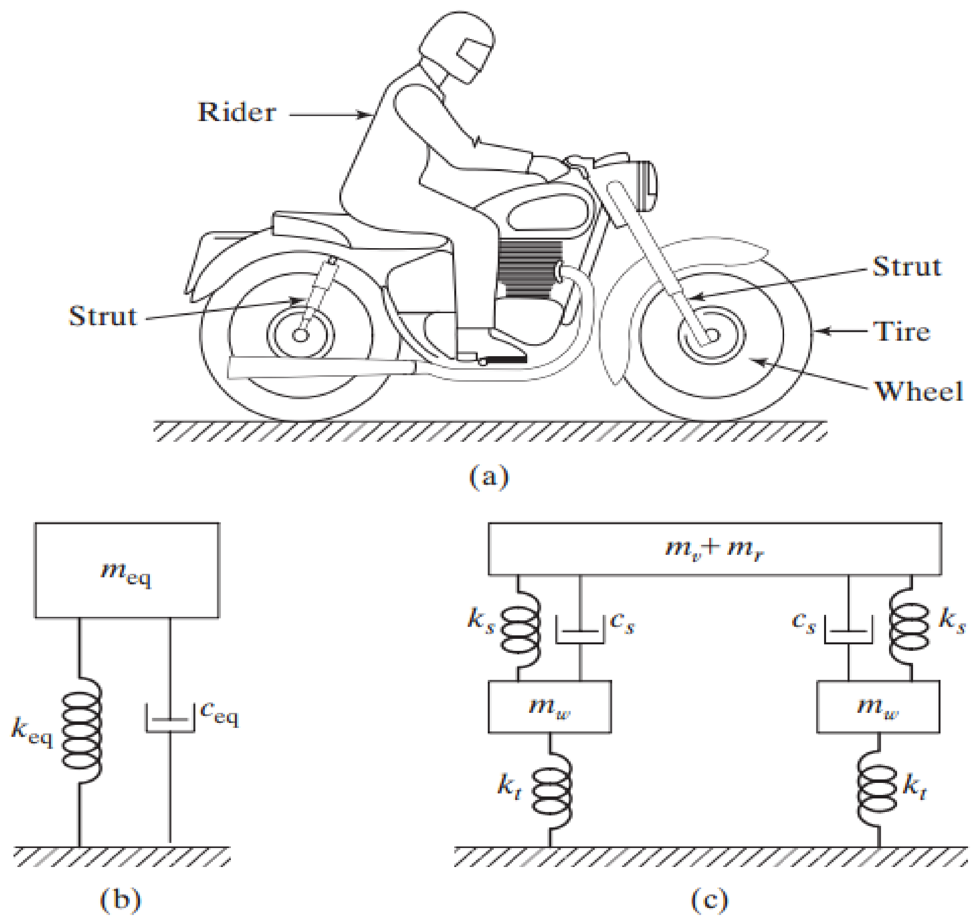
Step 4: Interpretation of the Results

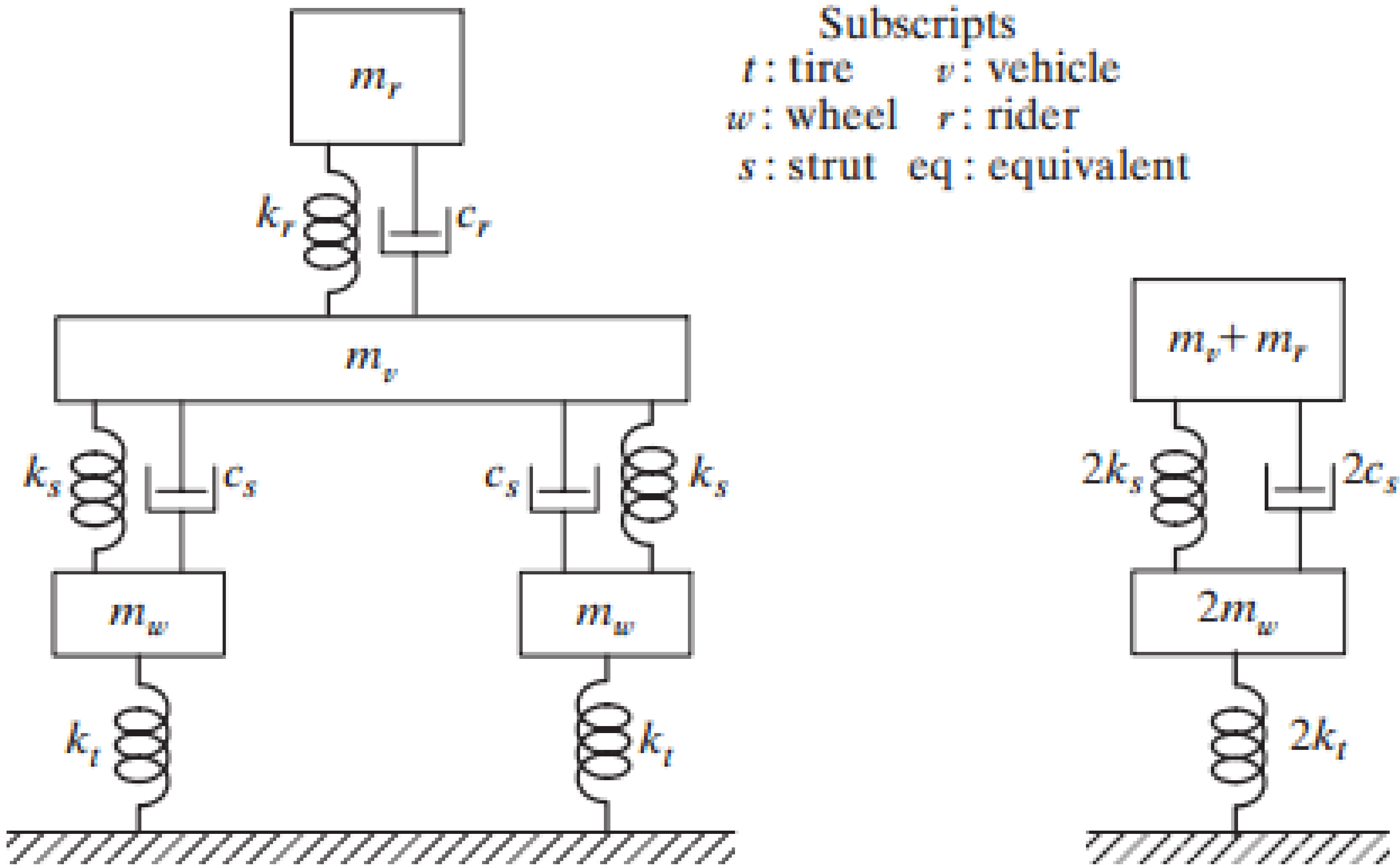
The solution of the governing equations gives the

- **displacements**
- **velocities**
- **accelerations** of the various masses of the system.

These results must be interpreted with a clear view of the purpose of the analysis and the possible design implications of the results.

E.g. Figure below shows a motorcycle with a rider. Develop a sequence of three mathematical models of the system for investigating vibration in the vertical direction. Consider the elasticity of the tires, elasticity and damping of the struts (in the vertical direction), masses of the wheels, and elasticity, damping, and mass of the rider.





Subscripts
t : tire v : vehicle
w : wheel r : rider
s : strut eq : equivalent
m_r
k_r
c_r
m_v
k_s
c_s
c_s
k_s
m_w
m_w
k_t
k_t
m_v + m_r
2k_s
2c_s
2m_w
2k_t

(d) (e)

Mechanical Elements

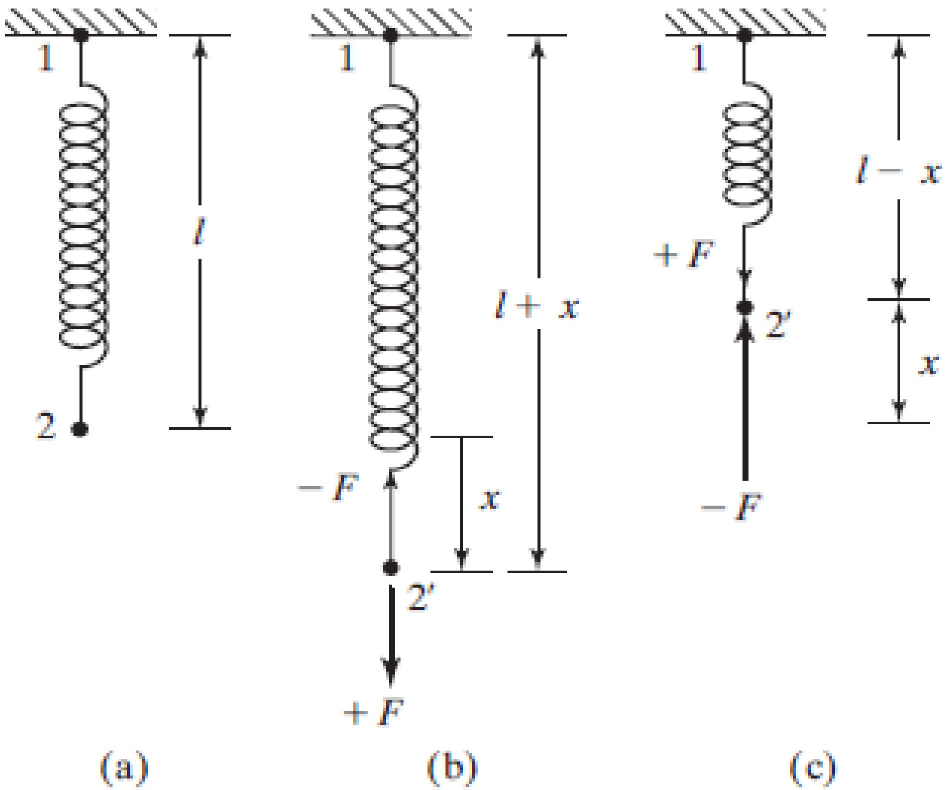
- The physical model of mechanical system consists of elements with known mechanical behavior.
- These elements are the **mass, spring, rod, beam, plate etc.,** and used as **building blocks** in the development of physical models of dynamic system.
- *The **physical model** is constructed then the **mathematical modeling** follows.*
- On the basis of physical model, the mathematical description is known as **equation of motion can be formulated.**

I. SPRING

- A spring is a type of mechanical link, which in most applications is assumed to have **negligible mass and damping.**
- The most common type of spring is the **helical-coil spring** used in retractable pens and pencils, staplers, and suspensions of freight trucks and other vehicles.
- Several other types of springs can be identified in engineering applications.
- In fact, any elastic or deformable body or member such as **a cable, bar, beam, shaft or plate** can be considered as a spring.

Contd.,

A spring is commonly represented as shown in Fig.



Contd.,

- **A spring is said to be linear** if the elongation or reduction in length x *is related to the* applied force F *as* where k *is a constant*, *known as the spring constant or spring stiffness or spring rate.*

$$F = kx$$

- The spring constant k is always positive and denotes the force (positive or negative) required to cause a unit deflection (elongation or reduction in length) in the spring.
- The work done (U) in deforming a spring is stored as strain or potential energy in the spring, and it is given by

$$U = \tfrac{1}{2}\, kx^2$$

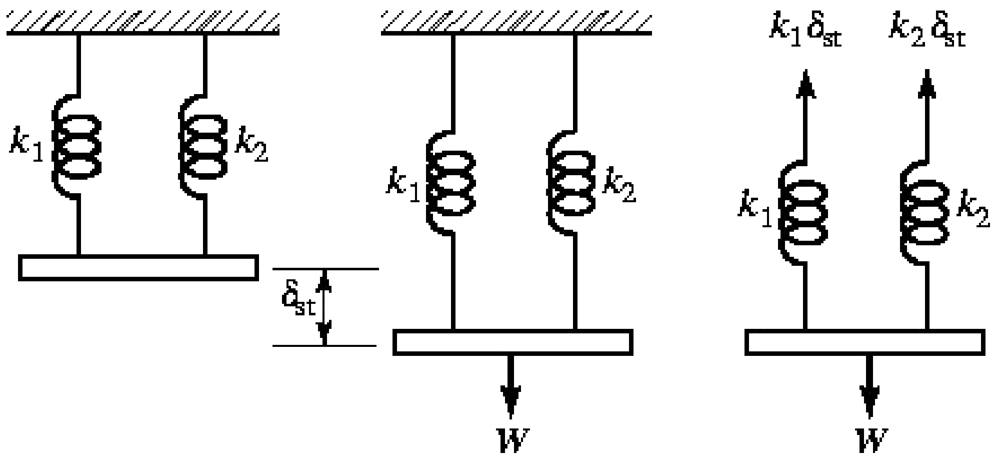
Contd.,

Combination of Springs:

Sometimes more than one spring is used in a vibrating system. The motion of the system will depend on the way the springs are connected.

1) Springs in parallel – if we have *n* spring constants k_1, k_2, …, k_n in *parallel*, then the equivalent spring constant k_{eq} is:

$$k_{eq} = k_1 + k_2 + \ldots + k_n$$

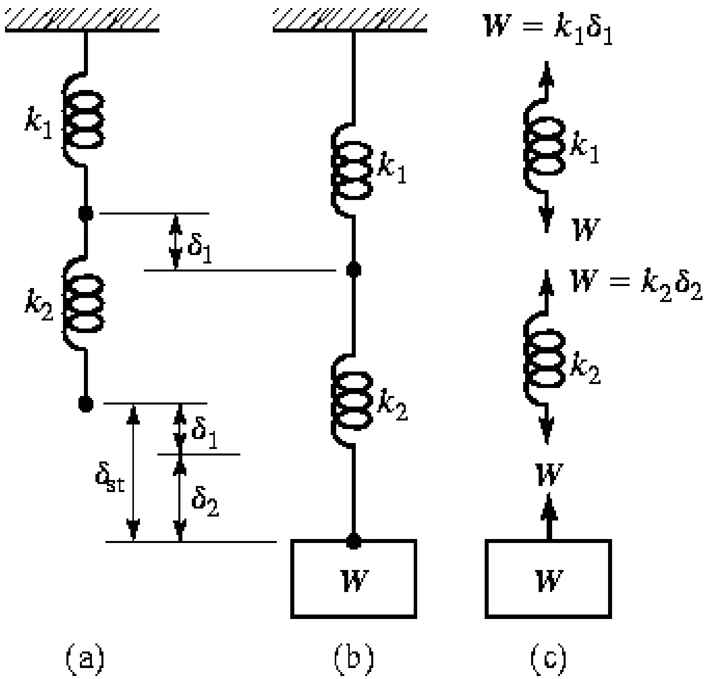


(a) (b) (c)

Contd.,

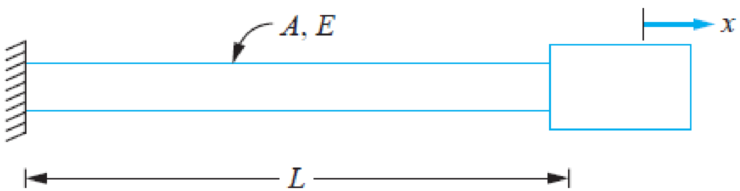
2) Springs in series – if we have *n* spring constants k_1, k_2, …, k_n in *series*, then the equivalent spring constant k_{eq} is:

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_n}$$

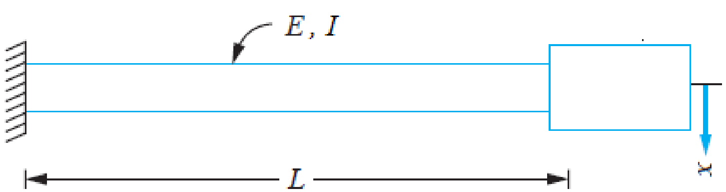


Elastic elements as springs

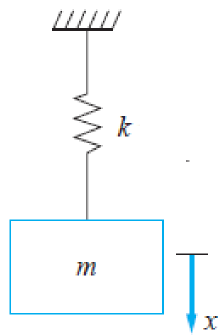
System | Stiffness Coeff. | SDOF Model

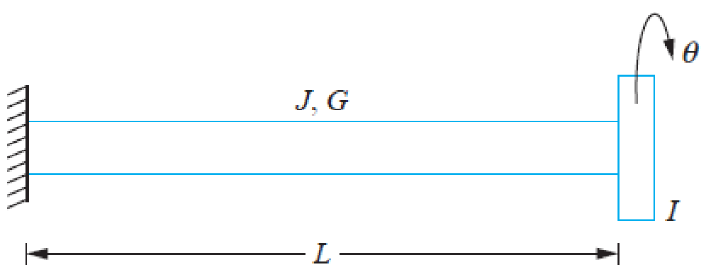


$$k = \frac{AE}{L}$$

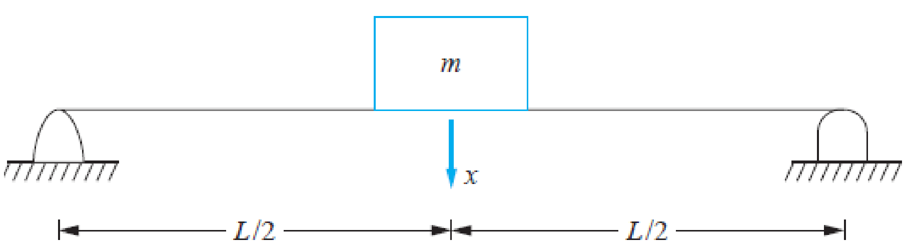


$$k = \frac{48EI}{L^3}$$





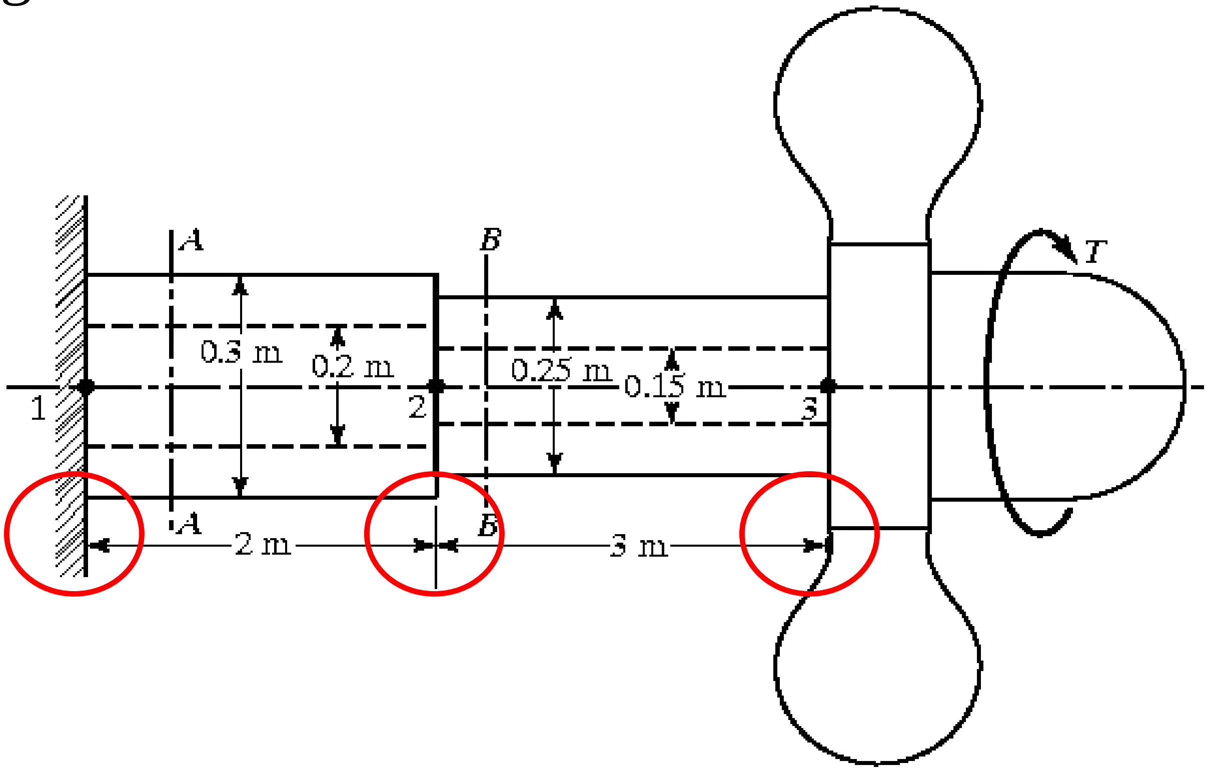
$$k = \frac{JG}{L}$$



$$k = \frac{3EI}{L^3}$$

Example 1

Determine the torsional spring constant of the speed propeller shaft shown in Fig.



Solution

We need to consider the segments 12 and 23 of the shaft as springs in combination. From Fig., the torque induced at any cross section of the shaft (such as *AA* or *BB*) can be seen to be equal to the torque applied at the propeller, *T*.

Hence, the elasticity (springs) corresponding to the two segments 12 and 23 are to be considered as series springs. The spring constants of segments 12 and 23 of the shaft (k_{t12} and k_{t23}) are given by:

$$k_{t_{12}} = \frac{GJ_{12}}{l_{12}} = \frac{G\pi(D_{12}^4 - d_{12}^4)}{32 l_{12}} = \frac{(80\times10^9)\pi(0.3^4 - 0.2^4)}{32(2)} = 25.5255\times10^6 \text{ N-m/rad}$$

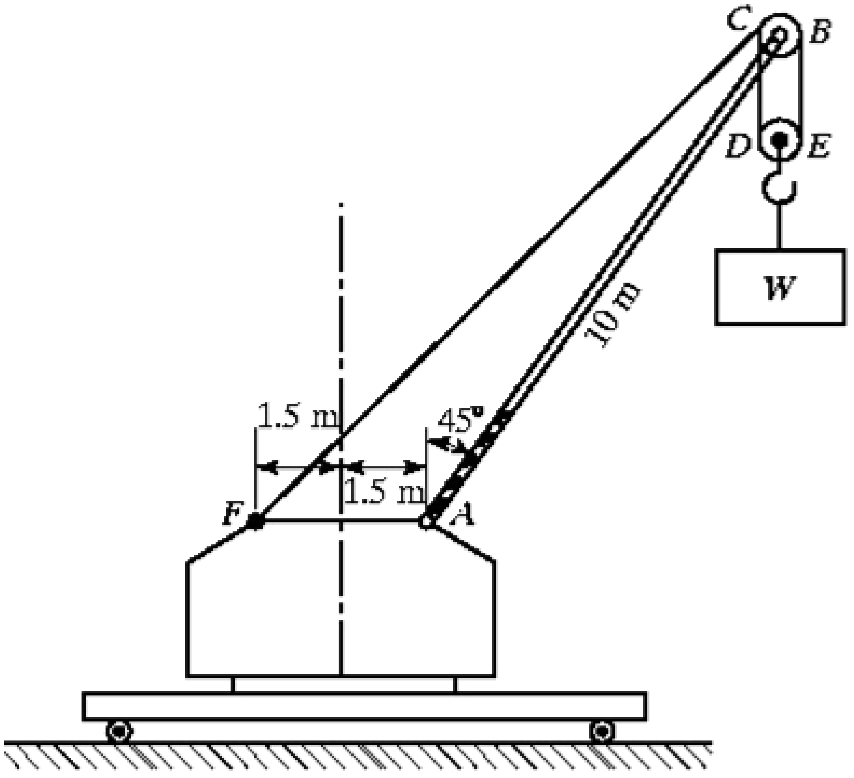
$$k_{t_{23}} = \frac{GJ_{23}}{l_{23}} = \frac{G\pi(D_{23}^4 - d_{23}^4)}{32 l_{23}} = \frac{(80\times10^9)\pi(0.25^4 - 0.15^4)}{32(3)} = 8.9012\times10^6 \text{ N-m/rad}$$

Since the springs are in series, Eq. gives

$$k_{t_{eq}} = \frac{k_{t_{12}} k_{t_{23}}}{k_{t_{12}} + k_{t_{23}}} = \frac{(25.5255\times10^6)(8.9012\times10^6)}{(25.5255\times10^6 + 8.9012\times10^6)} = 6.5997\times10^6 \text{ N-m/rad}$$

Example-2

The boom *AB* of crane is a uniform steel bar of length 10 m and x-section area of 2,500 mm^2. A weight *W* is suspended while the crane is stationary. Steel cable *CDEBF* has x-sectional area of 100 mm^2. Neglect effect of cable *CDEB*, find equivalent spring constant of system in the vertical direction.



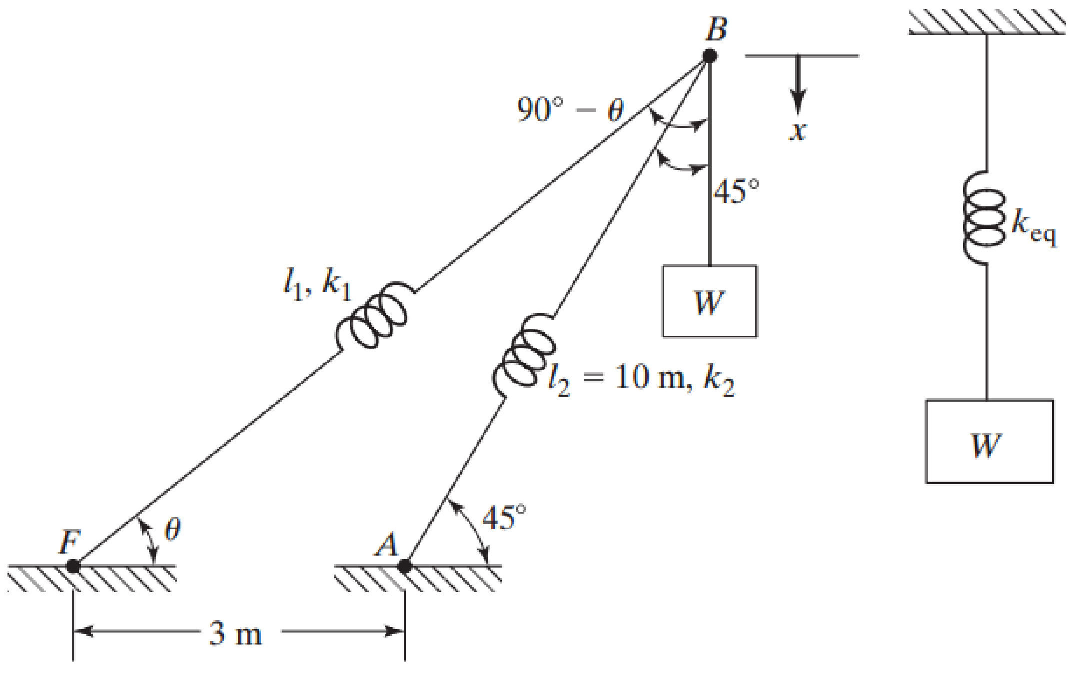
(a)

Solution

A vertical displacement x of pt B will cause the spring k_2 (boom) to deform by $x_2 = x \cos 45^\circ$ and the spring k_1 (cable) to deform by an amount $x_1 = x \cos (90^\circ - \theta)$. Length of cable FB, l_1 is as shown.

$$l_1^2 = 3^2 + 10^2 - 2(3)(10)\cos 135^\circ = 151.426$$

$$\therefore l_1 = 12.3055 \text{ m}$$



Solution

- The angle θ satisfies the relation:

$$l_1^2 + 3^2 - 2(l_1)(3)\cos\theta = 10^2$$

$$\cos\theta = 0.8184, \qquad \therefore \theta = 35.0736^\circ$$

- The total potential energy (U):

$$U = \frac{1}{2}k_1(x\cos 45^\circ)^2 + \frac{1}{2}k_2[x\cos(90^\circ - \theta)]^2 \qquad (E.1)$$

$$k_1 = \frac{A_1E_1}{l_1} = \frac{(100\times10^{-6})(207\times10^{9})}{12.0355} = 1.6822\times10^{6}\ \text{N/m}$$

$$k_2 = \frac{A_2E_2}{l_2} = \frac{(2500\times10^{-6})(207\times10^{9})}{10} = 5.1750\times10^{7}\ \text{N/m}$$

Solution

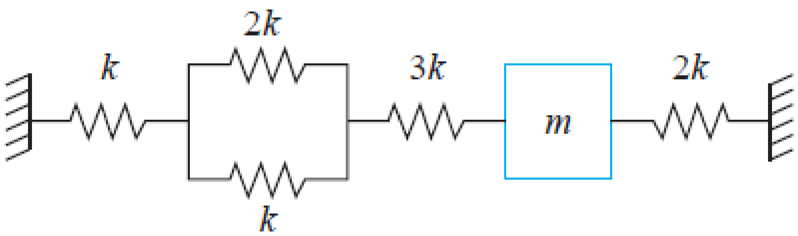
- Potential Energy of the equivalent spring is:

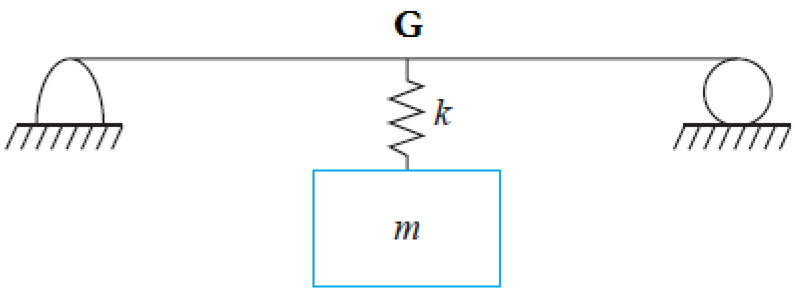
$$U_{eq} = \frac{1}{2} k_{eq} x^2 \qquad \text{(E.2)}$$

- By setting $U = U_{eq}$, we obtain the equivalent spring constant of the system as:

$$k_{eq} = k_1 \sin^2 \theta + k_2 \sin^2 45° = k_1 \sin^2 35.0736° + k_2 \sin^2 45° = 26.4304 \times 10^6 \text{ N/m}$$

What are the equivalent stiffnesses?





Do it yourself !

Do it yourself !

Example

A 200-kg machine is attached to the end of a cantilever beam of length L= 2.5 m, elastic modulus E= 200x10^9 N/m^2, and cross-sectional moment of inertia I = 1.8x10^{-6} m4. Assuming the mass of the beam is small compared to the mass of the machine, what is the stiffness of the beam?

Do it yourself !

Do it yourself !

II. Mass or inertia elements

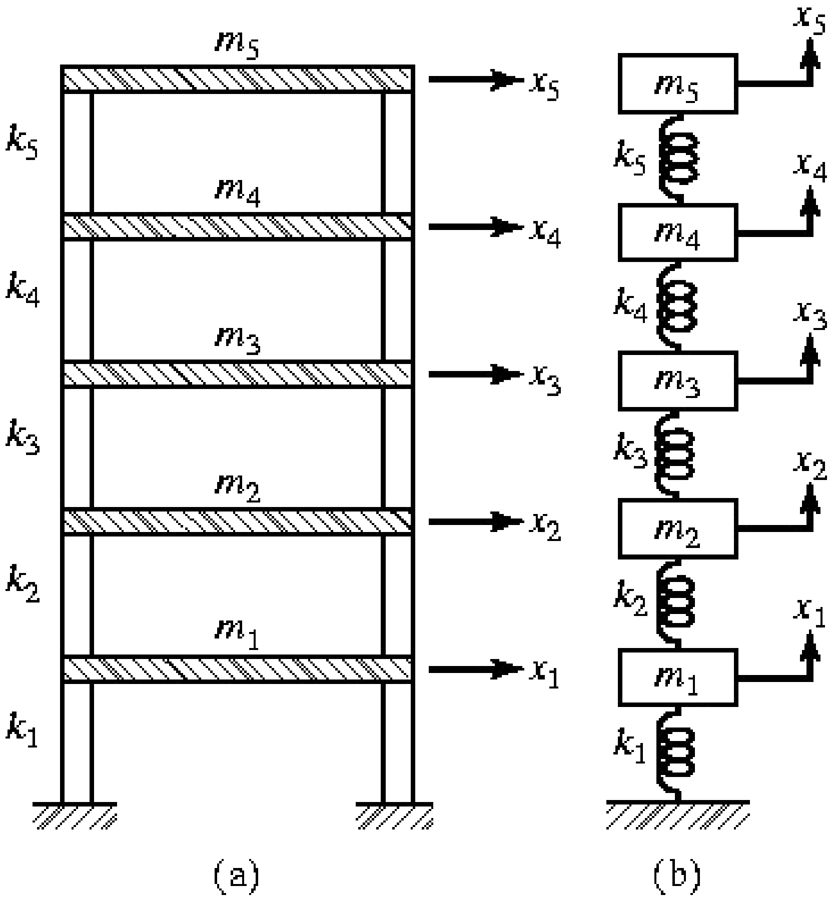
- The mass or inertia element is assumed to be a rigid body; it can gain or lose kinetic energy whenever the velocity of the body changes.
- From Newton's second law of motion, the product of the mass and its acceleration is equal to the force applied to the mass.

Ex: Idealization of a multi story building as a multi-degree-of-freedom system

In many practical applications, several masses appear in combination.

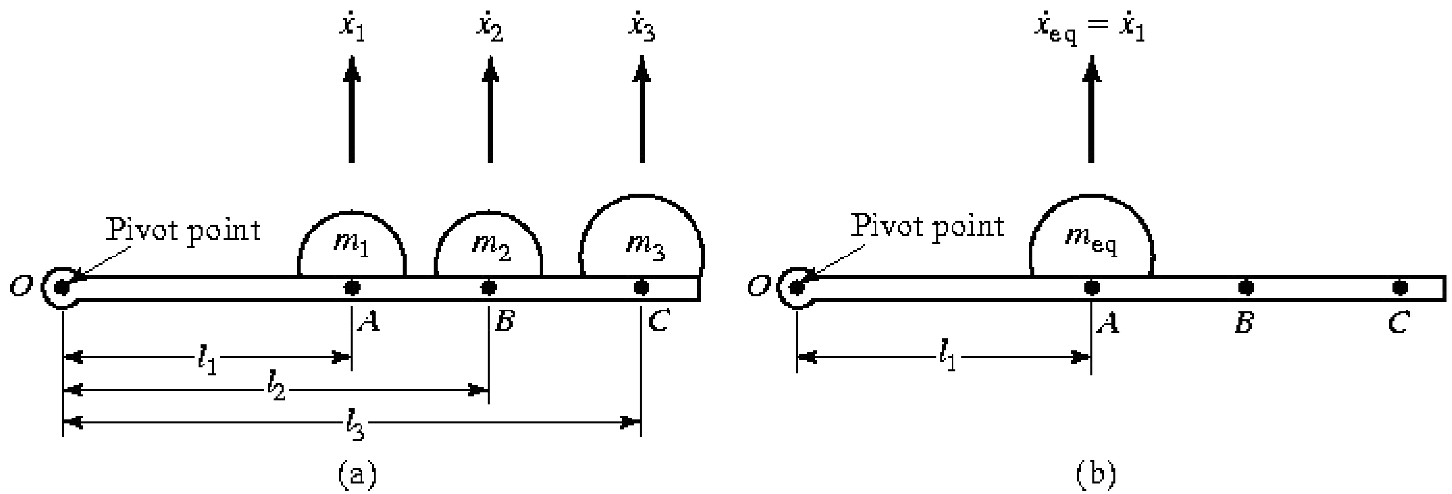
Combination of Masses

- Assume that the mass of the frame is negligible compared to the masses of the floors.
- The masses of various floor levels represent the mass elements, and the elasticity of the vertical members denote the spring elements.



Mass or Inertia Elements

- Case 1: Translational Masses Connected by a Rigid Bar



- Velocities of masses can be expressed as:

$$\dot{x}_2 = \frac{l_2}{l_1}\dot{x}_1 \qquad \dot{x}_3 = \frac{l_3}{l_1}\dot{x}_1 \qquad (1.18)$$

Mass or Inertia Elements

and,

$$\dot{x}_{eq} = \dot{x}_1 \qquad (1.19)$$

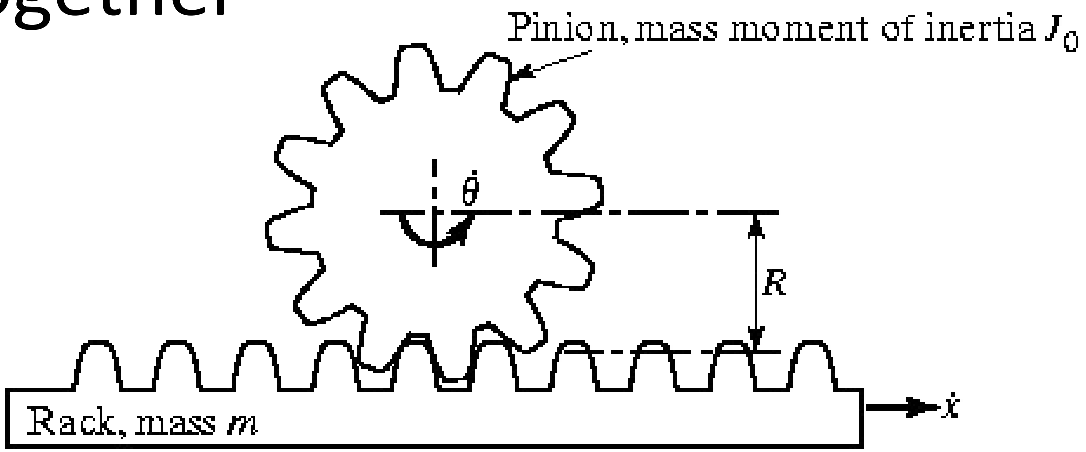
- By equating the kinetic energy of the system:

$$\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 = \frac{1}{2}m_{\text{eq}}\dot{x}_{\text{eq}}^2 \qquad (1.20)$$

$$m_{\text{eq}} = m_1 + \left(\frac{l_2}{l_1}\right)^2 m_2 + \left(\frac{l_3}{l_1}\right)^2 m_3 \qquad (1.21)$$

Mass or Inertia Elements

❑ Case 2: Translational and Rotational Masses Coupled Together



m_{eq} = single equivalent translational mass

$\dot{x}$ = translational velocity

$\dot{\theta}$ = rotational velocity

J_0 = mass moment of inertia

J_{eq} = single equivalent rotational mass

Mass or Inertia Elements

- Case 2: Translational and Rotational Masses Coupled Together

1. Equivalent translational mass:

   - Kinetic energy of the two masses is given by:

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 \qquad (1.22)$$

   - Kinetic energy of the equivalent mass is given by:

$$T_{\text{eq}} = \frac{1}{2} m_{\text{eq}} \dot{x}_{\text{eq}}{}^2 \qquad (1.23)$$

Case 2: Translational and Rotational Masses Coupled Together

- Since $\dot{\theta} = \dfrac{\dot{x}}{R}$ and $\dot{x}_{\text{eq}} = \dot{x}$, equating T_{eq} & T gives

$$m_{\text{eq}} = m + \frac{J_0}{R^2} \qquad (1.24)$$

2. Equivalent rotational mass:

- Here, $\dot{\theta}_{\text{eq}} = \dot{\theta}$ and $\dot{x} = \dot{\theta} R$, equating T_{eq} and T gives

$$\frac{1}{2} J_{\text{eq}} \dot{\theta}^2 = \frac{1}{2} m \left(\dot{\theta} R\right)^2 + \frac{1}{2} J_0 \dot{\theta}^2$$

$$or \qquad J_{\text{eq}} = J_0 + mR^2 \qquad (1.25)$$

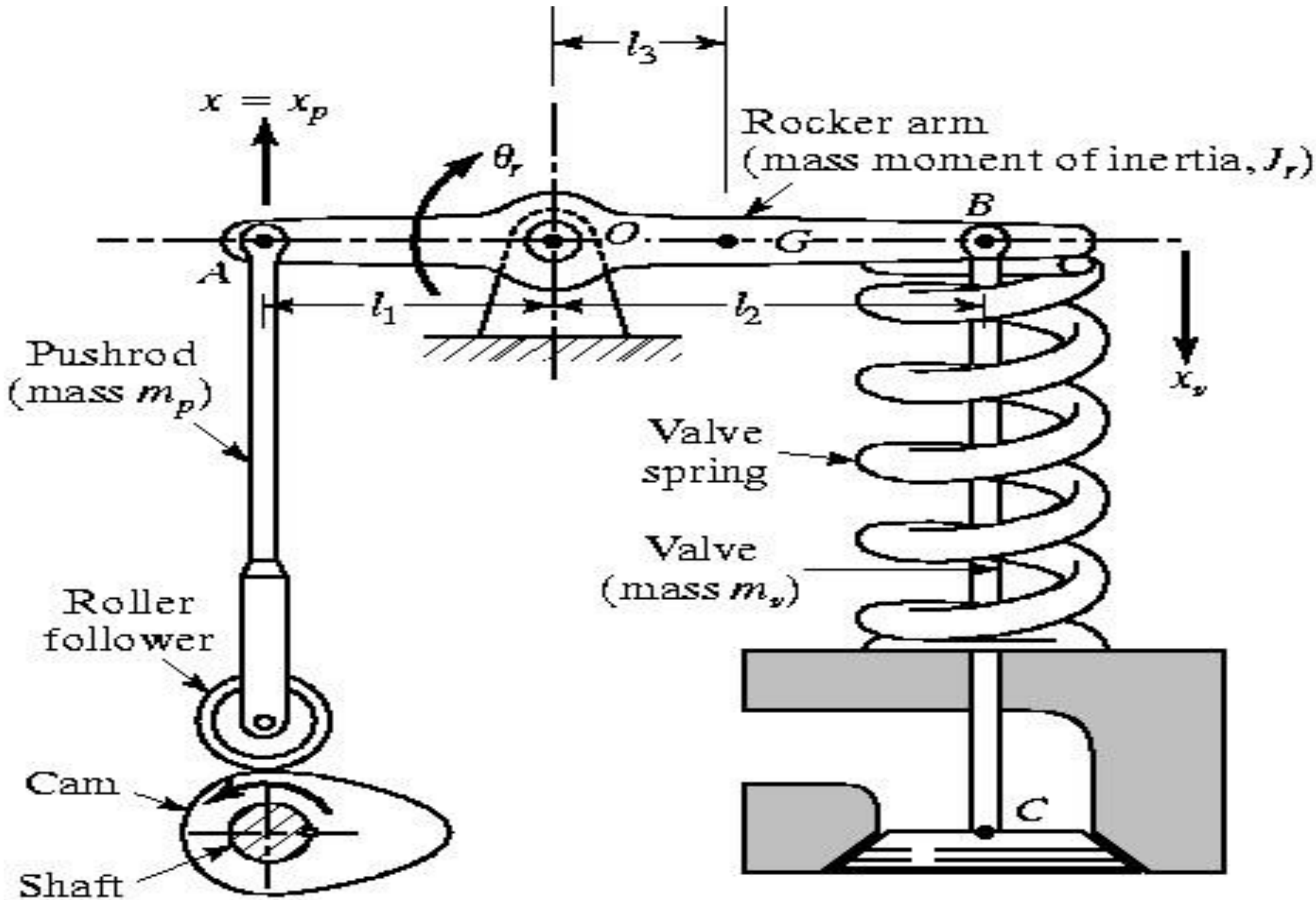
Example 1.7
Cam-Follower Mechanism

A cam-follower mechanism is used to convert the rotary motion of a shaft into the oscillating or reciprocating motion of a valve.

The follower system consists of a pushrod of mass m_p, a rocker arm of mass m_r, and mass moment of inertia J_r about its C.G., a valve of mass m_v, and a valve spring of negligible mass.

Find the equivalent mass (m_{eq}) of this cam-follower system by assuming the location of m_{eq} as (i) point A and (ii) point C.

Example 1.7
Cam-Follower Mechanism



Example 1.7
Solution

The kinetic energy of the system *(T)* is:

$$T = \frac{1}{2} m_p \dot{x}_p^2 + \frac{1}{2} m_v \dot{x}_v^2 + \frac{1}{2} J_r \dot{\theta}_r^2 + \frac{1}{2} m_r \dot{x}_r^2 \qquad (\text{E.1})$$

If m_{eq} denotes equivalent mass placed at point *A*, with $\dot{x}_{\text{eq}} = \dot{x}$

the kinetic energy equivalent mass system T_{eq} is:

$$T_{\text{eq}} = \frac{1}{2} m_{\text{eq}} \dot{x}_{\text{eq}}^2 \qquad (\text{E.2})$$

Example 1.7
Solution

By equating T and T_{eq}, and note that

$$\dot{x}_p = \dot{x},\ \dot{x}_v = \frac{\dot{x}l_2}{l_1},\ \dot{x}_r = \frac{\dot{x}l_3}{l_1},\ \text{and}\ \dot{\theta}_r = \frac{\dot{x}}{l_1}$$

$$m_{\text{eq}} = m_p + \frac{J_r}{l_1^2} + m_v\frac{l_2^2}{l_1^2} + m_r\frac{l_3^2}{l_1^2} \qquad \text{(E.3)}$$

Similarly, if equivalent mass is located at point C,
$\dot{x}_{\text{eq}} = \dot{x}_v$, hence,

$$T_{\text{eq}} = \frac{1}{2}m_{\text{eq}}\dot{x}_{\text{eq}}^2 = \frac{1}{2}m_{\text{eq}}\dot{x}_v^2 \qquad \text{(E.4)}$$

Example 1.7
Solution

Equating (E.4) and (E.1) gives

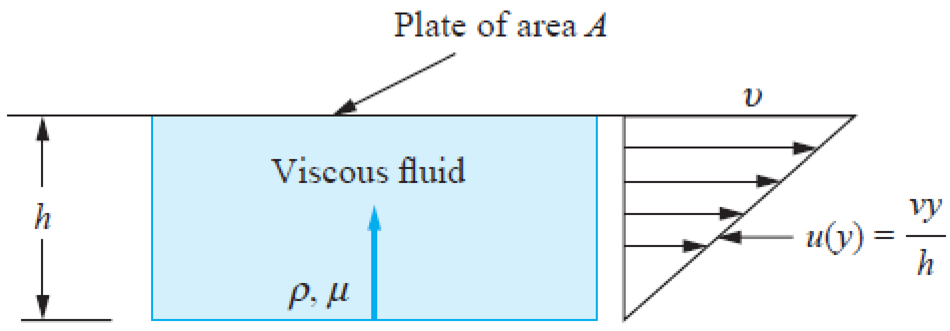
$$m_{\text{eq}} = m_v + \frac{J_r}{l_2^2} + m_p\left(\frac{l_1}{l_2}\right)^2 + m_r\left(\frac{l_3^2}{l_1^2}\right)^2 \qquad \text{(E.5)}$$

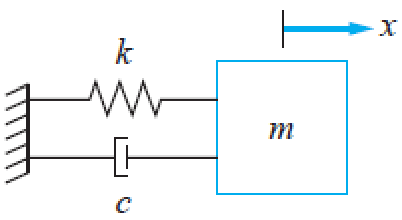
III. Damping Elements

- In many practical systems, the vibrational energy is gradually converted to heat or sound.
- Due to the reduction in the energy, the response, such as the displacement of the system, gradually decreases.
- The mechanism by which the vibrational energy is gradually converted into heat or sound is known as damping.
- Damping is the energy dissipation properties of a material or system under cyclic stress.
- Damping is modelled as one or more of the following types.
  1. **Viscous Damping**
  2. **Coulomb or Dry-Friction Damping**
  3. **Material or Solid or Hysteretic Damping**

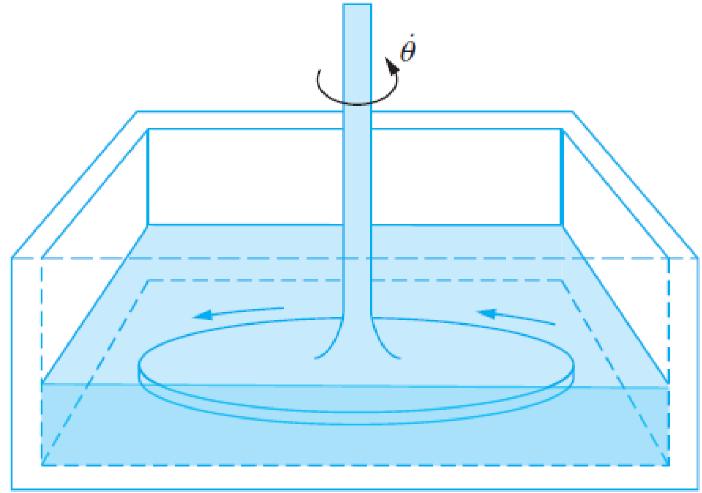
Damping

Viscous Damping





$$F = cv \qquad c = \frac{\mu A}{h}$$



$$M = c_t\dot{\theta}$$

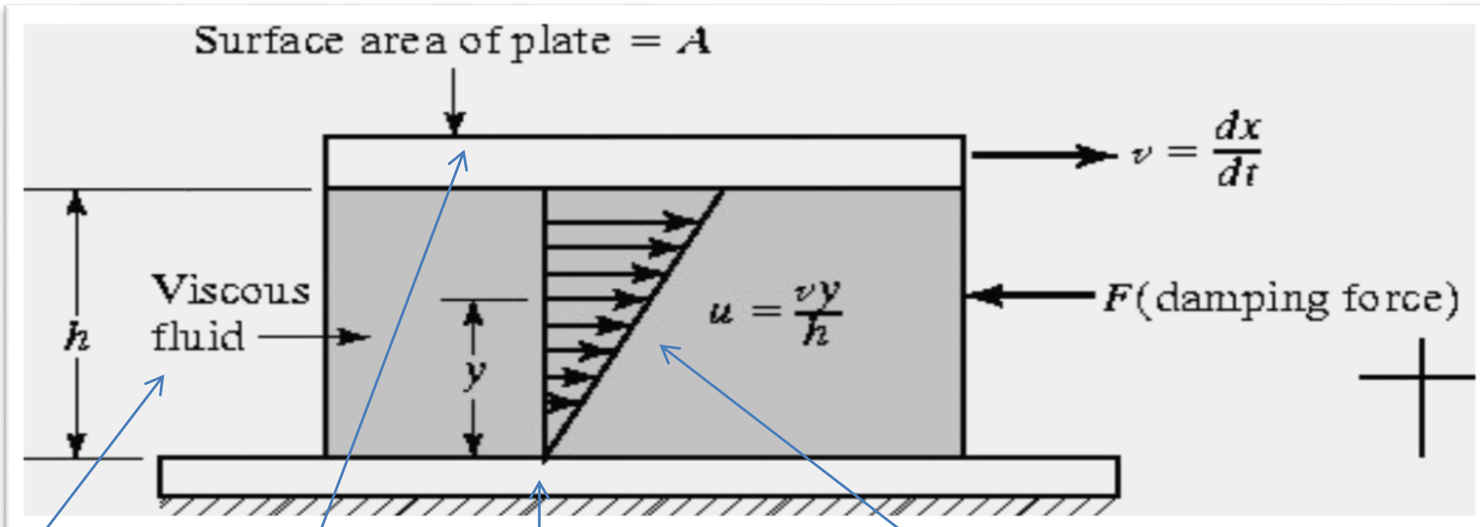
Damping Elements

- **Viscous Damping:**
  Damping force is proportional to the velocity of the vibrating body in a fluid medium such as air, water, gas, and oil.
- **Coulomb or Dry Friction Damping:**
  Damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body between dry surfaces.
- **Material or Solid or Hysteretic Damping:**
  Energy is absorbed or dissipated by material during deformation due to friction between internal planes.

Example
Damping Constant of Parallel Plates Separated by Viscous Fluid

- Consider two parallel plates separated by a distance h, with a fluid of viscosity μ between the plates. Derive an expression for the damping constant when one plate moves with a velocity v relative to the other as shown.

Construction of Viscous Dampers



μ

Fixed plane

Velocity of intermediate fluid layers are assumed to vary linearly

Plate be moved with a velocity v in its own plane

Solution

- *According to Newton's law of viscous flow, the shear Stress (τ)* developed in the fluid layer at a distance y from the fixed plate is:

$$\tau = \mu \frac{du}{dy} \qquad (1.26)$$

where $du/dy = v/h$ is the velocity gradient.

- *The Shear* or *Resisting Force (F)* developed at the bottom surface of the moving plate is:

$$F = \tau A = \mu \frac{Av}{h} = cv \qquad (1.27)$$

where A is the surface area of the moving plate and $c = \frac{\mu A}{h}$ is the damping constant.

Combination of Dampers

- In some dynamic systems, multiple dampers are used. In such cases, all the dampers are replaced by a single equivalent damper.
- For example, when two translational dampers, with damping constants c_1 and c_2 appear in combination, the equivalent damping constant can be:

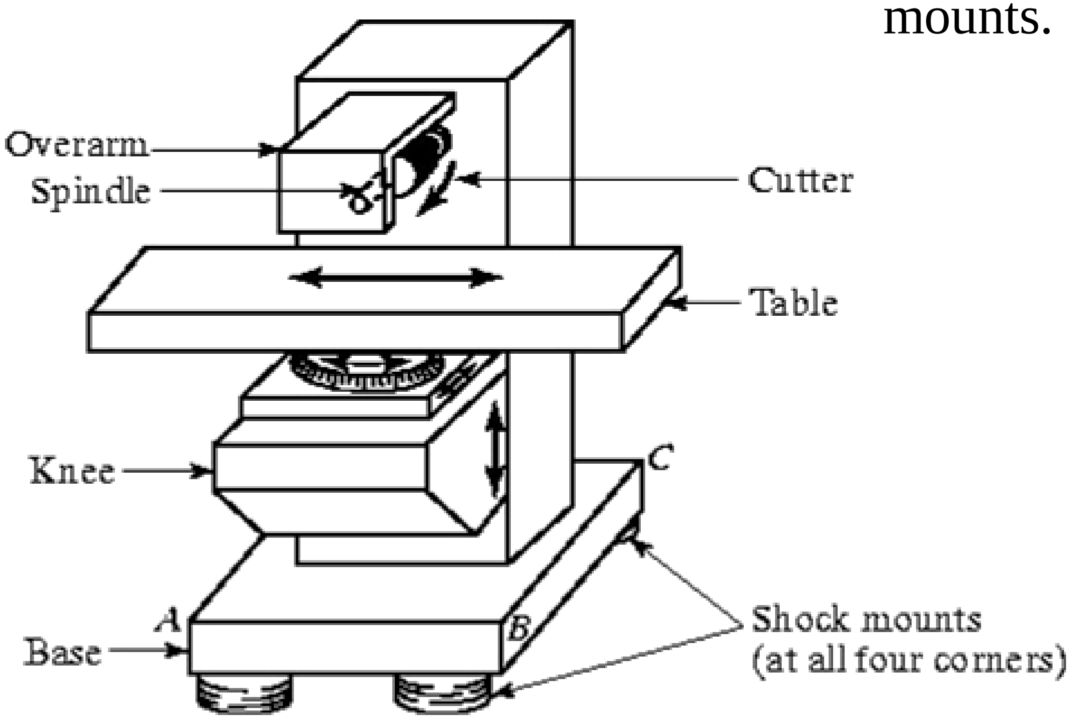
Parallel dampers: $c_{eq} = c_1 + c_2$

Series dampers: $\frac{1}{c_{eq}} = \frac{1}{c_1} + \frac{1}{c_2}$

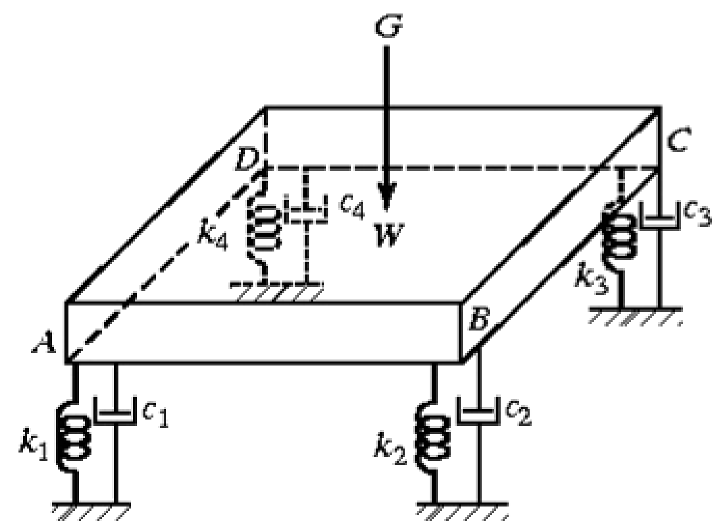
Example

Equivalent Spring and Damping Constants of a Machine Tool Support

A precision milling machine is supported on four shock mounts, as shown in Fig. The elasticity and damping of each shock mount can be modeled as a spring and a viscous damper, as shown in Fig. Find the equivalent spring constant, k_{eq}, and the equivalent damping constant, c_{eq}, of the machine tool support in terms of the spring constants (k_i) and damping constants (c_i) of the mounts.



(a)

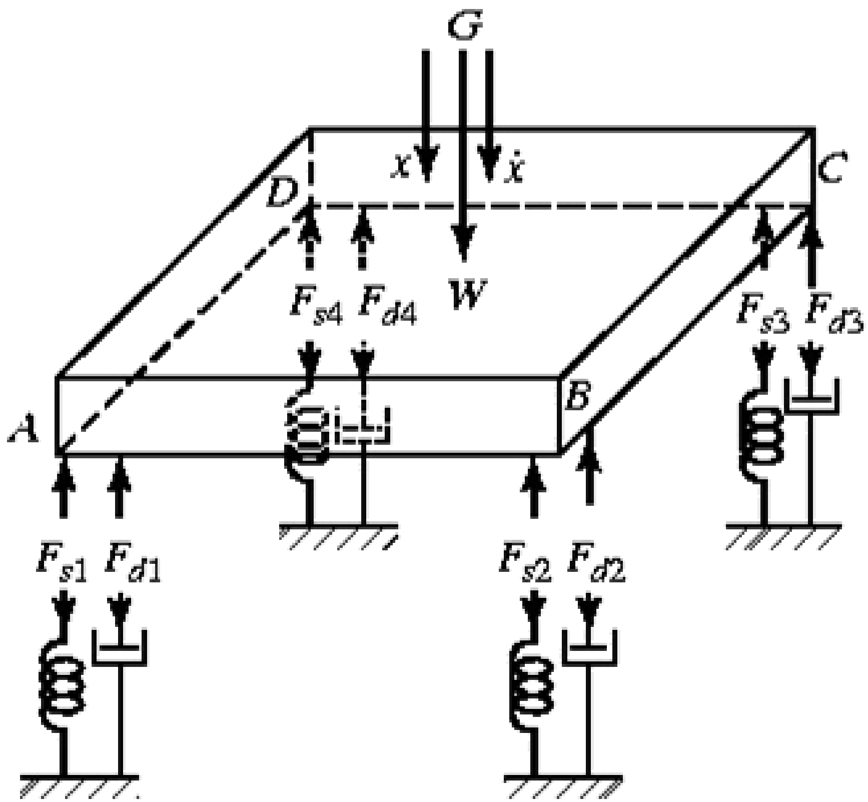


(b)

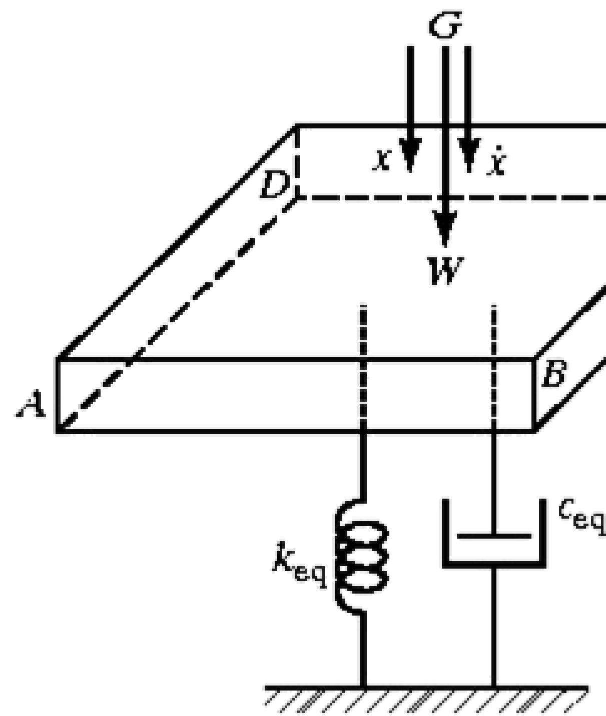
Solution

- The free-body diagrams of the four springs and four dampers are shown in Fig. Assuming that the center of mass, *G*, is located symmetrically with respect to the four springs and dampers, we notice that all the springs will be subjected to the same displacement x and all the dampers will be subject to the same relative velocity $\dot{x}$, where x and $\dot{x}$ denote the displacement and velocity, respectively, of the center of mass, *G*. Hence the forces acting on the springs (F_{si}) and the dampers (F_{di}) can be expressed as:

Solution



(c)



(d)

Solution

$$F_{si} = k_i x; \qquad i = 1,2,3,4$$

$$F_{di} = c_i \dot{x}; \qquad i = 1,2,3,4 \qquad \text{(E.1)}$$

Let the total forces acting on all the springs and all the dampers be F_s and F_d, respectively (see Fig. 1.37d). The force equilibrium equations can thus be expressed as

$$F_s = F_{s1} + F_{s2} + F_{s3} + F_{s4}$$

$$F_d = F_{d1} + F_{d2} + F_{d3} + F_{d4} \qquad \text{(E.2)}$$

Solution

where $F_s + F_d = W$, with W denoting the total vertical force (including the inertia force) acting on the milling machine. From Fig. 1.37(d), we have

$$F_s = k_{eq} x$$

$$F_d = c_{eq} \dot{x} \qquad \text{(E.3)}$$

Equation (E.2) along with Eqs. (E.1) and (E.3), yield

$$k_{eq} = k_1 + k_2 + k_3 + k_4 = 4k$$

$$c_{eq} = c_1 + c_2 + c_3 + c_4 = 4c \qquad \text{(E.4)}$$

Parallel

Where $k_i = k$ and $c_i = c$ for, $i = 1, 2, 3,4$.

The end