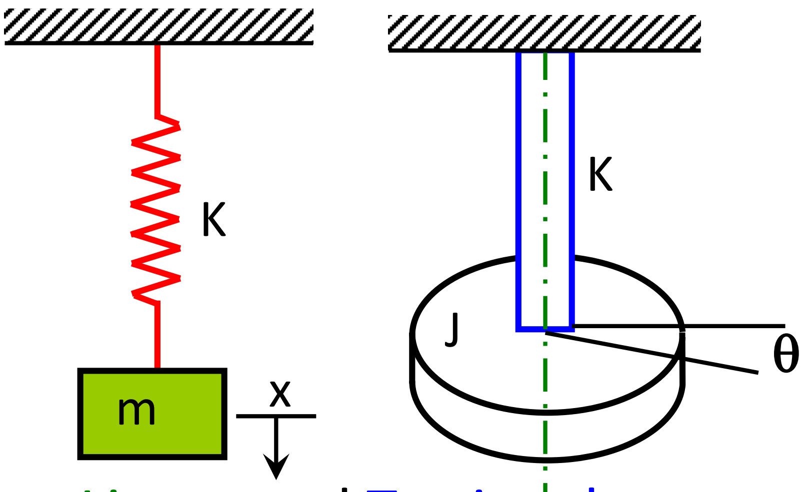
CHAPTER-V

Two Degree of Freedom

Introduction

In earlier classes we have discussed analysis of single degree of freedom systems



Linear and Torsional Vibratory systems

Degree of freedom:

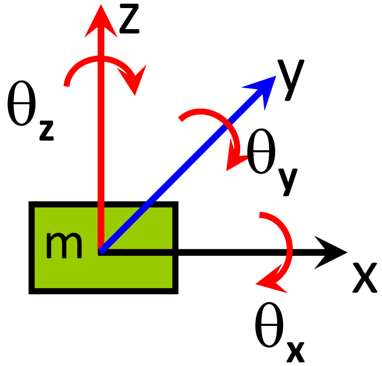
It is the number of independent co-ordinates required to describe the motion of a vibratory system

Systems with two degree of freedom

Introduction

Degree of freedom of a system =
(Number of masses in a system) **x** (number of possible types of motion of each mass

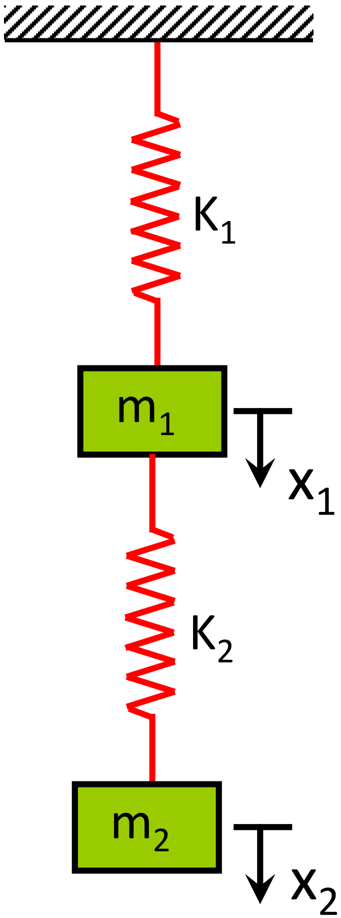
It is well known that each mass point has SIX degrees of freedom in space



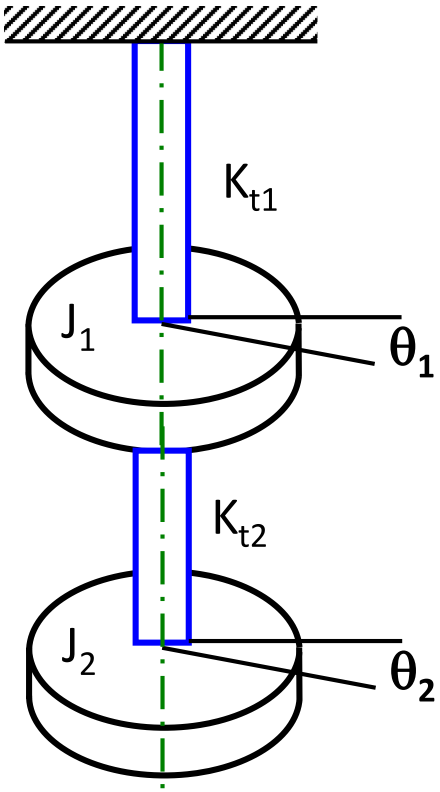
Translation: x, y and z direction
Rotation: x, y and z direction

Systems with two degree of freedom

Introduction



Linear and Torsional
2 DOF Vibratory systems



Systems with two degree of freedom

Introduction

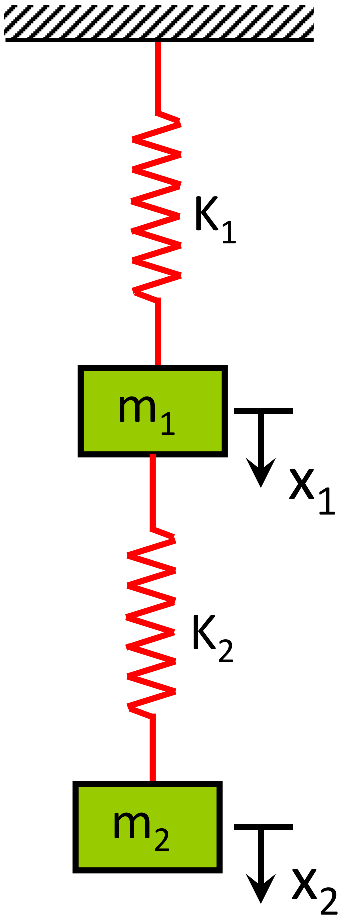
A two-degree freedom system is one that requires two coordinates to describe its motion.

These coordinates are called generalized coordinates when they are independent of each other.

A two DOF system has two equations of motion, which can be solved to obtain two natural frequencies

Systems with two degree of freedom

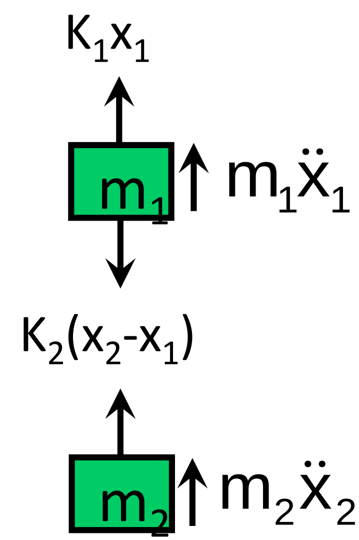
Model



Obtain the equations of the motion of the system

Governing equations

Newton's method

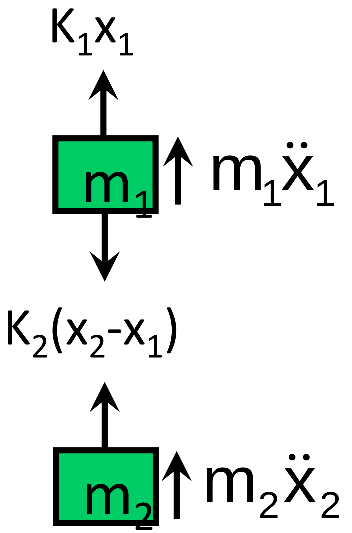


. Force equilibrium diagram

Systems with two degree of freedom

Governing equations

Newton's method



. From Force equilibrium diagram of mass m_1

$$m_1\ddot{x}_1 + K_1x_1 - K_2(x_2 - x_1) = 0$$

$$m_1\ddot{x}_1 + (K_1 + K_2)x_1 - K_2x_2 = 0$$

1st Eqn. of motion

. From Force equilibrium diagram of mass m_2

$$m_2\ddot{x}_2 + K_2(x_2 - x_1) = 0$$

$$m_2\ddot{x}_2 - K_2x_1 + K_2x_2 = 0$$

2nd Eqn. of motion

Systems with two degree of freedom

Governing equations

Matrix form

$$[M]\{\ddot{x}\} + [K]\{x\} = [0]$$

Mass matrix

generalized
force vector

generalized
acceleration vector

Stiffness
matrix

generalized
displacement vector

Systems with two degree of freedom

Solution of governing equations

It is possible to have pure harmonic free vibration for both the masses.

Let us assume

$$x_1 = A_1 \sin(\omega t + \varphi)$$

$$x_2 = A_2 \sin(\omega t + \varphi)$$

The above equations have to satisfy the governing equations of motions

$$m_1\ddot{x}_1 + (K_1 + K_2)x_1 - K_2 x_2 = 0$$ 1st Eqn. of motion

$$m_2\ddot{x}_2 - K_2 x_1 + K_2 x_2 = 0$$ 2nd Eqn. of motion

Systems with two degree of freedom

Solution of governing equations

$$\left[(K_1 + K_2) - m_1\omega^2\right]A_1\sin(\omega t + \varphi) - K_2A_2\sin(\omega t + \varphi) = 0$$

$$-K_2A_1\sin(\omega t + \varphi) - (K_2 - m_2\omega^2)A_2\sin(\omega t + \varphi) = 0$$

In above equations $\sin(\omega t + \varphi) \neq 0$

The above equations reduces to: (characteristic equation)

$$\left[(K_1 + K_2) - m_1\omega^2\right]A_1 - K_2A_2 = 0$$

$$-K_2A_1 - (K_2 - m_2\omega^2)A_2 = 0$$

Systems with two degree of freedom

Solution of governing equations

The solution of equations is:

$$\begin{vmatrix} (K_1 + K_2) - m_1\omega^2 & \cdots\cdots\cdots & -K_2 \\ -K_2 & & K_2 - m_2\omega^2 \end{vmatrix} = 0$$

The above equation is referred as a characteristic determinant

Solving, we get :

$$m_1 m_2 \omega^4 - [m_2(K_1 + K_2) + m_1 K_2]\omega^2 + K_1 K_2 = 0$$

$$\omega^4 - \left[\frac{(K_1 + K_2)}{m_1} + \frac{K_2}{m_2}\right]\omega^2 + \frac{K_1 K_2}{m_1 m_2} = 0$$

Frequency equation

Systems with two degree of freedom

Solution of governing equations

For simplification, let us consider : $K_1 = K_2 = K$

$m_1 = m_2 = m$

The frequency equation changes to:

$$\omega^4 - \left[\frac{2K}{m} + \frac{K}{m}\right]\omega^2 + \frac{K^2}{m^2} = 0$$

$$\omega^4 - \left[\frac{3K}{m}\right]\omega^2 + \frac{K^2}{m^2} = 0$$

put $\omega^2 = \lambda$, in the above equation:

Systems with two degree of freedom

Solution of governing equations

$$\lambda^2 - \left[\frac{3K}{m}\right]\lambda + \frac{K^2}{m^2} = 0 \qquad \lambda_1, \lambda_2 = \omega_1^2, \omega_2^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda_1, \lambda_2 = \omega_1^2, \omega_2^2 = \frac{\frac{3K}{m} \pm \sqrt{\left(\frac{3K}{m}\right)^2 - \frac{4K^2}{m^2}}}{2}$$

$$\omega_1^2, \omega_2^2 = \frac{\frac{3K}{m} \pm \sqrt{\frac{5K^2}{m^2}}}{2} \qquad \omega_1^2, \omega_2^2 = \frac{\left(3 \pm \sqrt{5}\right)\frac{K}{m}}{2}$$

Systems with two degree of freedom

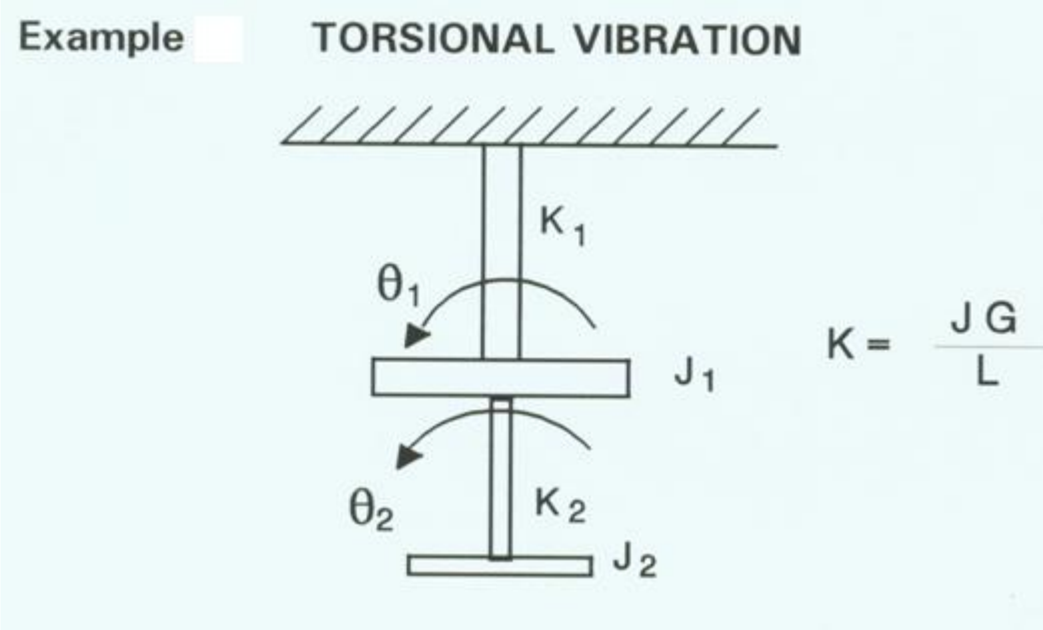
Solution of governing equations

$$\omega_1 = \sqrt{\frac{\left(3-\sqrt{5}\right)\frac{K}{m}}{2}} \qquad \omega_2 = \sqrt{\frac{\left(3+\sqrt{5}\right)\frac{K}{m}}{2}}$$

$$\omega_1 = 0.61\omega_n \qquad \omega_2 = 1.61\omega_n \text{ radians/sec}$$

As the system has two natural frequencies, under certain conditions it may vibrate with first or second frequency, which are referred as principal modes of vibration

Torsional Vibration



Equations of motion for θ_1:

$$J_1\ddot{\theta}_1 = -K_1\theta_1 - K_2(\theta_1 - \theta_2)$$

$$J_1\ddot{\theta}_1 + K_1\theta_1 + K_2(\theta_1 - \theta_2) = 0$$

Torsional Vibration

Equations of motion for θ_2:

$$J_2\ddot{\theta}_2 = -K_2(\theta_2 - \theta_1)$$

$$J_2\ddot{\theta}_2 + K_2(\theta_2 - \theta_1) = 0$$

Assume: $\theta_1 = A\,\mathrm{Cos}(\omega t - \phi)$ $\theta_2 = B\,\mathrm{Cos}(\omega t - \phi)$

Differentiating the above equations twice:

$$\ddot{\theta}_1 = -\omega^2 A\,\mathrm{Cos}(\omega t - \phi) \qquad \ddot{\theta}_2 = -\omega^2 B\,\mathrm{Cos}(\omega t - \phi)$$

Torsional Vibration

Substituting equations the displacement equations the EOM yields the following.

$$-J_1\omega^2 A + (K_1 + K_2)A - K_2 B = 0$$

$$-J_2\omega^2 B + K_2 B - K_2 A = 0$$

Find the solutions to the two independent frequencies by solving the determinant and finding the Eigenvalues.

$$\begin{bmatrix} (K_1 + K_2 - J_1\omega^2) & -K_2 \\ -K_2 & (K_2 - J_2\omega^2) \end{bmatrix} = 0$$

$$K_1 K_2 + K_2^2 - K_1 J_2\omega^2 - K_2 J_2\omega^2 - K_2 J_1\omega^2 + J_1 J_2\omega^4 - K_2^2 = 0$$

$$J_1 J_2\omega^4 - (K_1 + K_2)J_2\omega^2 - K_2 J_1\omega^2 + K_1 K_2 = 0$$

Torsional Vibration

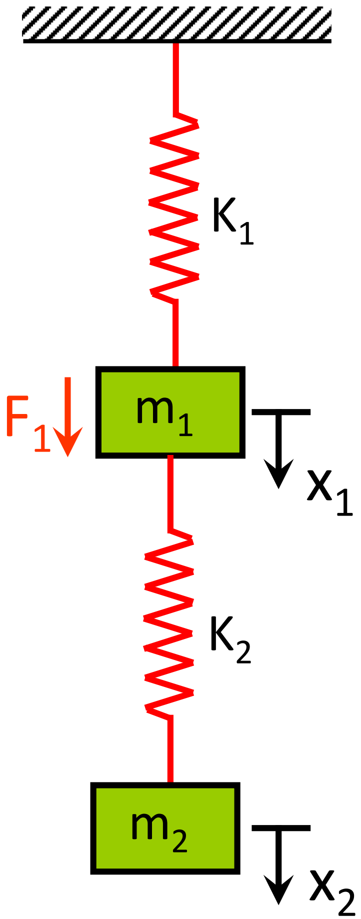
$$\omega^4 - \left[\frac{(K_1 + K_2)}{J_1} + \frac{K_2}{J_2}\right]\omega^2 + \frac{K_1 K_2}{J_1 J_2} = 0$$

Solve for the Eigen Values:

$$\lambda = \omega^2 \qquad \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Systems with two degree of freedom

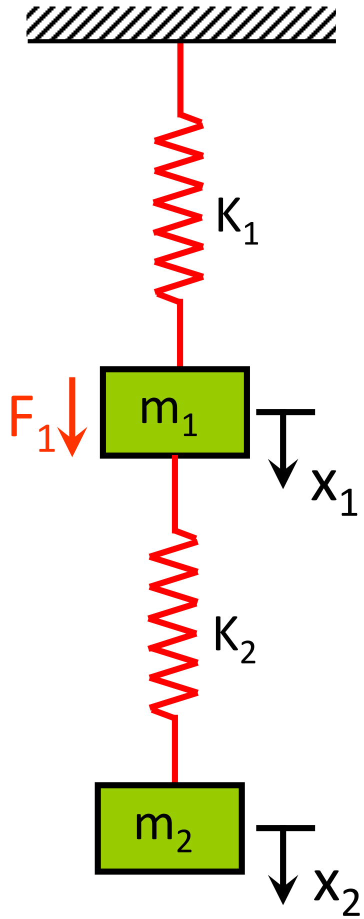
Forced Vibration



$$F_1 = F_0 \sin\omega t$$

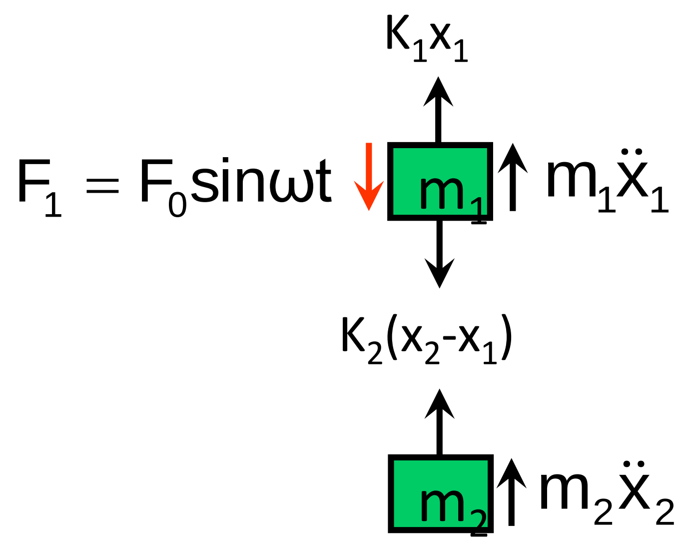
Systems with two degree of freedom

Problem-1



Governing equations

Newton's method

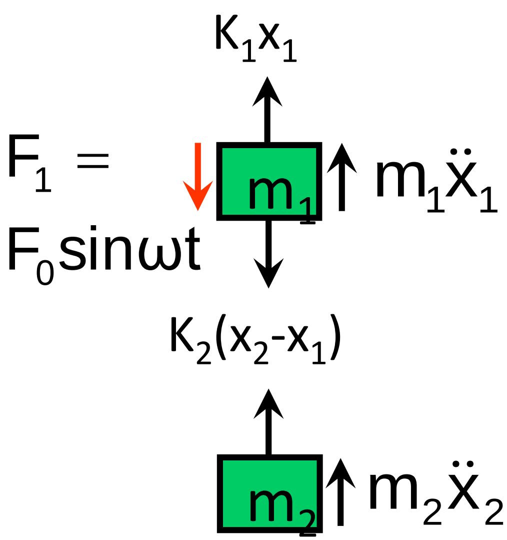


. Force equilibrium diagram

Systems with two degree of freedom

Governing equations

Newton's method



. From Force equilibrium diagram of mass m_1

$$m_1\ddot{x}_1 + K_1x_1 - K_2(x_2 - x_1) = F_0\sin\omega t$$

$$m_1\ddot{x}_1 + (K_1 + K_2)x_1 - K_2x_2 = F_0\sin\omega t$$

1st Eqn. of motion

. From Force equilibrium diagram of mass m_2

$$m_2\ddot{x}_2 + K_2(x_2 - x_1) = 0$$

$$m_2\ddot{x}_2 - K_2x_1 + K_2x_2 = 0$$

2nd Eqn. of motion

Systems with two degree of freedom

Solution of governing equations

It is possible to have pure harmonic free vibration for both the masses.

Let us assume

$$x_1 = A_1 \sin\omega t$$

$$x_2 = A_2 \sin\omega t$$

where ω is the forcing frequency

Be the forced response of the system, where A_1 and A_2 are the amplitudes to be obtained

The above equations have to satisfy the governing equations of motions

$$m_1\ddot{x}_1 + (K_1 + K_2)x_1 - K_2 x_2 = F_0 \sin\omega t$$

1st Eqn. of motion

$$m_2\ddot{x}_2 - K_2 x_1 + K_2 x_2 = 0$$

2nd Eqn. of motion

Systems with two degree of freedom

Solution of governing equations

$$\left[(K_1 + K_2) - m_1\omega^2\right]A_1\sin(\omega t + \varphi) - K_2A_2\sin(\omega t + \varphi) = F_0\sin\omega t$$

$$-K_2A_1\sin(\omega t + \varphi) - (K_2 - m_2\omega^2)A_2\sin(\omega t + \varphi) = 0$$

In above equations $\sin(\omega t + \varphi) \neq 0$

The above equations reduces to: (characteristic equation)

$$\left[(K_1 + K_2) - m_1\omega^2\right]A_1 - K_2A_2 = F_0\sin\omega t$$

$$-K_2A_1 - (K_2 - m_2\omega^2)A_2 = 0$$

Systems with two degree of freedom

Solution of governing equations

$$\left[(K_1 + K_2) - m_1\omega^2\right]A_1 - K_2A_2 = F_0\sin\omega t$$

$$-K_2A_1 - (K_2 - m_2\omega^2)A_2 = 0$$

Solution of governing equations

From above equations obtain A_1 and A_2 by Cramer's rule

$$A_1 = \frac{\begin{vmatrix} .F_0 & -K_2 \\ .0 \;\cdots\cdots & K_2 - m_2\omega^2 \end{vmatrix}}{\Delta}$$

$$A_2 = \frac{\begin{vmatrix} (K_1 + K_2) - m_1\omega^2 & F_0. \\ -K_2 & \cdots\cdots\; 0 \end{vmatrix}}{\Delta}$$

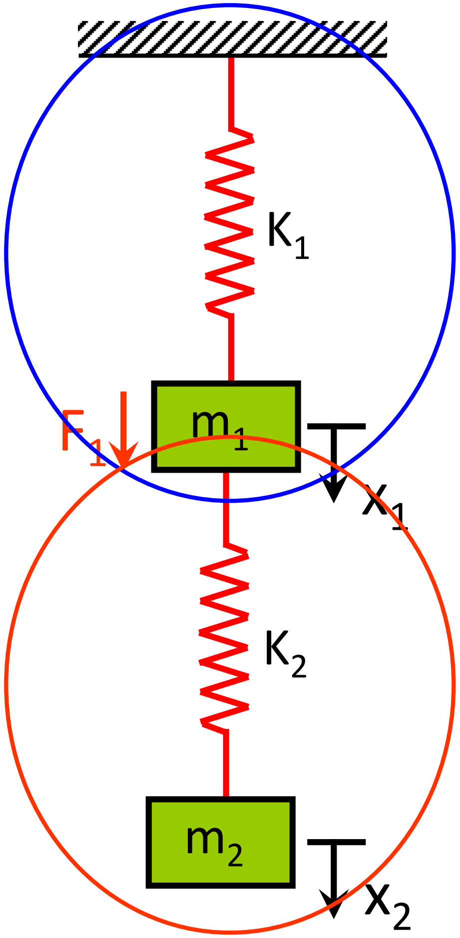
where Δ is the determinant of characteristic equations

$$\Delta = \begin{vmatrix} (K_1 + K_2) - m_1\omega^2 & -K_2 \\ -K_2 & \cdots\cdots\; K_2 - m_2\omega^2 \end{vmatrix} \neq 0$$

$$\Delta = \left\{(K_1 + K_2) - m_1\omega^2\right\}\left\{K_2 - m_2\omega^2\right\} - K_2^2$$

Systems with two degree of freedom

Dynamic vibration absorber



Main system having forced vibration

Absorber system

The system can be used as Dynamic vibration absorber (Tuned damper)

Choose K_2, m_2 such that vibrations of mass m_1 is minimized

Systems with two degree of freedom

Dynamic vibration absorber

Undesired maximum vibrations can occur only when main system is under resonance, or near it i.e $\omega_1 = \omega$

So, to reduce undesired vibration of main system mass m_1, we choose K_2, m_2 in such a way that $\omega_1 = \omega_2$

$$\sqrt{\frac{K_1}{m_1}} = \sqrt{\frac{K_2}{m_2}} \quad \textbf{OR} \quad \frac{K_1}{m_1} = \frac{K_2}{m_2}$$